Characterizing Intersection Graphs of Substars of a Star*

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Abstract

Starlike graphs are the intersection graphs of substars of a star. We describe different characterizations of starlike graphs, including one by forbidden subgraphs. In addition, we present characterizations for a natural subclass of it, the starlike-threshold graphs.

Key words: graph classes, forbidden subgraphs, intersection graphs, interval graphs, starlike graphs, starlike-threshold graphs.

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1 Introduction

Graph classes and intersection graphs are traditional topics in graph theory. In fact, theses studies have been receiving much attention, recently. For example, we mention the books by Brandstädt, Le and Spinrad [1] and by McKee and McMorris [12].

Chordal graphs form one of the most well studied classes of graphs. In particular, they can be considered as special intersection graphs (Buneman [2], Gavril [7], Walter [19]). Several subclasses of chordal graphs have been examined in the literature. In the present paper, we describe characterizations for two other classes of chordal graphs, namely starlike graphs and starlike-threshold graphs. Starlike graphs were introduced by Gustedt [8], in the study of the pathwidth problem for chordal graphs. Further, this class has been considered by Cerioli and Szwarcfiter [3], Hsieh [9], Moscarini et al. [14], Peng et al. [16]. See also McMorris and Shier [13] and Prisner [17]. Starlike-threshold graphs arise naturally when studying edge clique graphs of threshold graphs [3]. Furthermore, they generalize threshold graphs in a similar way as starlike graphs generalize split graphs.

In particular, we also present characterizations by forbidden subgraphs for these two considered classes. We remark that there are characterizations of this type for several subclasses of chordal graphs. For example, interval graphs (Lekkerkerker and Boland [11]), proper interval graphs (Roberts [18]), strongly chordal graphs (Farber [5]), split graphs (Földes and Hammer [6]), threshold graphs (Chvátal and Hammer [4]). More recently, such a characterization has been also described for directed path graphs (Panda [15]). Nevertheless, remain open the problems of finding forbidden subgraph characterizations for both undirected and rooted directed path graphs.

Finally, we also show that starlike-threshold graphs are interval graphs. Moreover, they admit an interval representation with at most two sizes of intervals.

All graphs considered are finite and simple. The vertex and edge sets of an undirected graph G are represented by $V(G)$ and $E(G)$, respectively. A vertex is ***universal*** if it is adjacent to all the other vertices. Two adjacent vertices which are adjacent exactly to the same vertices of G are called ***twins***. For $C \subseteq V(G)$, say that C is a ***clique*** when C induces a complete subgraph in G. A ***maximal clique*** is one not properly contained in any other. A vertex v is *simplicial* when the set of its adjacent vertices is a clique, i.e. when $N(v)$ is a clique. If C is a set of vertices, $N(C)$ denotes the set of all vertices that are not in C and are adjacent to some vertex in C. An (induced) path with three vertices is an *obstacle* in a graph when all its vertices are non simplicial.

A *chordal graph* is the intersection graph of a family of subtrees of some tree. A *star* S is a tree having one universal vertex, called the *center of* S. A star is *trivial* when it consists of a sole vertex. A subgraph of a star which is itself a star is called a *substar*. A *starlike graph* G is the intersection graph of a family of substars S_i of a star. In particular, if the vertex sets $V(S_i)$ of the non trivial substars S_i can be linearly ordered by inclusion, then G is a *starlike-threshold graph.*

Characterizations for the classes of starlike and starlike-threshold graphs are described in Sections 2 and 3, respectively.

2 Starlike Graphs

The following proposition describes characterizations of starlike graphs.

Theorem 1 *Let G be a graph. The following affirmatives are equivalent:*

(i) *G is starlike.*

(ii) *G admits a clique C, such that every two vertices outside C are either twins or non adjacent.*

(iii) *G contains neither a $2P_3$ nor an obstacle.*

(iv) *The set of non simplicial vertices of G is a clique.*

Proof:
(i) $\Rightarrow$ (ii): Let G be a starlike graph. By definition, G is the intersection graph of substars S_i of a star S. Let $c \in V(S)$ be the center of S and $v_i \in V(G)$ the vertex of G corresponding to S_i. Denote by C the subset of $V(G)$ corresponding to the subtrees S_i containing c. Clearly, C is a clique. We show that C satisfies condition (ii). Choose $v_j, v_k \in V(G) \setminus C$. If $v_j v_k \in E(G)$ then $V(S_j) \cap V(S_k) \neq \emptyset$. On the other hand, we know that $c \notin V(S_j) \cup V(S_k)$. Consequently, S_j and S_k must be trivial stars. Moreover, $S_j = S_k$. Therefore, v_j and v_k are twins.

