# Near polygons having a big sub near polygon isomorphic to $H^D(2n-1,4)$

Bart De Bruyn and Pieter Vandecasteele

**Department of Pure Mathematics and Computer Algebra, Ghent University, Galglaan 2, B-9000 Gent, Belgium, E-mail: bdb@cage.Ugent.be, pvdecast@cage.Ugent.be**

**Abstract**

We determine all spreads of symmetry of the dual polar space $H^D(2n-1,q^2)$. We use this to show the existence of glued near polygons of type $H^D(2n_1-1,q^2) \otimes H^D(2n_2-1,q^2)$. We also show that there exists a unique glued near polygon of type $H^D(2n_1-1,4) \otimes H^D(2n_2-1,4)$ for all $n_1, n_2 \geq 2$. The unique glued near polygon of type $H^D(2n-1,4) \otimes Q(5,2)$ has the property that it contains $H^D(2n-1,4)$ as a big geodetically closed sub near polygon. We will determine all dense near $(2n+2)$-gons, $n \geq 3$, which have $H^D(2n-1,4)$ as a big geodetically closed sub near polygon. We will prove that such a near polygon is isomorphic to either $H^D(2n+1,4)$, $H^D(2n-1,4) \otimes Q(5,2)$ or $H^D(2n-1,4) \times L$ for some line $L$ of size at least three.

## 1 Elementary notions and aim of the paper

In geometry, a *near polygon* is defined as a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point $p \in \mathcal{P}$ and for every line $L \in \mathcal{L}$ there exists a unique point on $L$ nearest to $p$. Here distances $\mathrm{d}(\cdot,\cdot)$ are measured in the point graph or collinearity graph $\Gamma$. If $\mathrm{diam}(\mathcal{S})$ denotes the diameter of $\Gamma$ (or of $\mathcal{S}$), then the near polygon is called a near $[2 \cdot \mathrm{diam}(\mathcal{S})]$-gon. There is a unique near 0-gon (one point, no lines), which we will denote by $\mathbb{O}$. The near 2-gons are precisely the lines. We will denote the unique line with $i \geq 2$ points by $\mathbb{L}_i$. The class of near quadrangles coincides with the class of generalized quadrangles (GQ's) which were introduced by Tits in [13]. Near polygons themselves were introduced by Shult and Yanushka in [12] because of their relationship with certain line systems in Euclidean spaces. In graph theory (see e.g. [2]), a *near polygon* is defined as a connected graph which satisfies the following properties: (i) every two adjacent vertices are contained in a unique maximal clique, (ii) for every vertex $x$ and every maximal clique $M$, there exists a unique vertex in $M$ nearest to $x$. This graph-theoretical definition is however equivalent with the geometrical one. If a graph $\Gamma$ is a near polygon, then the vertices and the maximal cliques of $\Gamma$ define a near polygon in the geometrical sense. If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a near polygon, then its point graph is a near polygon in the graph-theoretical sense. In the sequel, we will always adopt the geometrical point of view. The reader should however realize that each definition and result has an equivalent graph-theoretical version.

A nonempty set $X$ of points in a near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a *subspace* if every line meeting $X$ in at least two points is completely contained in $X$. A subspace $X$ is called *geodetically closed* if every point on a shortest path between two points of $X$ is as well contained in $X$. Having a subspace $X$, we can define a subgeometry $\mathcal{S}_X$ of $\mathcal{S}$ by considering only those points and lines of $\mathcal{S}$ which are completely contained in $X$. If $X$ is geodetically closed, then $\mathcal{S}_X$ clearly is a sub near polygon of $\mathcal{S}$. If a geodetically closed sub near polygon $\mathcal{S}_X$ is a nondegenerate generalized quadrangle, then $X$ (and often also $\mathcal{S}_X$) will be called a *quad.* (Recall that a generalized quadrangle is called *degenerate* if there exists a point which is incident with each line.) Sufficient conditions for the existence of quads were given in [12]. For every point $x$ of $\mathcal{S}$ we can define a partial linear space which is called *the local space at* $x$ and denoted by $\mathcal{L}(\mathcal{S}, x)$. Its points, respectively lines, are the lines, respectively quads, through $x$ with containment as incidence relation. Every nonempty set $X$ of points is contained in a unique minimal geodetically closed sub near polygon $\mathcal{C}(X)$, namely the intersection of all geodetically closed sub near polygons through $X$. We define $\mathcal{C}(\emptyset) = \emptyset$. If $X_1, \ldots, X_k$ are sets of points, then $\mathcal{C}(X_1 \cup \cdots \cup X_k)$ is also denoted by $\mathcal{C}(X_1, \ldots, X_k)$. If one of the arguments of $\mathcal{C}$ is a singleton $\{x\}$, we will often omit the braces and write $\mathcal{C}(\cdots, x, \cdots)$ instead of $\mathcal{C}(\cdots, \{x\}, \cdots)$.

If $A$ and $B$ are two sets of points, then $\mathrm{d}(A, B)$ denotes the minimal distance between a point of $A$ and a point of $B$. If $A = \{x\}$, then we also write $\mathrm{d}(x, B)$ instead of $\mathrm{d}(\{x\}, B)$. For every $i \in \mathbb{N}$, $\Gamma_i(A)$ denotes the set of all points $p$ for which $\mathrm{d}(p, A) = i$. If $A = \{x\}$, we also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$.

A near polygon is said to have *order* $(s, t)$ if every line is incident with exactly $s+1$ points and if every point is incident with exactly $t+1$ lines. A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. Dense near polygons satisfy several nice properties. By Lemma 19 of [3], every point of a dense near polygon $\mathcal{S}$ is incident with the same number of lines; we denote this number by $t_{\mathcal{S}} + 1$. If $x$ and $y$ are two points of a dense near polygon then, by Theorem 4 of [3], $\mathcal{C}(x,y)$ is the unique geodetically closed sub near $[2 \cdot d(x,y)]$-gon through $x$ and $y$. Since in a dense near polygon every two points at distance 2 are contained in a unique quad, all local spaces are linear. Geodetically closed sub near hexagons of a dense near polygon are called *hexes*.

A geodetically closed sub near polygon $\mathcal{F}$ of a near polygon $\mathcal{S}$ is called *big* if $\mathcal{F} \neq \mathcal{S}$ and if every point outside $\mathcal{F}$ is collinear with a unique point $\pi(x)$ in $\mathcal{F}$. If $x \in \mathcal{F}$, then we define $\pi(x) := x$. The map $\pi$ is called the *projection on* $\mathcal{F}$. Suppose now that each line of $\mathcal{S}$ is incident with exactly three points. Then for every point $x$ outside $\mathcal{F}$, the line $x\pi(x)$ contains a unique third point which we denote by $r(x)$. If $x \in \mathcal{F}$, then we define $r(x) := x$. The map $r$ is called the *reflection about* $\mathcal{F}$ and is an automorphism of $\mathcal{S}$.

The aim of the paper is to determine all dense near polygons which contain a big geodetically closed sub near polygon isomorphic to the classical near polygon $H^D(2n-1,4)$, $n \geq 3$, which we will define in Section 2.3. We will solve this problem in Section 5. This result, together with others, will allow us in [10] to determine all dense near $2n$-gons with three points on each line having the property that there exists a chain $F_0 \subset F_1 \subset \cdots \subset F_n$ of geodetically closed sub near polygons such that the sub near $2i$-gon $F_i$, $i \in \{0,\ldots,n-1\}$, is big in the sub near $2(i+1)$-gon $F_{i+1}$.

