The Vertex Linear Arboricity of an Integer Distance Graph with a Special Distance Set

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Abstract

The vertex linear arboricity $vla(G)$ of a graph G is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that each subset induces a subgraph whose connected components are paths. An integer distance graph is a graph $G(D)$ with the set of all integers as vertex set and two vertices $u, v \in Z$ are adjacent if and only if $| u - v | \in D$ where the distance set D is a subset of the positive integers set. Let $D_{m,k} = \{1, 2, \cdots, m\} - \{k\}$ for $m > k \geq 1$. In this paper, some upper and lower bounds of the vertex linear arboricity of the integer distance graph $G(D_{m,k})$ are obtained. Moreover, $vla(G(D_{m,1})) = \lceil \frac{m}{4} \rceil + 1$ for $m \geq 3$, $vla(G(D_{8l+1,2})) = 2l + 2$ for any positive integer l and $vla(G(D_{4q,2})) = q + 2$ for any integer $q \geq 2$.

Keywords integer distance graph; vertex linear arboricity; path coloring

1 Introduction

In this paper, R, Z, P, N denote the set of all the real numbers, all the integers, all the prime numbers, and all the positive integers, respectively. For $x \in R$, $\lfloor x \rfloor$ denotes the greatest integer not more than x; $\lceil x \rceil$ denotes the least integer not less than x. For a finite set S, its cardinality is denoted by $|S|$. G is called a *supergraph* of H if H is a subgraph of G (see [2]), denoted by $H \leq G$.

A *k-coloring* of a graph G is a mapping f from $V(G)$ to $\{1, 2, ..., k\}$. With respect to a given k-coloring, V_i denotes the set of all vertices of G colored with i. If V_i is an independent set for every $1 \leq i \leq k$, then f is called a proper k-coloring. The chromatic number $\chi(G)$ of a graph G

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is the minimum number k of colors for which G has a proper k-coloring. If V_i induces a subgraph whose connected components are paths, then f is called a *path k-coloring*. The *vertex linear arboricity* of a graph G, denoted by $vla(G)$, is the minimum number k of colors for which G has a path k-coloring.

Matsumoto [14] proved that for a finite graph G, $vla(G) \le \lceil\frac{\Delta(G)+1}{2}\rceil$, moreover, if $\Delta(G)$ is even, then $vla(G) = \lceil\frac{\Delta(G)+1}{2}\rceil$ if and only if G is a complete graph of order $\Delta(G)+1$ or a cycle. Goddard [9] and Poh [15] proved that $vla(G) \le 3$ for a planar graph G. Akiyama *etc.* [1] proved $vla(G) \le 2$ if G is an outerplanar graph. Fang and Wu [8] determined the vertex linear arboricity of complete multiple graphs and obtained an upper bound of Cartesian product of graphs.

If S is a subset of the set of real numbers and D is a subset of the set of positive real numbers, then the *distance graph* $G(S, D)$ is defined by the graph with vertex set S and two vertices x and y are adjacent if and only if $|x-y| \in D$ where the set D is called the *distance set*. In particular, if all elements of D are positive integers and $S = Z$, then the graph $G(Z, D) = G(D)$ is called the *integer distance graph* and the set D is called its *integer distance set*. The concept of the distance graph was introduced by Eggleton, Erdös and Skilton [6] in 1985. They proved $\chi(G(R, D)) = n+1$ if D is an interval between 1 and δ when $1 \le n-1 < \delta \le n$, $\chi(G(P)) = 4$. They also obtained some results on the chromatic number of $G(D_{m,k})$ where $D_{m,k} = \{1, 2, \cdots, m\} - \{k\}$. Recently, Chang, Liu and Zhu [3] determined completely the chromatic numbers of $G(D_{m,k})$. More results on the chromatic number of integer distance graphs, see [6], [7], [10], [16] and [17]. For the vertex linear arboricity of distance graphs, Zuo, Wu and Liu [18] obtained the following results: (1) $vla(G(R, D)) = n+1$ if D is an interval between 1 and δ when $1 \le n-1 < \delta \le n$; (2) $vla(G(D)) = 2$ if $|D| = 2$, or $|D| \ge 2$ and there is at most one even number in D; (3) $vla(G(D)) \le k$ if there is at most one multiple of k in D.