(ii) $\Rightarrow$ (iii): Let G be a graph containing a clique C, satisfying (ii). We show that the existence of either a $2P_3$ or an obstacle leads to a contradiction.

Suppose G contains a $2P_3$ formed by the paths v_i, v_j, v_k and v_a, v_b, v_c. Because v_j and v_b are non adjacent, one of them, say v_j, is not in C. Similarly, we conclude that either v_i and v_k, say v_i, is also not in C. In this situation, v_i and v_j are adjacent and not twins, a contradiction. Consequently, G does not contain $2P_3$'s.

For the second part of the assertion, suppose that G contains an obstacle v_i, v_j, v_k. Since v_i and v_k are not adjacent, they can not be both in C. Let $v_i \notin C$. Because v_i and v_j are adjacent and not twins, at least one of them must belong to C, so $v_j \in C$. In the sequel, suppose that v_i has a neighbour $v_a \neq v_j$. It follows that v_a and v_j must be adjacent. Because if $v_a \in C$ then $v_j \in C$ implies $v_a v_j \in E(G)$, whereas when $v_a \notin C$, v_a and v_i must be twins, again implying $v_a v_j \in E(G)$. In this situation, v_i is a simplicial vertex, a contradiction.

(iii) $\Rightarrow$ (iv): By hypothesis, G contains neither a $2P_3$ nor an obstacle. By contradiction, assume that the set C of non simplicial vertices of G is not a clique. Then we can choose vertices v_i and v_k which are both non adjacent and non simplicial.

First, consider the situation where v_i and v_k belong to distinct connected components of G. Since they are non simplicial, both are centers of P_3's. In this case, a P_3 containing v_i and another containing v_k form a $2P_3$, a contradiction.

In the sequel, examine the case where v_i and v_k belong to a same connected component. Let P be a shortest path between v_i and v_k. It follows that every internal vertex v_j of P must be non simplicial. Because, if v_j is simplicial then its neighbours in P would be adjacent, contradicting P being a shortest path. Since v_i and v_k are non adjacent, P has length ≥ 3. A contradiction arises because any three consecutive vertices of P form an obstacle.

Consequently, C is indeed a clique.

(iv) $\Rightarrow$ (i): By hypothesis G is a graph in which the subset $C \subseteq V(G)$ of its non simplicial vertices form a clique. Define a star S as follows. The center c of S corresponds to C, while every other vertex s' of S corresponds to a maximal clique C' of $G - C$. Define a family of subgraphs S_i of S as follows. There is a one-to-one correspondence between vertices $v_i \in V(G)$ and subgraphs S_i of S. Moreover, each S_i is formed by the vertices of S

which correspond to the cliques of G containing v_i. We show that S_i is a star. If the latter is false, it follows that $c \notin V(S_i)$ and S_i contains a pair of non adjacent vertices $s', s'' \in V(S)$. Then v_i is contained in a pair of distinct cliques c', c'' of G. The latter implies that v_i is not simplicial. Consequently, $v_i \in C$, contradicting $c \notin V(S_i)$. Hence S_i is a star.

It remains to show that G is the intersection graph of the substars S_i. Suppose $S_i \cap S_j \neq \emptyset$. Then v_i and v_j are contained in a common clique, implying $v_iv_j \in E(G)$, and conversely. This completes the proof. ■

The equivalence (i) ⇔ (iv) of the above theorem is central to next study.

The following theorem describes a characterization of starlike graphs by forbidden subgraphs.