# 2 Some classes of near polygons

In this section, we describe three important classes of near polygons. The constructions given in Sections 2.1 and 2.2 allow us to construct near polygons from other ones.

## 2.1 Product near polygons

For any near polygons $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$, a new near polygon $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ can be derived from $\mathcal{S}_1$ and $\mathcal{S}_2$. It is called the *direct product* of $\mathcal{S}_1$ and $\mathcal{S}_2$ and is denoted by $\mathcal{S}_1 \times \mathcal{S}_2$. We have: $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$, $\mathcal{L} = (\mathcal{P}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \mathcal{P}_2)$, the point $(x,y)$ of $\mathcal{S}_1 \times \mathcal{S}_2$ is incident with the line $(z,L) \in \mathcal{P}_1 \times \mathcal{L}_2$ if and only if $x = z$ and $y \, \mathrm{I}_2 \, L$, the point $(x,y)$ of $\mathcal{S}_1 \times \mathcal{S}_2$ is incident with the line $(M,u) \in \mathcal{L}_1 \times \mathcal{P}_2$ if and only if $x \, \mathrm{I}_1 \, M$ and $y = u$. If $\mathcal{S}_i$, $i \in \{1,2\}$, is a near $2d_i$-gon then $\mathcal{S}_1 \times \mathcal{S}_2$ is a near $2(d_1+d_2)$-gon. If $d_1, d_2 \geq 1$, then we call $\mathcal{S}_1 \times \mathcal{S}_2$ a *product near polygon*. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are dense, then also $\mathcal{S}_1 \times \mathcal{S}_2$ is dense and $t_{\mathcal{S}} = t_{\mathcal{S}_1} + t_{\mathcal{S}_2} + 1$.

**Lemma 1 (Lemma 4.5 of [1])** *Let $\mathcal{F}$ be a big geodetically closed sub near $2(n-1)$-gon of a dense near $2n$-gon $\mathcal{A}$, $n \geq 2$, then the following are equivalent:*

(a) $\mathcal{A} \cong \mathcal{F} \times L$ *for some line* $L$;

(b) $t_{\mathcal{A}} = t_{\mathcal{F}} + 1$;

(c) *every quad intersecting* $\mathcal{F}$ *in a line is a grid.*

**Definitions.** Since $\mathcal{S}_1 \times \mathcal{S}_2 \cong \mathcal{S}_2 \times \mathcal{S}_1$ and $\mathcal{S}_1 \times (\mathcal{S}_2 \times \mathcal{S}_3) \cong (\mathcal{S}_1 \times \mathcal{S}_2) \times \mathcal{S}_3$, also the direct product $\mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_k$ of $k \geq 3$ near polygons $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k$ is well-defined. If $A$ is a class of near polygons, then we denote by $A^{\times}$ the set $\{\mathbb{O}\} \cup A \cup B$, where $B$ is the set of all product near polygons $\mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_k$, where $k \geq 2$ and $\mathcal{S}_i \in A$, $i \in \{1, 2, \ldots, k\}$.

In [8], product near polygons are also called glued near polygons of type 0. In the following section we define the glued near polygons of type 1.

## 2.2 Glued near polygons (of type 1)

The construction which we will present in this section was first described in [5] for the case of generalized quadrangles and in [7] for the general case. We first give some relevant definitions.

### 2.2.1 Definitions

If $K$ and $L$ are two lines of a near polygon, then by Lemma 1 of [3], there are two possibilities. Either there exist unique points $k \in K$ and $l \in L$ such that $\mathrm{d}(K, L) = \mathrm{d}(k, l)$, or for every point $k \in K$, there exists a unique point $l \in L$ such that $\mathrm{d}(K, L) = \mathrm{d}(k, l)$. In the latter case $K$ and $L$ are called *parallel*. If $K$ and $L$ are parallel and $\mathrm{d}(K, L) = 1$, then we define $\{K, L\}^{\perp}$ as the set of all lines intersecting $K$ and $L$, and $\{K, L\}^{\perp\perp}$ as the set of all lines meeting every line of $\{K, L\}^{\perp}$. If $\{K, L\}^{\perp}$ and $\{K, L\}^{\perp\perp}$ cover the same set of points, then the pair $\{K, L\}$ is called *regular*.

A *spread* of a near polygon $\mathcal{A}$ is a set of lines partitioning the point set. If $\mathcal{A}$ is the direct product $\mathcal{B} \times L$ of a near polygon $\mathcal{B}$ and a line $L$, then the set $S = \{L_x | x \text{ is a point of } \mathcal{B}\}$ with $L_x := \{(x, y) | y \in L\}$ is a spread of $\mathcal{A}$. We call any such spread a *trivial spread*. A spread of $\mathcal{A}$ is called *admissible* if every two lines of it are parallel. An admissible spread $S$ of $\mathcal{A}$ is called *regular* if $\{K, L\}$ is regular and $\{K, L\}^{\perp\perp} \subseteq S$ for all $K, L \in S$ with $d(K, L) = 1$. A spread $S$ of $\mathcal{A}$ is called *a spread of symmetry* if for every line $K \in S$ and for every two points $k_1$ and $k_2$ on $K$ there exists an automorphism of $\mathcal{A}$ fixing each line of $S$ and mapping $k_1$ to $k_2$. Every spread of symmetry is regular. Every trivial spread is a spread of symmetry. Spreads of symmetry satisfy the following property.

**Lemma 2 (Theorem 5 of [7])** *Let $S$ be a spread of symmetry of a near polygon $\mathcal{A}$ and let $F$ be a geodetically closed sub near polygon of $\mathcal{A}$. Then the set $S_F$ of all lines of $S$ which are contained in $F$ is either empty or a spread of symmetry of $F$.*

### 2.2.2 Construction

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two near polygons both with constant line size $s+1$, and suppose that their respective diameters $d_1$ and $d_2$ are at least 2. Let $S_i = \{L_1^{(i)}, \ldots, L_{\alpha_i}^{(i)}\}$, $i \in \{1,2\}$, be an admissible spread of $\mathcal{A}_i$. In $S_i$, a special line $L_1^{(i)}$ is chosen which we will call the *base line* of the spread $S_i$. For every $i \in \{1,2\}$, for all $j,k \in \{1,\ldots,\alpha_i\}$ and for every $x \in L_j^{(i)}$, let $p_{j,k}^{(i)}(x)$ denote the unique point $L_k^{(i)}$ nearest to $x$. We put $\Phi_{j,k}^{(i)} := p_{k,1}^{(i)} \circ p_{j,k}^{(i)} \circ p_{1,j}^{(i)}$. For every $i \in \{1,2\}$, the group $\Pi_{S_i}(L_1^{(i)}) := \langle \Phi_{j,k}^{(i)} | 1 \leq j,k \leq \alpha_i \rangle$ is called the *group of projectivities of $L_1^{(i)}$ with respect to $S_i$*.

For every bijection $\theta$ between $L_1^{(1)}$ and $L_1^{(2)}$, we consider the following graph $\Gamma$ with vertex set $L_1^{(1)} \times S_1 \times S_2$. Two vertices $(x, L_{i_1}^{(1)}, L_{j_1}^{(2)})$ and $(y, L_{i_2}^{(1)}, L_{j_2}^{(2)})$ are adjacent if and only if exactly one of the following three conditions is satisfied:

(a) $L_{i_1}^{(1)} = L_{i_2}^{(1)}$, $L_{j_1}^{(2)} = L_{j_2}^{(2)}$ and $x \neq y$,

(b) $L_{j_1}^{(2)} = L_{j_2}^{(2)}$, $\mathrm{d}(L_{i_1}^{(1)}, L_{i_2}^{(1)}) = 1$ and $\Phi_{i_1,i_2}^{(1)}(x) = y$,

(c) $L_{i_1}^{(1)} = L_{i_2}^{(1)}$, $\mathrm{d}(L_{j_1}^{(2)}, L_{j_2}^{(2)}) = 1$ and $\Phi_{j_1,j_2}^{(2)} \circ \theta(x) = \theta(y)$.