It is easy to prove that $vla(G(D)) = \lceil\frac{m+1}{2}\rceil$ for $D = \{1, 2, \cdots, m\}$ (see [18]). It is interesting to determine $vla(G(D))$ when $D = D_{m,k} = \{1, 2, ..., m\} - \{k\}$ where $1 \le k < m$! In this paper, we will show: (1) $vla(G(D_{m,1})) = \lceil\frac{m}{4}\rceil + 1$ for $m \ge 3$; (2) $\lceil\frac{m+1}{4}\rceil + 1 \le vla(G(D_{m,2})) \le \lceil\frac{m}{4}\rceil + 2$ for $m \ge 6$; (3) $vla(G(D_{m,k})) = \lceil\frac{k}{2}\rceil$ for $m \le k + \lfloor\frac{k}{2}\rfloor - 1$; (4) $\lceil\frac{m+k+1}{4}\rceil \le vla(G(D_{m,k})) \le k(\lceil\frac{m}{4k}\rceil + 1)$ and $vla(G(D_{m,k})) \le d\lceil\frac{m+3k+1}{4d}\rceil$ for $m \ge 3k$ and $d = \gcd(k, m+3k+1)$, the greatest common divisor of k and $m+3k+1$. In particular, we also obtain $vla(G(D_{8l+1,2})) = 2l+2$ for any positive integer l, and $vla(G(D_{4q,2})) = q+2$ for any integer $q \ge 2$.

2 Main results

The following lemma can be found in [18].

Lemma 2.1. (1) *Let* $D = \{m+1, m+2, \cdots, m+h\}$ *with* $m \geq 0, h \geq 2$, *then* $vla(G(D)) = \lceil \frac{k+1}{2} \rceil$ *for* $m = 0$ *and* $vla(G(D)) \leq \lceil \frac{m+h}{m+3} \rceil + 1$ *for* $m \geq 1$.

(2) *If there is at most one multiple of* k *in distance set* D, *then we have* $vla(G(D)) \leq k$.

Clearly, $vla(G(D)) = 1$ if $|D| = 1$. If $|D| \geq 2$, then $vla(G(D)) \geq 2$ since the graph does not consist of paths only. It is obvious that $vla(G(D_2)) \leq vla(G(D_1))$ if $D_2 \subseteq D_1$. In the following, we will study the vertex linear arboricity of $G(D_{m,k})$ for $k = 1$, $k = 2$ and $k \geq 3$ respectively. First, we consider the case $k = 1$. Since $G(D_{2,1}) = G(\{2\})$, $vla(G(D_{2,1})) = 1$.

Theorem 2.2. *For each positive integer* $m \geq 3$, $vla(G(D_{m,1})) = \lceil \frac{m}{4} \rceil + 1$.

Proof. It follows from Lemma 2.1 that $vla(G(D_{3,1})) = vla(G(D_{4,1})) = 2$. We may assume in the following that $m \geq 5$. Let $n = \lceil m/4 \rceil$. Clearly, $n \geq 2$.

Firstly, we give a $(n+1)$-coloring f of $G(D_{m,1})$. For any integer i, let $f(4i) = f(4i+1) = f(4i+2) = f(4i+3) = i(\mod (n+1))$. Since every four such vertices induce a path $(4i+2, 4i, 4i+3, 4i+1)$ and every connected component of the subgraph induced by vertices received the same color is such a path, f is a path $(n+1)$-coloring of $G(D_{m,1})$. Hence, $vla(G(D_{m,1})) \leq n+1 = \lceil m/4 \rceil + 1$.

