A note on a conjecture of Gyárfás

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Abstract

This note proves that, given one member, T, of a particular family of radius-three trees, every radius-two, triangle-free graph, G, with large enough chromatic number contains an induced copy of T.

1 Introduction

A ground-breaking theorem by Erdős [1] states that for any positive integers χ and g, there exists a graph with chromatic number at least χ and girth at least g. This has an important corollary. Let H be a fixed graph which contains a cycle and let χ_0 be a fixed positive integer. Then there exists a G such that $\chi(G) > \chi_0$ and G does not contain H as a subgraph.

Gyárfás [2] and Sumner [9] independently conjectured the following:

Conjecture 1.1. *For every integer k and tree T there is an integer $f(k,T)$ such that every G with*

$$\omega(G) \leq k \qquad \text{and} \qquad \chi(G) \geq f(k,T)$$

contains an induced copy of T.

Of course, an acyclic graph need not be a tree. But, Conjecture 1.1 is the same if we replace T, by F where F is a forest. Suppose $F = T_1 + \cdots + T_p$ where each T_i is a tree, then we can see by induction on both k and p that

$$f(k,F) \leq 2p + |V(F)| f(k-1,F) + \max_{1 \leq i \leq p} \{f(k,T_i)\}.$$

A similar proof is given in [4]. Thus, it is sufficient to prove Conjecture 1.1 for trees, as stated.

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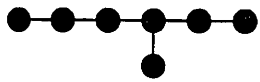

Figure 1: Kierstead-Penrice's T

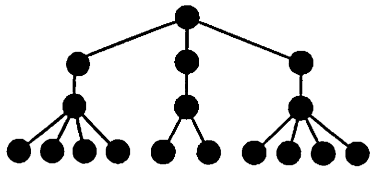

Figure 2: A radius three tree covered in [7].

1.1 Current Progress

The first major progress on this problem came from Gyárfás, Szemerédi and Tuza [3] who proved the case when $k = 3$ and T is either a radius two tree or a so-called "mop." A mop is a graph which is path with a star at the end. Kierstead and Penrice [4] proved the conjecture for $k = 3$ and when T is the graph in Figure 1.

The breakthrough for $k > 3$ came through Kierstead and Penrice [5], where they proved that Conjecture 1.1 is true if T is a radius two tree and k is any positive integer. This result contains the one in [3]. Furthermore, Kierstead and Zhu [7] prove the conjecture true for a certain class of radius three trees. These trees are those with all vertices adjacent to the root having degree 2 or less. A good example of such a tree is in Figure 2. The paper [7] contains the result in [4].

Scott [8] proved the following theorem:

Theorem 1.2 (Scott). *For every integer k and tree T there is an integer $f(k,T)$ such that every G with $\omega(G) \leq k$ and $\chi(G) \geq f(k,T)$ contains a subdivision of T as an induced subgraph.*

Theorem 1.2 results in an easy corollary:

Corollary 1.3 (Scott). *Conjecture 1.1 is true if T is a subdivision of a star and k is any positive integer.*

Kierstead and Rodl [6] discuss why Conjecture 1.1 does not generalize well to directed graphs.

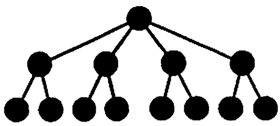

Figure 3: $T(4,2)$

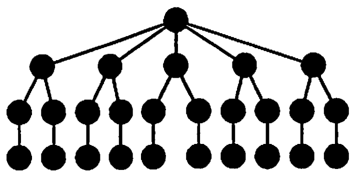

Figure 4: $T(5,2,1)$

2 The Theorem

In order to prove the theorem, we must define some specific trees. In general, let $T(a,b)$ denote the radius two tree in which the root has a children and each of those children itself has exactly b children. (Thus, $T(a,b)$ has $1+a+ab$ vertices.) In particular, $T(t,2)$ is the radius two tree for which the root has t children and each neighbor of the root has 2 children. Figure 3 gives a drawing of $T(4,2)$. Let $T(t,2,1)$ be the radius three tree in which the root has t children, each neighbor of the root has 2 children, each vertex at distance two from the root has 1 child and each vertex at distance three from the root is a leaf. Figure 4 gives a drawing of $T(5,2,1)$.