By [7], the graph $\Gamma$ has diameter $d_1+d_2-1$ and every two adjacent vertices are contained in a unique maximal clique (of size $s+1$). Considering these maximal cliques as lines, we obtain a partial linear space $\mathcal{S}$. If $\mathcal{S}$ is a near polygon, then it is called a *glued near polygon of type* 1, or shortly a *glued near polygon.* (Glued near polygons of type $\delta \geq 2$ can also be defined, see [8], but we will not need them in this paper.) The following lemma gives necessary and sufficient conditions for $\mathcal{S}$ to be a near polygon.

**Lemma 3 (Theorem 14 of [7])** *The partial linear space $\mathcal{S}$ is a near polygon if and only if the commutator $[\Pi_{S_1}(L_1^{(1)}), \theta^{-1}\Pi_{S_2}(L_1^{(2)})\theta]$ is the trivial group of permutations of $L_1^{(1)}$.*

The following lemma explains why the notion spread of symmetry is important in the theory of glued near polygons.

**Lemma 4 (Theorem 16 of [7])** *If $\mathcal{S}$ is a near polygon and if none of the spreads $S_1$ and $S_2$ is trivial, then $S_1$ and $S_2$ are spreads of symmetry.*

If $s$ is equal to 2, then we can say more.

**Lemma 5 (Theorems 11 and 16 of [7])** *If each line of $\mathcal{A}_1$ and $\mathcal{A}_2$ is incident with three points and if none of the spreads $S_1$ and $S_2$ is trivial, then $\mathcal{S}$ is a near polygon (for an arbitrary choice of the base lines and the bijection $\theta$ between these base lines) if and only if $S_1$ and $S_2$ are spreads of symmetry.*

A near polygon $\mathcal{S}$ is said to be of *type* $\mathcal{A}_1 \otimes \mathcal{A}_2$ if it can be obtained in the above-described way from two near polygons $\mathcal{A}_1$ and $\mathcal{A}_2$. If there is no confusion about the spreads $S_1$ and $S_2$, the base lines in these spreads and the map $\theta$, then we will denote $\mathcal{S}$ also by $\mathcal{A}_1 \otimes \mathcal{A}_2$. We will also use this notation for $\mathcal{S}$ if all near polygons of type $\mathcal{A}_1 \otimes \mathcal{A}_2$ are isomorphic. This latter property is not always satisfied, see for instance [6] for counterexamples. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are dense near polygons, then every glued near polygon of type $\mathcal{A}_1 \otimes \mathcal{A}_2$ is also dense.

### 2.2.3 Characterization

If we consider a tuple $(\mathcal{A}_1, \mathcal{A}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$ which gives rise to a glued near polygon $\mathcal{A}_1 \otimes \mathcal{A}_2$ and if we define

$$T_1 := \left\{ \{(x, L, M) | x \in L_1^{(1)}, L \in S_1\} | M \in S_2 \right\},$$

$$T_2 := \left\{ \{(x, L, M) | x \in L_1^{(1)}, M \in S_2\} | L \in S_1 \right\},$$

then $T_i$, $i \in \{1, 2\}$, is a partition of $\mathcal{A}_1 \otimes \mathcal{A}_2$ in geodetically closed sub near polygons isomorphic to $\mathcal{A}_i$. Now, if we have an arbitrary near polygon $\mathcal{A}$, then with every tuple which gives rise to a glued near polygon isomorphic to $\mathcal{A}$, there corresponds a pair $\{T_1, T_2\}$ of partitions of $\mathcal{A}$ in geodetically closed sub near polygons. We denote by $\Delta_1(\mathcal{A})$ the set of all pairs $\{T_1, T_2\}$ arising in this way. If $\{T_1, T_2\} \in \Delta_1(\mathcal{A})$, then:

- all elements of $T_i$, $i \in \{1, 2\}$, are isomorphic;
- every element of $T_1$ intersects every element of $T_2$ in a line;
- if $x$ denotes an arbitrary point of $\mathcal{A}$ and if $F_i$, $i \in \{1, 2\}$, denotes the unique element of $T_i$ through $x$, then every line through $x$ different from $F_1 \cap F_2$ is contained in either $F_1$ or $F_2$.

These properties will be used in the following characterization result.

**Lemma 6 (Theorem 10 of [8])** *Let $\mathcal{A}$ be a dense near polygon and let $T_1$ and $T_2$ denote two partitions of $\mathcal{A}$ in geodetically closed sub near polygons. For every point $x$ of $\mathcal{S}$ and every $i \in \{1,2\}$, let $F_i(x)$ denote the unique element of $T_i$ through $x$ and put $F_i'(x) := \mathcal{C}(\Gamma_1(x) \setminus F_{3-i}(x))$. Suppose that* $\mathrm{diam}(F_1(x) \cap F_2(x)) = 1$ *and $\Gamma_1(x) \subseteq F_1(x) \cup F_2(x)$ for every point $x$ of $\mathcal{A}$. Then we have the following:*

- *If $F_1'(x) = F_1(x)$ and $F_2'(x) = F_2(x)$ for every point $x$ of $\mathcal{A}$, then $\mathcal{A}$ is glued and $\{T_1, T_2\} \in \Delta_1(\mathcal{A})$.*
- *If there exists a point $x^*$ such that $F_1'(x^*) \cap F_2(x^*) = \{x^*\}$, then $\mathcal{A} \cong F_1'(x^*) \times F_2(x^*)$.*
- *If there exists a point $x^*$ such that $F_2'(x^*) \cap F_1(x^*) = \{x^*\}$, then $\mathcal{A} \cong F_1(x^*) \times F_2'(x^*)$.*

## 2.3 Classical near polygons

If $(x, Q)$ is a point-quad pair of a near polygon $\mathcal{S}$, then by Proposition 2.6 of [12], one of the following possibilities occurs.

- There exists a unique point $x'$ in $Q$ nearest to $x$, and $\mathrm{d}(x,y) = \mathrm{d}(x,x') + \mathrm{d}(x',y)$ for every point $y \in Q$. In this case the pair $(x, Q)$ is called *classical.*
- The points in $Q$ nearest to $x$ form an ovoid of $Q$, i.e. a set of points of $Q$ intersecting each line in exactly one point. In this case the pair $(x, Q)$ is called *ovoidal.*
- The generalized quadrangle $Q$ is a dual grid, i.e. a complete bipartite graph $K_{n,m}$ with $n, m \geq 2$, and the set of points of $Q$ nearest to $x$ is a proper subset of size at least 2 of one of the two ovoids of $Q$. In this case the pair $(x, Q)$ is called *thin ovoidal.*

**Remark.** More generally, we call a pair $(x, F)$ with $x$ a point and $F$ a geodetically closed sub near polygon classical if there exists a unique point $x'$ in $F$ such that $\mathrm{d}(x,y) = \mathrm{d}(x,x') + \mathrm{d}(x',y)$ for every point $y$ of $F$.