Conversely, suppose that $G(D_{m,1})$ has a path n-coloring f. Then f is also a path n-coloring of the subgraph G_m of $G(D_{m,1})$ induced by vertices $0, 1, 2, ..., 4n$. Since $|V(G_m)| = 4n+1$, there must be at least 5 vertices in G_m, say a, b, c, d and e, received the same color α, where $0 \leq a < b < c < d < e \leq 4n$.

Claim 2.1. $m = 4n - 3$.

Suppose that $m \geq 4n-2$. If $c < d-1$, then $ca, cd, ce \in E(G_m)$, that is, vertices a, c, d and e induce a $K_{1,3}$, a contradiction. So $c = d-1$. If $a < b-1$, then $ba, bd, be \in E(G_m)$, a contradiction. So $a = b-1$. Similarly, we have $b = c-1, d = e-1$. Therefore $a+2 = b+1 = c = d-1 = e-2$. But under the condition, a, c and e form a cycle of length 3, a contradiction, too. The contradiction proves Claim 2.1.

Claim 2.2. $b < 2$ and $d > 4n-2$, that is, $a = 0, b = 1, d = 4n-1$ and $e = 4n$.

Here we only give the proof of the case $b < 2$. The case $d > 4n-2$ can be settled similarly. Assume, on the contrary, $b \geq 2$.

Suppose $b-a \geq 2$. If $b \geq 3$ or $e < 4n$, then $e-b \leq 4n-3 = m$, that is, $ba, bd, be \in E(G_m)$, a contradiction. So $b=2$ and $e=4n$. It follows that $a=0$. We also have $c=b+1=3$, for otherwise, if $c > b+1$, then $ba, bc, bd \in E(G_m)$, a contradiction. If $d > c+1$, then $ca, cd, ce \in E(G_m)$, a contradiction. So $d = c+1 = 4$. But under the condition, $ab, bd, ad \in E(G_m)$, a, b and d induce a cycle of length 3, a contradiction, too.

Suppose $b-a=1$. Then $a \geq 1$ and $c \geq 3$. If $c > b+1$ then $ca, cb, ce \in E(G_m)$, a contradiction. If $d > c+1$ then $ca, cd, ce \in E(G_m)$, a contradiction. If $e > d+1$ then $da, db, de \in E(G_m)$, a contradiction. So $a+2=b+1=c=d-1=e-2$, $ac, ce, ae \in E(G_m)$, a contradiction, too.

Hence, Claim 2.2. holds.

It follows from Claim 2.1 and Claim 2.2 that $c=2$ or $c=4n-2$, for otherwise, if $3 \leq c \leq 4n-3$, then $ac, bc, ec \in E(G_m)$, a contradiction. At the same time, vertices 2 and $4n-2$ receive different colors, for otherwise, vertices $0, 2, 4n-2$ and $4n-1$ induce a $K_{1,3}$, a contradiction, too.

Without loss of generality, assume that the vertex $4n-2$ is colored with α. This implies that except α, any other color just colors four vertices in G_m and these four vertices must be consecutive since the difference of any two such vertices is less than m. That is to say, vertices $2, 3, 4$ and 5 receive one color, vertices $6, 7, 8$ and 9 receive another color and so on.

Now let us come back to analyze the coloring of vertices $4n+1$ and $4n+2$ of $G(D_{m,1})$. Suppose $f(4n+1) = \beta \neq \alpha$. Then there exists $l (2 \leq l \leq 4n-6)$ such that $f(l) = f(l+1) = f(l+2) = f(l+3) = \beta$. Then vertices $l, l+1, l+3$ and $4n+1$ induce a $K_{1,3}$ since $4 \leq 4n+1-(l+3) \leq m$ for any integer $2 \leq l \leq 4n-6$, a contradiction. So $f(4n+1) = \alpha$. Similarly, $f(4n+2) = \alpha$.

It follows that $f(4n-2) = f(4n-1) = f(4n) = f(4n+1) = f(4n+2) = \alpha$ which is impossible. This contradiction implies that $vla(G(D_{m,1})) \geq n+1 = \lceil \frac{m}{4} \rceil + 1$. □

Second, we consider the case $k=2$. By Lemma 2.1, $vla(G(D_{3,2})) = vla(G(D_{4,2})) = vla(G(D_{5,2})) = 2$. For $m \geq 6$, we have the following result.