This allows us to state the theorem:

Theorem 2.1. *Let t be a positive integer. There exists a function f, such that if G is a radius two graph with no triangles and $\chi(G) > f(t)$, then G must have $T(t,2,1)$ as an induced subgraph.*

Proof. We will let r be the root of G and let $S_1 = S(r,1)$ be the neighbors of r and $S_2 = S(r,2)$ be the second neighborhood of r. We will try to create a $T(t,2,1)$ with a root r vertex by vertex. We look for a $v_1 \in S_1$ with the property that there exist $w_{1a}, w_{1a} \in N_{S_2}(v_1)$ as well as $x_{1a} \in N_{S_2}(w_{1a}) \setminus N_{S_2}(w_{1b}) \neq \emptyset$ and $x_{1b} \in N_{S_2}(w_{1b}) \setminus N_{S_2}(w_{1a}) \neq \emptyset$ such that $x_{1a} \not\sim x_{1b}$. So, clearly, $\{v_1, w_{1a}, w_{1b}, x_{1a}, x_{1b}\}$ induce the tree $T(2,1)$. Let us remove the following vertices from G to create G_2:

$$\{v_1, w_{1a}, w_{1b}, x_{1a}, x_{1b}\} \cup N_{S_2}(v_1) \cup N(w_{1a}) \cup N(w_{1b}) \cup N(x_{1a}) \cup N(x_{1b}).$$

Since G has no triangles, the graph induced by these vertices has chromatic number at most 4.[1] Thus, $\chi(G_2) \geq \chi(G) - 4$.

[1]One such coloring is (1) $N_{S_2}(w_{1a}) \cup N_{S_2}(x_{1a})$, (2) $N_{S_2}(w_{1b}) \cup N_{S_2}(x_{1b})$, (3) $N_{S_2}(v_1)$ and (4) $\{v_1, x_{1a}, x_{1b}\}$.

We continue to find $v_2, \ldots, v_s$ from each of $G_2, \ldots, G_s$ in the same manner with $s < t$ so that G has an induced $T(s, 2, 1)$ rooted at r. We also have a G_{s+1} so that $\chi(G_{s+1}) \geq \chi(G) - 4s$. If we can continue this process to the point that $s = t$, we have our $T(t, 2, 1)$ rooted at r. So, let us suppose that the process stops for some $s < t$. From this point forward, S_1 will actually denote $S_1 \cap V(G_{s+1})$ and S_2 will denote $S_2 \cap V(G_{s+1})$.

Furthermore, in the graph G_{s+1}, each vertex $v_1 \in S_1$ has the following property: For any $w_{1a}, w_{1b} \in N(v_1)$, the pair

$$(N_{S_2}(w_{1a}) \setminus N_{S_2}(w_{1b}), N_{S_2}(w_{1b}) \setminus N_{S_2}(w_{1a}))$$

induces a complete bipartite graph. If this were not the case, then we could find the x_{1a} and x_{1b} that we need.

Consider this property in reverse. Let $v \in S_1$ and $z_1, z_2 \in S_2 \setminus N_{S_2}(v)$. Then the two sets $N_{S_2}(v) \cap N(z_1)$ and $N_{S_2}(v) \cap N(z_2)$ have the property that one is inside the other or they are disjoint. As a result, $N_{S_2}(v)$ has two nonempty subsets such that any $z \in S_2 \setminus N_{S_2}(v)$ has the property that $N_{S_2}(v) \cap N(z)$ contains either one subset or the other.