**Definitions.** (a) A near polygon $\mathcal{S}$ is called *classical* if every two points at distance two are contained in a unique quad and if every point-quad pair is classical. Clearly, the near 0-gon $\mathbb{O}$, the lines $\mathbb{L}_i$ $(i \geq 2)$ and the nondegenerate generalized quadrangles are classical. The direct product of classical near polygons is again classical.

(b) For every polar space $P$ of rank at least 2 a dual polar space $P^D$ can be defined. The points, respectively lines, of $P^D$ are the maximal, respectively next-to-maximal, totally isotropic subspaces of $P$ with reverse containment as incidence relation.

Cameron ([4]) proved that the class of the classical near $2n$-gons, $n \geq 2$, coincides with the class of the *dual polar spaces* of rank at least 2. We will now emphasize on the case were each line is incident with exactly three points. The quadrics $Q(2n, 2)$, $n \geq 2$, and the hermitian varieties $H(2n-1, 4)$, $n \geq 2$, give rise to two classes $Q^D(2n, 2)$ and $H^D(2n-1, 2)$ of dual polar spaces. By the classification of polar spaces ([14]), we known that the classical dense near polygons with three points on each line are precisely the near polygons from the set $(\{\mathbb{L}_3\} \cup \{Q^D(2n,2) \mid n \geq 2\} \cup \{H^D(2n-1,4) \mid n \geq 2\})^{\times}$. For generalized quadrangles, which are always classical, we have the following three well-known examples:

- The $3 \times 3$-grid $\mathbb{L}_3 \times \mathbb{L}_3$ of order $(2, 1)$.
- The generalized quadrangle $Q^D(4, 2)$ of order 2 (i.e. order $(2, 2)$). This GQ is isomorphic to $W(2)$, the GQ whose points and lines are the totally isotropic points and lines of a symplectic polarity in PG$(3, 2)$ (natural incidence).
- The generalized quadrangle $H^D(5, 4)$ of order $(2, 4)$. This GQ is isomorphic to $Q(5, 2)$, the GQ whose points and lines are the points and lines lying on an elliptic quadric in PG$(5, 2)$ (natural incidence).

In the sequel, the generalized quadrangles $\mathbb{L}_3 \times \mathbb{L}_3$, $W(2)$ and $Q(5, 2)$ will often occur as quads in other near polygons. We will refer to them as *grid-quads*, $W(2)$-quads and $Q(5, 2)$-quads, respectively.

# 3 The near polygon $H^D(2n-1, 4) \otimes Q(5, 2)$

Consider the hermitian variety $H := H(2n-1, q^2)$ in $\Pi :=$ PG$(2n-1, q^2)$, $n \geq 2$, and let $H^D = H^D(2n-1, q^2)$ denote its associated dual polar space.

**Lemma 7** *Let $\mathcal{L}$ be the linear space whose points are the points of $H$ and whose lines are the lines of $\Pi$ which are not tangent to $H$. Then the subspaces of $\mathcal{L}$ are precisely the intersections of $H$ with subspaces of $\Pi$.*

**Proof.** For every set $X$ of points on $H$, let $\langle X\rangle$ denote the subspace of $\Pi$ generated by all points of $X$ and let $\overline{X}$ denote the smallest subspace of $\mathcal{L}$ through $X$. Since $\langle X\rangle \cap H$ is a subspace of $\mathcal{L}$, we have $\overline{X} \subseteq \langle X\rangle \cap H$. We will now prove that $\overline{X} = \langle X\rangle \cap H$ $(*)$. This property obviously holds if $m := \dim(\langle X\rangle) \leq 1$, and since every intersection of a hermitian variety with a plane is either the plane itself, a line in this plane, a unital or a cone $pB$ with $p$ a point and $B$ a Baer subline, it also holds if $m = 2$. So, suppose that $m \geq 3$ and consider a subset $Y$ of $X$ such that $\dim(\langle Y \rangle) = m - 1$. We may suppose that $\langle Y \rangle \cap H = \overline{Y} \subset \overline{X}$ (otherwise use induction). Now consider a fixed point $x$ in $X \setminus \langle Y \rangle$. For every point $x'$ of $\langle X \rangle \cap H$ different from $x$, the line $xx'$ intersects $\langle Y \rangle$ in a point $x''$ and one of the following possibilities occurs.

- There exists a Baer subline $L$ in $\langle Y \rangle$ through the point $x''$. Since property (*) holds if $m = 2$, we have $x' \in \langle x, L \rangle \cap H = \overline{L \cup \{x\}} \subseteq \overline{X}$.

- Every line of $\langle Y \rangle$ through the point $x''$ is a tangent line. Then $x''$ is a singular point of $\langle Y \rangle \cap H$. Hence $x'' \in H$ and $x' \in xx'' \subseteq \overline{X}$.

In any case, we have $x' \in \overline{X}$. Since $x'$ was an arbitrary point of $\langle X \rangle \cap H$ different from $x$ and since $x \in \overline{X}$, we have $\langle X \rangle \cap H \subseteq \overline{X}$ and hence $\overline{X} = \langle X \rangle \cap H$. As a consequence the subspaces of $\mathcal{L}$ are precisely the intersections of $H$ with subspaces of $\Pi$. □

There exists a bijective correspondence between the geodetically closed sub near polygons of $H^D$ and the subspaces on $H$. If $\pi$ is a subspace on $H$, then the set of all generators of $H$ through $\pi$ determines a geodetically closed sub near polygon of $H^D$ which we denote by $\pi^\phi$. We have $\dim(\pi) + \operatorname{diam}(\pi^\phi) = n - 1$. If $\pi_1$ and $\pi_2$ are two subspaces on $H$, then $\pi_1^\phi \subseteq \pi_2^\phi$ if and only if $\pi_2 \subseteq \pi_1$.

**Theorem 1** *If $V$ is the set of all $(n-2)$-dimensional subspaces of $H$ which lie in a nontangent hyperplane $\Pi_\infty$, then $V^\phi := \{\alpha^\phi | \alpha \in V\}$ is a spread of symmetry of $H^D$. Conversely, every spread of symmetry is obtained in this way.*

**Proof.** (a) Every generator of $H$ contains a unique element of $V$, or equivalently, every point of $H^D$ is incident with a unique line of $V^\phi$. So, $V^\phi$ is a spread. We can choose our reference system in such a way that $H$ has equation $X_0^{q+1} + \cdots + X_{2n-1}^{q+1} = 0$ and that $\Pi_\infty$ has equation $X_{2n-1} = 0$. The group $G := \{\theta_\lambda : (x_0, \ldots, x_{2n-2}, x_{2n-1}) \to (x_0, \ldots, x_{2n-2}, \lambda x_{2n-1}) \,|\, \lambda^{q+1} = 1\}$ of automorphisms of $\Pi$ fixes $H$ setwise and $\Pi_\infty$ pointwise. So, $G$ determines a group $G^\phi$ of automorphisms of $H^D$ which fixes every element of $V^\phi$. This group $G^\phi$ acts regularly on each line of $V^\phi$, proving that $V^\phi$ is a spread of symmetry.

(b) Now consider a spread of symmetry $S$ of $H^D$ and let $X$ denote the set of the points $x$ of $H$ for which the geodetically closed sub near $2(n-1)$-gon $x^\phi$ contains a line of $S$. Take two different points $x_1$ and $x_2$ in $X$, then one of the following possibilities occurs:

- $|x_1x_2 \cap H| = q^2+1$. Let $y$ denote an arbitrary point of $x_1^\phi \cap x_2^\phi = (x_1x_2)^\phi$. By Lemma 2, the unique line of $S$ through $y$ is contained in $x_1^\phi$ and $x_2^\phi$ and hence in $x_1^\phi \cap x_2^\phi$. As a consequence, each of the $q^2+1$ geodetically closed sub near $2(n-1)$-gons through $(x_1x_2)^\phi$ belongs to $X^\phi$, or equivalently, each of the $q^2+1$ points of $x_1x_2$ belongs to $X$.