Theorem 2.3. *For each positive integer $m \geq 6$,*

$$\lceil \frac{m+1}{4} \rceil + 1 \leq vla(G(D_{m,2})) \leq \lceil \frac{m}{4} \rceil + 2.$$

In particular, $vla(G(D_{8l+1,2})) = 2l+2 = \lceil \frac{8l+2}{4} \rceil + 1$ and $vla(G(D_{4q,2})) = q+2 = \lceil \frac{4q}{4} \rceil + 2$.

Proof. Firstly, we prove the upper bound. Suppose that $m = 8l+j$ with $4 < j \leq 8$ and $0 \leq h < 8(\lceil \frac{m}{8} \rceil + 1)$. Let

$$f(h) = \begin{cases} 2u+1, & \text{for } h = 8u, 8u+2, 8u+4, 8u+6, \\ 2u+2, & \text{for } h = 8u+1, 8u+3, 8u+5, 8u+7. \end{cases}$$

For each $t \in Z$, let $f(8t(\lceil \frac{m}{8} \rceil + 1) + h) = f(h)$. Since the vertices that received same color in one connected component are only vertices $8u, 8u+2, 8u+4$ and $8u+6$, or $8u+1, 8u+3, 8u+5$ and $8u+7$, and they induce a path $(8u+4, 8u, 8u+6, 8u+2)$ or $(8u+5, 8u+1, 8u+7, 8u+3)$, f is a path coloring of $G(D_{m,2})$. Therefore $vla(G(D_{m,2})) \leq 2\lceil \frac{m}{8} \rceil + 2 = \lceil \frac{m}{4} \rceil + 2$.

Suppose $m = 8l + j$ for $1 \leq j \leq 4$ and $0 \leq h < 8\lceil \frac{m}{8} \rceil$. Let

$$\bar{f}(h) = \begin{cases} 2u+1, & \text{for } h = 8u, 8u+2, 8u+4, 8u+6, \\ 2u+2, & \text{for } h = 8u+1, 8u+3, 8u+5, 8u+7, \end{cases}$$

$\bar{f}(8l+8) = \bar{f}(8l+9) = \bar{f}(8l+10) = 2l+3$ and $\bar{f}((8l+10)t+h) = \bar{f}(h)$ for any $t \in Z$. Then $\bar{f}$ is a path coloring of $G(D_{m,2})$. Hence, $vla(G(D_{m,2})) \leq 2l+3 = \lceil \frac{m}{4} \rceil + 2$.

Secondly, we will prove $vla(G(D_{m,2})) \geq \lceil \frac{m+1}{4} \rceil + 1$. For sake of simplicity, we only give a proof of the case $m = 4q \geq 6$, and other cases can be settled similarly. Now $\lceil \frac{m+1}{4} \rceil + 1 = q + 2$. Let $n = q + 1$. Assume, to the contrary, that $G(D_{m,2})$ has a path n-coloring f. Consider the subgraph H_m of $G(D_{m,2})$ induced by vertices $0, 1, 2, \cdots, 4n = m + 4$. Since $|V(H_m)| = 4n + 1$, there must be at least 5 vertices in H_m, say a, b, c, d and e, such that $f(a) = f(b) = f(c) = f(d) = f(e) = \alpha$, where $0 \leq a < b < c < d < e \leq 4n$.

Claim 3.1. $\min\{d - a, e - b\} > m$.