Theorem 2 *A graph G is starlike if and only if G does not contain any of the six graphs of Figure 1 as an induced subgraph.*

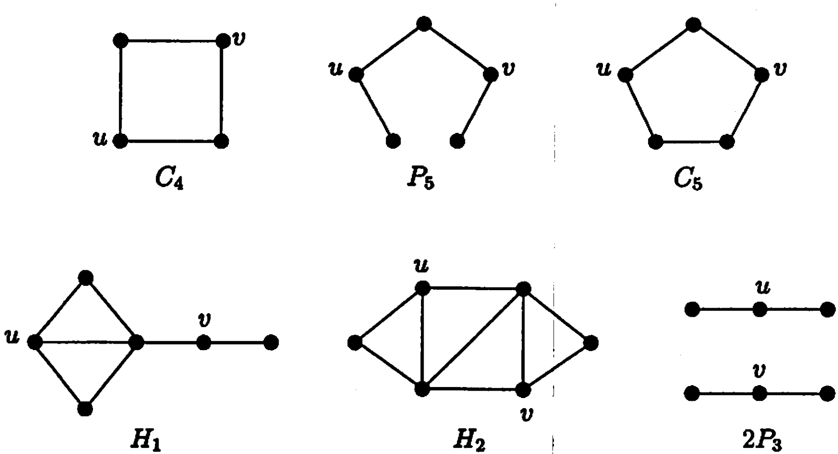

Figure 1: The forbidden subgraphs for starlike graphs.

Proof: Let G be a starlike graph. By Theorem 1, the set of its non simplicial vertices form a clique of G. However, u and v are non simplicial and non adjacent vertices in all the six graphs of Figure 1. Consequently, they can not be induced subgraphs of G.

Conversely, let G be a graph which does not contain any of the graphs of Figure 1, as an induced subgraph. Toward a contradiction assume G is not starlike. By Theorem 1, G contains a pair of vertices v_j, v_b which are both non adjacent and non simplicial. The latter implies that v_b and v_j are centers of two P_3's of G denoted v_i, v_j, v_k and v_a, v_b, v_c, respectively.

Next, we analyse the possibilities of the subgraphs of G induced by the subset $H = \{v_i, v_j, v_k, v_a, v_b, v_c\}$. We know that $v_iv_j, v_jv_k, v_av_b, v_bv_c \in E(G)$ while $v_jv_b, v_iv_k, v_av_c \notin E(G)$. Denote by F the subset formed by the remaining pairs of vertices of H. We consider the possible inclusion as edges of G of the pairs belonging to F. We also consider the possible identification of vertices, except that we know that v_i, v_j, v_k are distinct, v_a, v_b, v_c are also distinct, $v_j \neq v_a, v_b, v_c$, and $v_b \neq v_i, v_j, v_k$. The latter discussion leads to the three non isomorphic cases, depicted in Figure 2. In each of these cases, we show that the inclusion in $E(G)$ of any subset of pairs of F leads to a forbidden graph of Figure 1. This contradiction implies that G is indeed starlike.

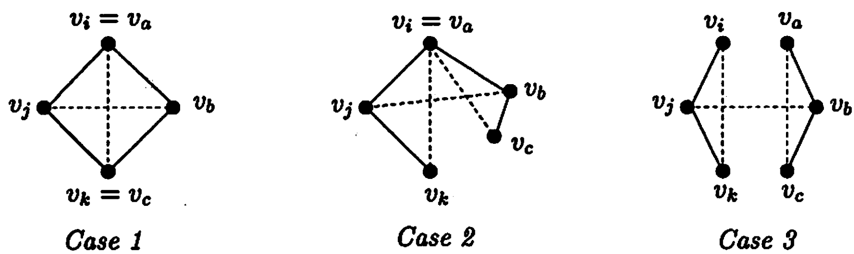

Figure 2: Subgraphs containing a pair of non adjacent non simplicial vertices.

Case 1: $v_i = v_a$ and $v_k = v_c$.

So $F = \emptyset$ and $\{v_i, v_j, v_k, v_b\}$ induces a C_4.

Case 2: $v_i = v_a$ and $v_k \neq v_c$.

$F = \{v_jv_c, v_kv_b, v_kv_c\}$. If $v_jv_c \in E(G)$ then $\{v_i, v_j, v_b, v_c\}$ induces a C_4. On the other hand, $\{v_i, v_j, v_k, v_b\}$ induces a C_4 when $v_kv_b \in E(G)$. Then $v_jv_c, v_kv_b \notin E(G)$. In this situation, $\{v_i, v_j, v_k, v_b, v_c\}$ induces a C_5 or a P_5 according whether $v_kv_c \in E(G)$ or $v_kv_c \notin E(G)$, respectively.

Case 3: $v_i \neq v_a$ and $v_k \neq v_c$.