- $|x_1x_2 \cap H| = q+1$. In this case $x_1^\phi$ and $x_2^\phi$ are two disjoint geodetically closed sub near $2(n-1)$-gons of $H^D$. Put $x_1x_2 \cap H = \{x_1, x_2, \ldots, x_{q+1}\}$. Let $L$ denote an arbitrary line of $S$ contained in $x_1^\phi$ and let $Q$ denote the unique quad through $L$ which intersects each of the sub near polygons $x_i^\phi$, $i \in \{2, \ldots, q+1\}$, in a line. Now, let $y$ denote an arbitrary point of $Q \cap x_2^\phi$. The unique line of $S$ through $y$ is contained in $Q$ and in $x_2^\phi$ and hence coincides with the line $Q \cap x_2^\phi$. Since $Q \cap (x_1^\phi \cup x_2^\phi \cup \cdots \cup x_{q+1}^\phi)$ is a subgrid of $Q$ and since $S$ is a regular spread, we now see that each line $Q \cap x_i^\phi$, $i \in \{1, 2, \ldots, q+1\}$, belongs to $S$. Hence $x_1, x_2, \ldots, x_{q+1} \in X$.

As a consequence, the set $X$ is a subspace of $\mathcal{L}$ and hence the intersection of $H$ with a subspace $\pi$. The elements of $S$ are precisely the elements $\alpha^\phi$, where $\alpha$ is an $(n-2)$-dimensional subspace contained in $\pi \cap H$. [If $L \in S$, then by Lemma 2, every point of $L^{\phi^{-1}}$ belongs to $X$ and hence $L^{\phi^{-1}}$ is contained in $\pi \cap H$. Conversely, let $\alpha$ be an $(n-2)$-dimensional subspace contained in $\pi \cap H$, let $x_1, \ldots, x_{n-1}$ denote $n-1$ points of $X$ generating $\alpha$ and let $u$ denote an arbitrary point of $\alpha^\phi$. By Lemma 2 the unique line $K$ of $S$ through $u$ is contained in each geodetically closed sub near polygon $x_i^\phi$ and hence coincides with the line $\alpha^\phi = x_1^\phi \cap \cdots \cap x_{n-1}^\phi$.] If $\pi \cap H$ contains a subspace $\beta$ of dimension $n-1$, then every line through the point $\beta^\phi$ would belong to $S$, which is impossible. As a consequence $n-2$ is the maximal dimension of the subspaces contained in $\pi \cap H$. If $x$ is a singular point of $\pi \cap H$, then $x$ is contained in all $(n-2)$-dimensional subspaces of $\pi \cap H$ and so all lines of $S$ would be contained in the geodetically closed sub near $2(n-1)$-gon $x^\phi$, a contradiction. So, $\pi \cap H$ is a nonsingular hermitian variety of type $H(2n-2, q^2)$ or $H(2n-3, q^2)$, but since $S$ must have the right amount of lines, i.e. $\frac{|H^D|}{q+1}$, we know that $\pi \cap H$ is of type $H(2n-2, q^2)$ and that $\pi$ is a nontangent hyperplane. □

**Lemma 8** *Let $S$ be a spread of symmetry of $H^D(2n-1,4)$, $n \geq 2$, and let $K$ denote an arbitrary line of $S$. Then the group of automorphisms of $H^D(2n-1,4)$ which fix $K$ and $S$ induces the full group of permutations of the line $K$.*

**Proof.** We choose our reference system in PG$(2n-1,4)$ in such a way that

- $H(2n-1,4)$ has equation $X_0^3 + X_1^3 + \cdots + X_{2n-1}^3 = 0$,
- $S$ is the spread determined by the hyperplane $X_{2n-1} = 0$,
- the line $K$ corresponds with the subspace on $H(2n-1,4)$ generated by the $n-1$ points $\langle \bar{e}_{2i} + \bar{e}_{2i+1} \rangle$, $0 \leq i \leq n-2$.

The elements $(x_0, \ldots, x_{2n-2}, x_{2n-1}) \to (x_0^\theta, \ldots, x_{2n-2}^\theta, \lambda x_{2n-1}^\theta)$, $\lambda \in \mathrm{GF}(4)^*$ and $\theta \in \mathrm{Aut}(\mathrm{GF}(4))$, of $\mathrm{P\Gamma L}(2n,4)$ determine a group $G$ of six automorphisms of $H^D(2n-1,4)$. The group $G$ fixes $K$ and $S$ and induces the full group of permutations of the line $K$. □

**Theorem 2** *For every prime power $q$ and all $n_1, n_2 \in \mathbb{N} \setminus \{0,1\}$, there exists a glued near polygon of type $H^D(2n_1-1,q^2) \otimes H^D(2n_2-1,q^2)$.*

**Proof.** Let $S_i$, $i \in \{1,2\}$ be a spread of symmetry of $H^D(2n_i-1,q^2)$, and let $L_1^{(i)}$ be an arbitrary base line in $S_i$. By the proof of Lemma 1, we know that there exists a group $G_i \cong C_{q+1}$ of automorphisms of $H^D(2n_i-1,q^2)$ fixing each element of $S_i$. In Theorem 10 of [7], the relationship between the groups $G_i$ and $\Pi_{S_i}(L_1^{(i)})$ is explained. It follows that $\Pi_{S_1}(L_1^{(1)}) \cong \Pi_{S_2}(L_1^{(2)}) \cong C_{q+1}$. So, there exists a bijection $\theta$ between $L_1^{(1)}$ and $L_1^{(2)}$ such that $\Pi_{S_1}(L_1^{(1)}) = \theta^{-1}\Pi_{S_2}(L_1^{(2)})\theta$. With these choices of $S_1$, $S_2$, $L_1^{(1)}$, $L_1^{(2)}$ and $\theta$, we obtain a glued near polygon of type $H^D(2n_1-1,q^2) \otimes H^D(2n_2-1,q^2)$ by Lemma 3. □

**Theorem 3** *For all $n_1, n_2 \in \mathbb{N} \setminus \{0,1\}$, there exists a unique glued near polygon of type $H^D(2n_1-1,4) \otimes H^D(2n_2-1,4)$. In particular, for every $n \in \mathbb{N} \setminus \{0,1\}$, there exists a unique glued near polygon of type $H^D(2n-1,4) \otimes Q(5,2)$.*

**Proof.** By Theorem 1, all spreads of symmetry of $H^D(2n_i-1,4)$, $i \in \{1,2\}$, are isomorphic. We may therefore fix arbitrary spreads of symmetry $S_1$ and $S_2$ in $H^D(2n_1-1,4)$ and $H^D(2n_2-1,4)$, respectively. By [7], every near polygon which can be obtained for a certain choice of the base lines can always be obtained for any other choice of the base lines (by changing the map $\theta$ accordingly). Hence we may also fix arbitrary base lines $L_1^{(1)} \in S_1$ and $L_1^{(2)} \in S_2$. By Lemma 5, every bijection $\theta$ between $L_1^{(1)}$ and $L_1^{(2)}$, gives rise to a glued near polygon of type $H^D(2n_1-1,4) \otimes_\theta H^D(2n_2-1,4)$. By reasons of symmetry, all these near polygons are isomorphic if the group of automorphisms of $H^D(2n_1-1,4)$ which fix $S_1$ and the base line $L_1^{(1)} \in S_1$

induces the full group of permutations on this base line. But this is precisely what we have shown in Lemma 8. □

**Remark.** The uniqueness of $Q(5,2) \otimes Q(5,2)$ was already shown in [1].

The near polygon $H^D(2n-1,4) \otimes Q(5,2)$ has $H^D(2n-1,4)$ as a big geodetically closed sub near polygon. In section 5, we will determine all dense near polygons which have $H^D(2n-1,4)$ as a big geodetically closed sub near polygon. During the classification, we will use the classification of dense near hexagons with three points per line.