Suppose $d - a \leq m$. Then $ad \in E(H_m)$. If $b - a \neq 2$, then $ab \in E(H_m)$ and $ac \notin E(H_m)$. So $c = a+2 = b+1$ and $bc \in E(H_m)$. Thus $bd \notin E(H_m)$, that is, $d - b = 2$ and then $cd \in E(H_m)$, a, b, c and d induce a cycle of length 4, a contradiction. Hence, $b - a = 2$. It is similar to prove $c-b = d-c = 2$. So $d = c+2 = b+4 = a+6$ and then $ac, bd, ad \in E(H_m)$. Since $d = a + 6 \geq 6$, $e - d < m$. It follows from $bd, ad \in E(H_m)$ that $e - d = 2$ and $ae, ce \in E(H_m)$, so vertices a, c, e induce a cycle of length 3, a contradiction. Hence, $d - a > m$. Similarly, we obtain that $e - b > m$. Claim 3.1 is proved.

By Claim 3.1, if vertices $u, u+2, u+4, u+6$ receive the same color β, then any other vertex v which received color β would satisfy $\min\{|v - u|, |v - (u+6)|\} > m$.

Claim 3.2. There are at most five vertices receiving color α in H_m.

If there are six vertices receiving the same color α, it is not difficult to verify that they would be vertices $0, 1, 2, 4q+2, 4q+3$ and $4q+4 = 4n$. Then $4q - 1$ vertices $3, 4, \cdots, 4q+1 = m+1$ receive q colors where one color β colors three vertices g, h, l and any other color colors four vertices. Any four vertices with the same color would be $u, u+2, u+4, u+6$ for some integer $u \geq 3$. Thus vertices $4n+1, 4n+2, 4n+3$ and $4n+4$ would be colored β, but these vertices induce a cycle of length 4, a contradiction.

Therefore, there are just five vertices a, b, c, d and e in H_m receiving color α.

Claim 3.3. $\min\{c-a, e-c\} = 2$.

If $c-a=3$ and $e-c \geq 3$, then $ac, cd \in E(H_m)$ because of $m-3 < d-a-(c-a) = d-c \leq (4n-1)-3 = 4q = m$ by Claim 3.1. Thus $bc, ce \notin E(H_m)$ and then $ab \in E(H_m)$, that is, $b = a+1$ and $e-c > m$. Hence, $a=0, b=1, c=3$ and $e=4n$.

If $d = 4q+3$, then $4q$ vertices $2, 4, 5, \cdots, 4q+2$ receive q colors, and any color just colors four vertices that are $h, h+2, h+4, h+6$ in $\{2, 4, 5, \cdots, 4q+2\}$ for some $h \geq 2$. Consider vertex $h_1 = 4n+2$ in $G(D_{m,2})$, by Claim 3.1, h_1 would be colored α since $h_1 - (h+6) \leq m$ for any $4q-4 \geq h \geq 2$, but $dc, de, dh_1 \in E(G(D_{m,2}))$, thus we have a contradiction.

Suppose $d = 4q+2$. Then $4q$ vertices $2, 4, 5, \cdots, 4q+1, 4q+3$ receive q colors. If $f(2) \neq f(4q+3)$, then any four vertices with the same color in $\{2, 4, 5, \cdots, 4q+1, 4q+3\}$ would be $v, v+2, v+4, v+6$ for some integer $4q-3 \geq v \geq 2$. Thus $f(4n) = f(4n+1) = f(4n+2) = f(4n+3) = \alpha$ which is impossible. If $2, g_1, g_2, g_3 = 4q+3$ are colored β, then $g_1g_3 \in E(H_m)$ or $g_2g_3 \in E(H_m)$, and any other four vertices with the same color would be $v, v+2, v+4, v+6$ for some integer $4q-5 \geq v \geq 4$. So $|\{u | f(u) = \beta, 4n+2 \leq u \leq 4n+5\}| \leq 1$ and $|\{u | f(u) = \alpha, 4n+2 \leq u \leq 4n+5\}| \geq 3$. But e and any three vertices in $\{4n+2, 4n+3, 4n+4, 4n+5\}$ would induce a subgraph which contains a cycle, a contradiction, too.

If $d \leq 4q+1$, then $d = 4q+1$ by Claim 3.1, so $bd, cd, ed \in E(H_m)$, a contradiction, too.