So, for each $v \in S_2$, there exists some (not necessarily unique and not necessarily distinct) pair of vertices, $w_a(v), w_b(v) \in N_{S_2}(v)$ such that for all $z \in S_2$, if z is adjacent to some member of $N_{S_2}(v)$ then either $z \sim w_a(v)$ or $z \sim w_b(v)$ or both.

For every $v \in S_1$, find such vertices and label them, arbitrarily as $w_a(v)$ or $w_b(v)$, recognizing that a vertex can have many labels. Now form the graph H^* induced by vertices from among those labelled as some $w_a(v)$ or $w_b(v)$. Find a minimal induced subgraph H so that if $h^* \in V(H^*)$, then there exists $h \in V(H)$ such that $N_{S_2}(h^*) \subseteq N_{S_2}(h)$.

We have a series of claims that end the proof:

Claim 1. $\chi(H) = \chi(S_2)$.
Proof of Claim 1. Since H is a subgraph of S_2, $\chi(H) \leq \chi(S_2)$. If we properly color H with $\chi(H)$ colors, then we can extend this to a coloring of S_2. We do this by giving $z \in S_2$ the same color as that of some $h \in V(H)$ with the property that $N_{S_2}(z) \subseteq N_{S_2}(h)$.

This is possible first because there must be some $h^* = w_A(v)$ or $h^* = w_B(v)$ in H^* with $N_{S_2}(z) \subseteq N_{S_2}(h^*)$. Further, there is an h such that $N_{S_2}(h^*) \subseteq N_{S_2}(h)$. So, $N_{S_2}(z) \subseteq N_{S_2}(h)$. Now suppose z_1 and z_2 are given the same color but are adjacent. Let h_1 and h_2 be the vertices in H whose neighborhoods dominate those of z_1 and z_2, respectively and whose colors z_1 and z_2 inherit. Because $z_1 \sim z_2$, $h_1 \sim z_2$ and $h_2 \sim z_1$. But then it must also be the case that

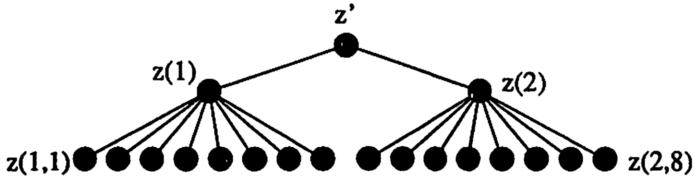

Figure 5: $T(2,8)$ with some vertices labelled

$h_1 \sim h_2$. Thus, h_1 and h_2 cannot receive the same color, a contradiction. ■

Claim 2. H induces a $T(2t+1,8)$.
Proof of Claim 2. Because S_1 is an independent set, $\chi(S_2) \geq \chi(G_{s+1}) - 1$. Because $\chi(G)$, hence $\chi(G_{s+1})$, is large, Claim 1 ensures that $\chi(H)$ is large. Claim 2 results from [3], because $T(2t+1,8)$ is a radius-two tree. ■

Let the tree T, guaranteed by Claim 2, have root z', its children be labelled $z(1),\ldots,z(2t+1)$ and the children of each $z(i)$ be labelled $z(i,1),\ldots,z(i,8)$. Figure 5 shows one such tree.

Claim 3. If $v \in S_1$ is adjacent to $z(i,j)$, then v cannot be adjacent to any other vertices of T except one other vertex $z(i,j')$ or z'.
Proof of Claim 3. If $v \in S_1$ is adjacent to, say, $z(1,1)$, then $v \not\sim z(i,j)$ if $i \neq 1$. This is because $N_{S_2}(w_A(v)) \Delta N_{S_2}(w_B(v))$ induces a complete bipartite graph which would imply an edge between $z(1)$ and $z(i)$.