# 4 Dense near hexagons with three points on each line

By [1], there are 11 dense near hexagons of order $(2,*)$. These near hexagons are listed in the table below using the notation of [9]. We refer to [1] or [9] for a description of the near hexagons $\mathbb{H}_3$, $\mathbb{G}_3$, $\mathbb{E}_1$, $\mathbb{E}_2$ and $\mathbb{E}_3$. In the table we also mention what the local spaces are. These local spaces are all isomorphic for the same near hexagon and provide information about the configuration of quads through a fixed point of the near hexagon. In $Q^D(6,2)$, $\mathbb{E}_2$ or $H^D(5,4)$, the local spaces are nondegenerate projective spaces. In $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $W(2) \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$ or $Q(5,2) \otimes Q(5,2)$, the local spaces are *crosses*. An $(i,j)$-cross $C_{i,j}$ $(i,j \geq 2)$ is the unique linear space on $i+j-1$ points containing a line of length $i$, a line of length $j$, and $(i-1)(j-1)$ additional lines of length 2. The local spaces of $\mathbb{E}_1$ are isomorphic to the complete graph $K_{12}$ on 12 points (regarded as linear space). The remaining local spaces are related to sets of points in projective planes. For every set $P$ of points in a projective plane $\mathrm{PG}(2,q)$, we define $\mathcal{L}_P$ as the linear space whose points are the points of $P$ and whose lines are the lines of $\mathrm{PG}(2,q)$ containing at least two points of $P$. The local spaces of $\mathbb{H}_3$, $\mathbb{G}_3$ and $\mathbb{E}_3$ respectively correspond with a set $X_1$ of six points of $\mathrm{PG}(2,2)$, the union $X_2$ of three non-concurrent lines in $\mathrm{PG}(2,4)$ and the complement $X_3$ of a hyperoval in $\mathrm{PG}(2,4)$.

| near hexagon $\mathcal{S}$ | # points | $t_{\mathcal{S}}$ | local spaces |
|---|---|---|---|
| $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$ | 27 | 2 | $C_{2,2}$ |
| $W(2) \times \mathbb{L}_3$ | 45 | 3 | $C_{2,3}$ |
| $Q(5,2) \times \mathbb{L}_3$ | 81 | 5 | $C_{2,5}$ |
| $\mathbb{H}_3$ | 105 | 5 | $\mathcal{L}_{X_1}$ |
| $Q^D(6,2)$ | 135 | 6 | PG(2, 2) |
| $Q(5,2) \otimes Q(5,2)$ | 243 | 8 | $C_{5,5}$ |
| $\mathbb{G}_3$ | 405 | 11 | $\mathcal{L}_{X_2}$ |
| $\mathbb{E}_1$ | 729 | 11 | $K_{12}$ |
| $\mathbb{E}_2$ | 759 | 14 | PG(3, 2) |
| $\mathbb{E}_3$ | 567 | 14 | $\mathcal{L}_{X_3}$ |
| $H^D(5,4)$ | 891 | 20 | PG(2, 4) |

**Lemma 9** *Let $x$ and $y$ denote two points of a dense near polygon $\mathcal{S}$ of order $(2,t)$. If $x$ is contained in a quad of order $(2,i)$, $i \in \{1,2,4\}$, then $y$ is also contained in a quad of order $(2,i)$.*

**Proof.** By connectedness of $\mathcal{S}$, it suffices to prove the lemma for collinear points $x$ and $y$. If $Q$ denotes an arbitrary quad through $x$, then $\mathcal{C}(Q,x)$ is either the quad $Q$ itself or a hex $\mathcal{H}$. In any case, there exists a quad through $y$ of the same order as $Q$ (Recall that all local spaces in a hex are isomorphic). □

# 5 $H^D(2n-1,4)$ as big geodetically closed sub near polygon

In this section, we will prove the following result.

**Theorem 4** *Let $\mathcal{S}$ be a dense near $2n$-gon, $n \geq 4$, containing a big geodetically closed sub near polygon $\mathcal{F}$ isomorphic to $H^D(2n-3,4)$. Then either $\mathcal{S} \cong H^D(2n-1,4)$, $\mathcal{S} \cong H^D(2n-3,4) \otimes Q(5,2)$ or $\mathcal{S} \cong H^D(2n-3,4) \times L$ for some line $L$ of size at least 3.*

The proof of this theorem happens in several lemmas.

**Lemma 10** *Let $G$ be a big geodetically closed sub near polygon of a dense near polygon $\mathcal{A}$. Then every geodetically closed sub near polygon $G'$ which meets $G$ either is contained in $G$ or intersects $G$ in a big geodetically closed sub near polygon of $G'$.*

**Proof.** Suppose that $G'$ is not contained in $G$ and let $x$ denote a common point of $G$ and $G'$. Every point $y$ not contained in $G$ is collinear with a unique point $y'$ of $G$. If $d(x,y) \leq d(x,y')$, then the line $yy'$ contains a point $y''$ at distance $d(x,y')-1$ from $x$. Since $G$ is geodetically closed it follows that $y'' \in G$. But then $y'y'' \subseteq G$, contradicting $y \notin G$. Hence, $d(x,y) = d(x,y')+1$ and $y'$ is on a shortest path between $x$ and $y$. In particular, if $y \in G'$, then since $x, y \in G'$, also $y' \in G'$. Hence, every point in $G'$ has distance at most 1 from a point of $G \cap G'$. This proves the lemma. □

**Lemma 11** *If not all lines of $\mathcal{S}$ are incident with 3 points, then $\mathcal{S}$ is isomorphic to $H^D(2n-3,4) \times L$ for some line $L$ of size at least 4.*

**Proof.** Let $K$ be a line with more than three points. Since every line disjoint with $\mathcal{F}$ contains as many points as its projection on $\mathcal{F}$, $K$ intersects $\mathcal{F}$ in a point $x$. Suppose that $t_{\mathcal{S}} > t_{\mathcal{F}}+1$, then there exists a line $L \neq K$ through $x$ which is not contained in $\mathcal{F}$. By Lemma 10, the quad $Q := \mathcal{C}(K,L)$ intersects $\mathcal{F}$ in a line $M$, so $t_Q \geq 2$. Since $t_Q \geq 2$, every line of $Q$ is incident with the same number of points, a contradiction, since $K$ has more points than $M$. Hence, $t_{\mathcal{S}} = t_{\mathcal{F}}+1$. The lemma now follows from Lemma 1. □

From now on we suppose that each line of $\mathcal{S}$ is incident with exactly three points.