By inspection, $F = \{v_iv_a, v_iv_b, v_iv_c, v_jv_a, v_jv_c, v_kv_a, v_kv_b, v_kv_c\}$. First, observe that v_iv_b and v_kv_b can not be simultaneously edges of G, otherwise $\{v_i, v_j, v_k, v_b\}$ would induce a C_4. For a similar reason, v_jv_a and v_jv_c also can not be at the same time in G. Consequently, G contains at most two edges of the subset $F' = \{v_jv_a, v_jv_c, v_iv_b, v_kv_b\}$. Consider the three possible alternatives.

Case 3.1: $|E(G) \cap F'| = 2$.

From the above observation, we know that one of the edges of $E(G) \cap F'$ is incident to v_j and the other one to v_b. Without loss of generality, we can restrict to discuss only the alternative $v_jv_a, v_kv_b \in E(G)$ with $v_jv_c, v_iv_k \notin E(G)$. It follows that $v_kv_a \in E(G)$, otherwise $\{v_j, v_k, v_a, v_b\}$ induces a C_4. On the other hand, $v_iv_c \in E(G)$ and $v_kv_c \in E(G)$ can not occur simultaneously, otherwise $\{v_i, v_j, v_k, v_c\}$ induces a C_4. The following then may occur:

Case 3.1.1: $v_iv_c \in E(G)$.

Then $v_kv_c \notin E(G)$ and this implies that $\{v_i, v_j, v_k, v_b, v_c\}$ induces a C_5.

Case 3.1.2: $v_kv_c \in E(G)$.

Then $v_iv_c \notin E(G)$. If $v_iv_a \in E(G)$ then $\{v_i, v_j, v_k, v_a, v_b, v_c\}$ induces the graph H_2, while $v_iv_a \notin E(G)$ implies that $\{v_i, v_j, v_a, v_b, v_c\}$ induces a P_5.

Case 3.1.3: $v_iv_c, v_kv_c \notin E(G)$.

Then $\{v_i, v_j, v_k, v_b, v_c\}$ induces a P_5.

Case 3.2: $|E(G) \cap F'| = 1$.

Without loss of generality, $v_jv_a \in E(G)$ and $v_jv_c, v_iv_b, v_kv_b \notin E(G)$. Again, v_iv_c and v_kv_c can not be both edges of G.

Case 3.2.1: $v_iv_c \in E(G)$.

Then $v_kv_c \notin E(G)$ and this implies that $\{v_b, v_c, v_i, v_j, v_k\}$ induces a P_5.

Case 3.2.2: $v_kv_c \in E(G)$.

Then $v_iv_c \notin E(G)$ and $\{v_i, v_j, v_k, v_c, v_b\}$ induces a P_5.

Case 3.2.3: $v_iv_c, v_kv_c \notin E(G)$.

Then $v_kv_a \in E(G)$, otherwise $\{v_k, v_j, v_a, v_b, v_c\}$ induces a P_5. Suppose $v_iv_a \in E(G)$, hence $\{v_i, v_j, v_k, v_a, v_b, v_c\}$ induces the graph H_1. When $v_iv_a \notin E(G)$, it follows that $\{v_i, v_j, v_a, v_b, v_c\}$ induces a P_5.

Case 3.3: $|E(G) \cap F'| = 0$.

As in Cases 3.1 and 3.2, at most one of v_iv_c, v_kv_c may be an edge of G.

Case 3.3.1: $v_iv_c \in E(G)$.

As in the Case 3.2.1, $v_kv_c \notin E(G)$ and $\{v_b, v_c, v_i, v_j, v_k\}$ induces a P_5.

Case 3.3.2: $v_kv_c \in E(G)$.

As in the Case 3.2.2, $v_iv_c \notin E(G)$ and $\{v_i, v_j, v_k, v_c, v_b\}$ induces a P_5.

Case 3.3.3: $v_iv_c, v_kv_c \notin E(G)$.

If $v_iv_a \in E(G)$ then $\{v_j, v_i, v_a, v_b, v_c\}$ induces a P_5. When $v_iv_a \notin E(G)$ the subset $\{v_i, v_j, v_k, v_a, v_b, v_c\}$ induces a P_6 or a $2P_3$, depending whether $v_kv_a \in E(G)$ or not, respectively. ∎

3 Starlike-threshold Graphs

In this section we describe characterizations for starlike-threshold graphs and give a property of this class, related to interval graphs.