It can be shown, for similar reasons, that if $v \sim z(1,1)$, then $v \not\sim z(i)$ for any $i \neq 1$. Also, $v \not\sim z(1)$ because G is triangle-free. ■

Claim 4. We may assume that there is a $v_1 \in S_1$ that is adjacent to (without loss of generality) $z(1,1)$ as well as z'.
Proof of Claim 4. We prove this by contradiction. Applying Claim 3 to every leaf of T, we see that since Claim 4 is not true, then for $i = 1,\ldots,2t+1$, we can find a set of 4 vertices of the form $z(i,j)$ and 4 vertices from S_1 so that they induce a perfect matching. Furthermore, the $4(2t+1)$ vertices from S_1 are each adjacent to no other vertices of T, because of Claim 3. Hence, we have our induced $T(t,2,1)$, a contradiction. ■

Because our definition of H guaranteed that vertices had neighborhoods that were not nested, there must be some $z'' \in S_2$ that is adjacent to $z(1,1)$ but not z'. Call this vertex z''.

Claim 5. For any $z(i,j)$ with $i \neq 1$ and any $v \in S_1$ adjacent to $z(i,j)$, v cannot be adjacent to both z' and z''.
Proof of Claim 5. We again proceed by contradiction, supposing that $v \sim z(i,j), z', z''$. There is, without loss of generality, $w_a(v) \in N_{S_2}(v)$ such that

$N_{S_2}(z'') \subseteq N_{S_2}(w_a(v))$. Thus, either $N_{S_2}(z') \subseteq N_{S_2}(w_a(v))$ or $N_{S_2}(z(i,j)) \subseteq N_{S_2}(w_a(v))$. But if $w_a(v)$ were deleted from H^* to form H, either z' or $z(i,j)$ would have been deleted as well.

Therefore, either $w_a(v) = z'$ or $w_a(v) = z(i,j)$. So, $N_{S_2}(z'') \subseteq N_{S_2}(z')$ or $N_{S_2}(z'') \subseteq N_{S_2}(z(i,j))$. We can conclude that either $z' \sim z(1,1)$ or $z(i,j) \sim z(1,1)$. This contradicts the fact that T is an induced subtree. ■

Claim 6. For all $i \neq 1$, z'' is adjacent to $z(i)$ but no vertex $z(i,j)$.
Proof of Claim 6. Note that $z(2), \ldots, z(2t+1)$ are adjacent to z' but not $z(1,1)$. Because of the condition that $N_{S_2}(z') \Delta N_{S_2}(z(1,1))$ induces a complete bipartite graph, z'' must be adjacent to $z(2), \ldots, z(2t+1)$. Because G is triangle-free, z'' cannot be adjacent to any vertex of the form $z(i,j)$ where $i \neq 1$. ■

Now we construct the tree we need. For each $z(i,j)$, $i \neq 1$, find a vertex $v(i,j) \in S_1$ to which $z(i,j)$ is adjacent. According to Claim 3, no $v(i,j)$ vertex can be adjacent to any vertex of $V(T) \setminus \{z'\}$ and, according to Claim 5, it is adjacent to at most one of $\{z', z''\}$.

For each $i \in \{2, \ldots, 2t+1\}$, the majority of $\{v(i,1), \ldots, v(i,8)\}$ have that $v(i,j)$ is either nonadjacent to z' or nonadjacent to z''. Without loss of generality, we conclude that z' has the property that, for $i = 2, \ldots, t+1$, the vertices $v(i,1), \ldots, v(i,4)$ fail to be adjacent to z'.

Since any vertex of S_1 can be adjacent to at most two vertices of H, then for $i = 2, \ldots, t+1$, $|\{v(i,1), \ldots, v(i,4)\}| \geq 2$. Therefore, we assume that for each $i \in \{2, \ldots, t+1\}$, $v(i,1)$ and $v(i,2)$ are distinct. But now the vertex set

$$\{z'\} \cup \bigcup_{i=2}^{t+1} (\{z(i), z(i,1), z(i,2), v(i,1), v(i,2)\})$$

induces $T(t,2,1)$. ■

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