**Lemma 12** *The near polygon $\mathcal{S}$ does not contain quads isomorphic to $W(2)$.*

**Proof.** Suppose the contrary, then by Lemma 9, there exists a $W(2)$-quad $Q$ through a point $x$ of $\mathcal{F}$. By Lemma 10, this quad $Q$ intersects $\mathcal{F}$ in a line $K$. Let $L$ be a line of $Q$ through $x$ different from $K$ and let $H \cong H^D(5,4)$ be a hex of $\mathcal{F}$ through $K$. Let $X$ denote the set of points of $\mathcal{L}(H,x)$ (i.e. the set of lines of $H$ through $x$) which are contained in a $W(2)$-quad together with $L$. We will now show that $|U \cap X| \in \{0,3\}$ for every line $U$ of $\mathcal{L}(H,x)$. Suppose that $|U \cap X| \geq 1$ and let $Q_U$ denote the $Q(5,2)$-quad corresponding with $U$. Since $|U \cap X| \geq 1$, there exists a $W(2)$-quad $R$ through $L$ which intersects $Q_U$ in a line. By Section 4, $\mathcal{C}(Q_U,R)$ is isomorphic to either $\mathbb{G}_3$ or $\mathbb{E}_3$. In any case, exactly three lines of $Q_U$ through $x$ are contained in a $W(2)$-quad together with $L$, or equivalently, $|U \cap X| = 3$. Since $H$ contains exactly five $Q(5,2)$-quads through $K$, we have $|X| = 1+5\cdot 2 = 11$. Hence $X$ is a set of 11 points in PG(2,4) such that every line meets it in either 0 or 3 points, but such a set does not exist (Otherwise, every point of PG(2,4) outside $X$ would be contained in $\frac{11}{3}$ lines meeting $X$). □

For every line $L$ intersecting $\mathcal{F}$ in a point $x$, let $\mathcal{A}_L$ be the set of lines $M$ of $\mathcal{F}$ through $x$ such that $\mathcal{C}(L,M) \cong Q(5,2)\}$. By Lemmas 10 and

12, $\mathcal{C}(L, L') \cong Q(5,2)$ for every line $L' \neq L$ intersecting $\mathcal{F}$ in $x$. Hence $t_{\mathcal{S}} - t_{\mathcal{F}} = 3|\mathcal{A}_L| + 1$. As a consequence, $|\mathcal{A}_L|$ is independent from the choice of $L$ and equal to $\alpha := \frac{t_{\mathcal{S}} - t_{\mathcal{F}} - 1}{3}$.

**Lemma 13** *$\mathcal{A}_L$ is a subspace of $\mathcal{L}(\mathcal{F}, x)$.*

**Proof.** Let $K_1$ and $K_2$ be two different lines of $\mathcal{A}_L$ and let $K_3$ be an arbitrary line through $x$ contained in $\mathcal{C}(K_1, K_2)$. Since the hex $\mathcal{C}(L, K_1, K_2)$ has three $Q(5,2)$-quads through the same point $x$ and no $W(2)$-quads, it must be isomorphic to $H^D(5,4)$. Hence, $\mathcal{C}(L, K_3) \cong Q(5,2)$ and $K_3 \in \mathcal{A}_L$. This proves the lemma. □

**Lemma 14** $\alpha \in \{0, 1, t_{\mathcal{F}} + 1\}$.

**Proof.** We suppose that $2 \leq \alpha \leq t_{\mathcal{F}}$ and derive a contradiction.

(a) Suppose first that $n = 4$, so $\mathcal{F} \cong H^D(5,4)$. Let $x$ be an arbitrary point of $\mathcal{F}$ and let $L$ be an arbitrary line through $x$ not contained in $\mathcal{F}$. Since $\mathcal{A}_L$ is a subspace and $2 \leq |\mathcal{A}_L| \leq 20$, there exists a $Q(5,2)$-quad $Q_x$ through $x$ such that $\mathcal{A}_L$ is the set of five lines of $Q_x$ through $x$. Also, $\alpha = 5$, $t_{\mathcal{S}} = t_{\mathcal{F}} + 3\alpha + 1 = 36$ and the hex $H_x := \mathcal{C}(L, Q_x)$ is isomorphic to $H^D(5,4)$. If $L'$ is a line through $x$ different from $L$ and not contained in $\mathcal{F}$, then $\mathcal{C}(L, L')$ is isomorphic to $Q(5,2)$ and hence intersects $\mathcal{F}$ in a line of $Q_x$. It follows that every line through $x$ not contained in $\mathcal{F}$ is contained in $H_x$. Since $H_x \cong H^D(5,4)$, we then have that $H_x := \mathcal{C}(\Gamma_1(x) \setminus \mathcal{F})$. So, $H_x$ and $Q_x = H_x \cap \mathcal{F}$ only depend on $x$ and not on the line $L$. If $y \in Q_x$, then $H_y = H_x$ and $Q_y = Q_x$. Hence, the quads $Q_x$, $x \in \mathcal{F}$, partition the point set of $\mathcal{F}$, and the hexes $H_x$, $x \in \mathcal{F}$, partition the point set of $\mathcal{S}$. Since $\mathcal{F} \cong H^D(5,4)$ is big in $\mathcal{S}$, every hex isomorphic to $H^D(5,4)$ is big. [For, the number of points at distance 0 or 1 from $\mathcal{F}$ equals the number of points at distance 0 or 1 from any hex isomorphic to $\mathcal{F}$.] In particular, each of the hexes $H_x$ is big. The total number of quads $Q_x$, $x \in \mathcal{F}$, equals $\frac{|H^D(5,4)|}{|Q(5,2)|} = 33$. Let $X$ denote the set of 33 points of $H(5,4)$ corresponding to the quads $Q_x$, $x \in \mathcal{P}$. If $u$ is a point of $X$, then we denote the $Q(5,2)$-quad of $\mathcal{F}$ corresponding with it by $u^\phi$ (see Section 3) and the unique hex of $\mathcal{S}$ intersecting $\mathcal{F}$ in $u^\phi$ by $H(u)$. We will now show that $X$ is a subspace of the linear space $\mathcal{L}$ defined in Lemma 7. Consider two different points $u_1$ and $u_2$ in $X$. Since $u_1^\phi$ and $u_2^\phi$ are disjoint, $u_1u_2 \cap H(5,4)$ is a Baer subline $\{u_1, u_2, u_3\}$ and $u_3^\phi$ is the reflection of $u_1^\phi$ about $u_2^\phi$ (in $\mathcal{F}$). The reflection of $H(u_1)$ about $H(u_2)$ (in $\mathcal{S}$) is a hex which meets $\mathcal{F}$ in the quad $u_3^\phi$ and hence coincides with $H(u_3)$. As a consequence $u_3 \in X$. This proves that $X$ is a subspace of the linear space $\mathcal{L}$. Hence, there exists a subspace $\pi$ such that $X = \pi \cap H(5,4)$. Since no two points of $X$ are collinear on $H(5,4)$, we have $|X| \leq |H(2,4)| = 9$, contradicting $|X| = 33$.