Theorem 3 *Let G be a graph and C the set of simplicial vertices of it. The following affirmatives are equivalent:*

(i) *G is starlike-threshold.*

(ii) *G is starlike and does not contain an induced P_4.*

(iii) *G is starlike and the subsets $C \cap N(v_i)$ can be linearly ordered by inclusion, where $v_i \in V(G) \setminus C$.*

Proof:
(i) $\Rightarrow$ (ii): Let G be a starlike-threshold graph. Then G is the intersection graph of substars S_i of a star S and the vertex sets $V(S_i) \subseteq V(S)$ of the non trivial substars can be linearly ordered by inclusion. Denote by $v_i \in V(G)$ the vertex of G corresponding to S_i. By definition, G is starlike. Suppose the theorem false. Then G contains a induced path v_1, v_2, v_3, v_4. By Theorem 1, the set of non simplicial vertices of G is a clique. Because v_2 and v_3 are not simplicial, $v_2, v_3 \in C$. The latter implies $v_1, v_4 \notin C$. Consequently, S_1 and S_4 are trivial substars, while S_2 and S_3 are not. Let $V(S_1) = \{c_1\}$ and $V(S_4) = \{c_4\}$. Then $c_1 \in V(S_2) \setminus V(S_3)$, while $c_4 \in V(S_3) \setminus V(S_2)$. The latter implies that $V(S_2)$ and $V(S_3)$ can not be linearly ordered by inclusion, a contradiction. Therefore G does not contain induced P_4's.

(ii) $\Rightarrow$ (iii): By hypothesis G is a starlike graph and contains no induced P_4. Let C be the set of non simplicial vertices of G. Suppose the assertion false. Then there exist subsets $C \cap N(v_i)$ and $C \cap N(v_j)$ such that neither $C \cap N(v_i) \subseteq C \cap N(v_j)$ nor $C \cap N(v_j) \subseteq C \cap N(v_i)$, for $v_i, v_j \in V(G) \setminus C$. We can then choose $v_k \in C \cap (N(v_i) \setminus N(v_j))$ and $v_l \in C \cap (N(v_j) \setminus N(v_i))$. By Theorem 1, C is a clique. Consequently, $v_kv_l \in E(G)$. In this situation,

$\{v_i, v_k, v_l, v_j\}$ induces either a C_4 or a P_4, according whether v_i, v_j are adjacent or not, respectively. In any case, a contradiction arises, implying that (iii) holds.

(iii) $\Rightarrow$ (i): Apply a construction similar to that of the proof (iv) $\Rightarrow$ (i) of Theorem 1. ∎

Corollary 1 *A graph is starlike-threshold if and only if it does not contain any of the graphs of Figure 3 as an induced subgraph.*

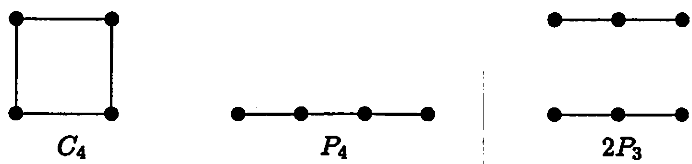

Figure 3: The forbidden subgraphs for starlike-threshold graphs.

Threshold graphs are interval graphs which admit an interval representation whose intervals have at most two distinct sizes [10] (c.f. [12]). By definition the class of starlike-threshold graphs contains that of threshold graphs [4]. The following proposition extends the result of [10] to the more general class of starlike-threshold graphs.

Corollary 2 *Starlike-threshold graphs are interval graphs. Moreover they admit an interval representation having at most two distinct interval sizes.*

Proof: It follows from Theorem 3 that the vertices of a starlike-threshold graph can be partitioned into cliques $C, C_1, \ldots, C_q$, such that C is the set of non simplicial vertices of G and the vertices of each C_i are twins satisfying $N(C_i) \subseteq N(C_j)$, for $1 \leq i < j \leq q$.

In the sequel, we associate an open interval I_v to each vertex $v \in V(G)$, as follows:

(i) $v \in C_i \Rightarrow I_v = (i, i+1)$

(ii) $v \in N(C_i) \setminus N(C_{i-1}) \Rightarrow I_v = (i, i+q)$, where $N(C_0) = \emptyset$.

It is simple to verify that the above family of intervals form an interval representation for the graph. In addition, at most two interval sizes have been used. ∎

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