(b) We suppose that $n \geq 5$. Let $x$ denote an arbitrary point of $\mathcal{F}$ and let $L$ denote an arbitrary line through $x$ not contained in $\mathcal{F}$. Since $2 \leq \alpha \leq t_{\mathcal{F}}$, there exist lines $K_1$, $K_2$ and $K_3$ through $x$ such that $K_1, K_2 \in \mathcal{A}_L$, $K_1 \neq K_2$ and $K_3 \notin \mathcal{A}_L$. Now, the near octagon $\mathcal{C}(L, K_1, K_2, K_3)$ contradicts (a). □

**Lemma 15** *If $\alpha = 0$, then $\mathcal{S} \cong H^D(2n-3,4) \times \mathbb{L}_3$. If $\alpha = 1$, then $\mathcal{S} \cong H^D(2n-3,4) \otimes Q(5,2)$. If $\alpha = t_{\mathcal{F}} + 1$, then $\mathcal{S} \cong H^D(2n-1,4)$.*

**Proof.** If $\alpha = 0$, then $t_{\mathcal{S}} - t_{\mathcal{F}} = 1$ and hence $\mathcal{S} \cong H^D(2n-3,4) \times \mathbb{L}_3$ by Lemma 1. If $\alpha = t_{\mathcal{F}}+1$, then no grid-quad intersects $\mathcal{F}$ and hence all quads are isomorphic to $Q(5,2)$ by Lemma 9. Since the generalized quadrangle $Q(5,2)$ has no ovoids (see e.g. Theorem 3.4.1 of [11]), $\mathcal{S}$ must then be classical and isomorphic to $H^D(2n-1,4)$, see Section 2.3. ($H^D(2n-1,4)$ is the only classical near $2n$-gon in which all quads are isomorphic to $Q(5,2)$.)

Suppose now that $\alpha = 1$, then $t_{\mathcal{S}} = t_{\mathcal{F}} + 4$. If $x$ is an arbitrary point of $\mathcal{F}$ and if $L$ and $M$ denote two lines through $x$ not contained in $\mathcal{F}$, then $\mathcal{C}(K, L)$ is a $Q(5,2)$-quad. Hence, every point $x \in \mathcal{F}$ is contained in a unique $Q(5,2)$-quad $Q_x$ which intersects $\mathcal{F}$ in a line $L_x$. The intersection lines $L_x$, $x \in \mathcal{F}$, determine a spread of $\mathcal{F}$ and the set $T_1 := \{Q_x | x \in \mathcal{F}\}$ determines a partition of $\mathcal{S}$ in quads.

Now, consider a point $x$ of $\mathcal{S}$ not contained in $\mathcal{F}$, let $y$ be the unique point of $\mathcal{F}$ collinear with $x$, and let $A := xy$, $B$, $C$, $D$ and $E$ be the lines of $Q := Q_y$ through $x$. Let $L$ be a line intersecting $Q$ in $x$ and consider the hex $H := \mathcal{C}(L, Q)$. The hex $H$ contains a grid-quad $\mathcal{C}(L, A)$, no $W(2)$-quads, and at least two $Q(5,2)$-quads through the line $L_y = Q \cap \mathcal{F}$ (namely $Q$ and $H \cap F$). It follows from Section 4 that $H \cong Q(5,2) \otimes Q(5,2)$. Hence, exactly one line of $Q$ through $x$, say $B$, is contained in a $Q(5,2)$-quad with $L$.

Now, consider a geodetically sub near $2(n-1)$-gon $\mathcal{F}'$ of $\mathcal{S}$ containing $Q' := \mathcal{C}(B, L)$ and intersecting $Q$ in $B$. We will show that every quad of $\mathcal{F}'$ is isomorphic to $Q(5,2)$. Let $L_1$ and $L_2$ be two lines of $\mathcal{F}'$ through $x$. If $L_1 \neq B \neq L_2$, then the hex $H' := \mathcal{C}(A, L_1, L_2)$ contains grid-quads (namely $\mathcal{C}(A, L_1)$ and $\mathcal{C}(A, L_2)$), a $Q(5,2)$-quad (namely $\mathcal{C}(A, L_1, L_2) \cap \mathcal{F}$) and no $W(2)$-quads and hence must be isomorphic to either $Q(5,2) \times \mathbb{L}_3$ or $Q(5,2) \otimes Q(5,2)$, see Section 4. In any case, $\mathcal{C}(L_1, L_2)$ is isomorphic to $Q(5,2)$. If $L_1 = B$ and if $L_2$ is not contained in $Q'$, then $\mathcal{C}(L_2, M) \cong Q(5,2)$ for every line $M \neq B$ through $x$ contained in $Q'$. This is only possible if $\mathcal{C}(L_2, Q')$ is isomorphic to $H^D(5,4)$, see Section 4. Hence, also $\mathcal{C}(L_1, L_2) = \mathcal{C}(B, L_2)$ is isomorphic to $Q(5,2)$. It now follows that every quad of $\mathcal{F}'$ through $x$ is isomorphic to $Q(5,2)$. By Lemma 9 it then follows that every quad of $\mathcal{F}'$ is isomorphic to $Q(5,2)$. As before, we then know that $\mathcal{F}'$ is classical and isomorphic to $H^D(2n-3,4)$. Obviously, $\mathcal{F}'$ is the only geodetically closed sub near polygon through $x$ isomorphic to

$H^D(2n-3,4)$.

Repeating the above construction for every point $x$ outside $\mathcal{F}$, we obtain a partition $T_2$ of $\mathcal{S}$ in geodetically closed sub near polygons isomorphic to $H^D(2n-3,4)$. We can now apply Lemma 6 and conclude that $\mathcal{S}$ is a glued near polygon of type $H^D(2n-3,4) \otimes Q(5,2)$. In Section 3 we have shown that there exists a unique such glued near polygon. □

# Acknowledgements

The first author is a postdoctoral researcher of the Fund for Scientific Research - Flanders (Belgium). The second author's research is supported by the Flemish Institute for the Promotion of Scientific and Technological Research in Industry (IWT), grant No. IWT/SB/001022/Vandecasteele.

# References

[1] A. E. Brouwer, A. M. Cohen, J. I. Hall, and H. A. Wilbrink. Near polygons and Fischer spaces. *Geom. Dedicata*, 49:349–368, 1994.

[2] A. E. Brouwer, A. M. Cohen and A. Neumaier. Distance-Regular Graphs. SpringerVerlag, Berlin, 1989.

[3] A. E. Brouwer and H. A. Wilbrink. The structure of near polygons with quads. *Geom. Dedicata*, 14:145–176, 1983.

[4] P. J. Cameron. Dual polar spaces. *Geom. Dedicata*, 12:75–86, 1982.

[5] B. De Bruyn. On near hexagons and spreads of generalized quadrangles. *J. Alg. Comb.*, 11:211–226, 2000.

[6] B. De Bruyn. On the number of nonisomorphic glued near hexagons. *Bull. Belg. Math. Soc.*, 7:493–510, 2000.

[7] B. De Bruyn. The glueing of near polygons. *Bull. Belg. Math. Soc. Simon Stevin*, 9:621–630, 2002.

[8] B. De Bruyn. Decomposable near polygons. *Ann. Comb.*, 8:251–267, 2004.

[9] B. De Bruyn and P. Vandecasteele. Two conjectures regarding dense near polygons with three points on each line. *Eur. J. Comb.*, 24:631–647, 2003.

[10] B. De Bruyn and P. Vandecasteele. Near polygons with a nice chain of sub near polygons. *J. Combin. Theory Ser. A*, 108:297–311, 2004.

[11] S. E. Payne and J. A. Thas. *Finite Generalized Quadrangles*, volume 110 of *Research Notes in Mathematics*. Pitman, Boston, 1984.

[12] E. E. Shult and A. Yanushka. Near $n$-gons and line systems. *Geom. Dedicata*, 9:1–72, 1980.

[13] J. Tits. Sur la trialité et certains groupes qui s'en déduisent. *Inst. Hautes Etudes Sci. Publ. Math.*, 2:14–60, 1959.

[14] J. Tits.8 Classification of buildings of spherical type and Moufang polygons: a survey. In *Teorie Combinatorie, Proc. Intern. Colloq. (Roma 1973)*, volume I, pages 229–246. Accad. Naz. Lincei, 1976.