It is similar to get a contradiction when $e-c=3, c-a \geq 3$. If $c-a > 3, e-c > 3$, then $ce, ac \in E(H_m)$ and $bc, cd \notin E(H_m)$. Hence, $b = c-2, d = c+2$. So $c-a = d-2-a \geq m-1, e-c = e-(b+2) = e-b-2 \geq m-1$ because of $d-a > m, e-b > m$, and then $e-a \geq 2m-2 > m+4$ which is impossible since $e \leq 4n = m+4$.

Therefore, we obtain that $c-a=2$ or $e-c=2$.

Claim 3.4. $c=2$ or $c=4q+2$.

Assume, on the contrary, that $3 \leq c \leq 4q+1$. Suppose $a = c-2$. Then $ab, bc \in E(H_m)$ and $bd \notin E(H_m)$, that is, $d-b > m$, so $b=2, c=3, d=4q+3, e=4n$ and $a=1$, and vertex 0 could not be colored α. Hence, three vertices with the color of vertex 0 would be u_1, u_2, and $u_3 = 4q+1$ or $u_3 = 4q+2$, for otherwise, if $u_3 \leq 4q$, then vertex 0 will be adjacent to vertices u_1, u_2 and u_3, a contradiction. Clearly, $u_1u_3 \in E(H_m)$ or $u_2u_3 \in E(H_m)$ because of $\min\{u_1, u_2\} \geq 4$. Without loss of generality, suppose $u_1u_3 \in E(H_m)$. Any other four vertices that received the same color must be $v, v+2, v+4, v+6$ for some integer $4q-4 \geq v > 3$. Thus, $f(4n+2) = f(4n+3) = f(0)$, and vertices $4n+2, 4n+3$ are all adjacent to u_3. So vertices $u_1, u_3, 4n+2$ and $4n+3$ induce a $K_{1,3}$, a contradiction.

Suppose $e = c+2$. Then $e \leq 4q+3, d \leq 4q+2$ and $cd, de \in E(H_m)$. Thus $bd \notin E(H_m)$ and then $d - b > m$ (for otherwise, if $d - b = 2, bc, be \in E(H_m)$, vertices b, c, d and e induce a cycle, a contradiction), so $b = 1, c = 4q+1, d = 4q+2, e = 4q+3$ and $a = 0$. Then $4q$ vertices $2, 3, 4, 5, \cdots, 4q = m, 4q+4 = m+4$ receive q colors. If $f(m+4) \notin \{f(2), f(3)\}$, then any four vertices with the same color in $\{2, 3, 4, 5, \cdots, 4q = m, m+4\}$ would be $v, v+2, v+4, v+6$ for some integer $v \geq 2$, but it is impossible for any v with $2 \leq v \leq m-2$ such that $m = v+4$ and $m+4 = v+6$. If vertices $2 < g_1 < g_2 < g_3 = 4q+4$ are colored β, then $4 \leq g_2 \leq 4q$, so $g_2 g_3 \in E(H_m)$. Thus $|\{u | f(u) = \beta, 4n+2 \leq u \leq 4n+4\}| \leq 2$. Because any other four vertices with the same color would be $v, v+2, v+4, v+6$ for some integer $v \geq 3$, $|\{u | f(u) = \alpha, 4n+2 \leq u \leq 4n+4\}| \geq 1$, but c, d, e and any one vertex in $\{4n+2, 4n+3, 4n+4\}$ would induce a subgraph which contains a cycle, a contradiction. It is similar to get a contradiction if $3 < g_1 < g_2 < g_3 = 4q+4$ are colored β.

Therefore, Claim 3.4 is proved.

Without loss of generality, suppose $c = 2$. Then $a = 0, b = 1$ and $d \geq 4q+2$, $e = 4q+3$ or $e = 4n$. If $d = 4q+3$, then $e = 4n$, so $4q$ vertices $3, 4, \cdots, 4q+2$ in H_m receive q colors where any of which colors four vertices $v, v+2, v+4, v+6$ for some integer $v \geq 3$. Hence, vertices $4n+1, 4n+2$ would be colored α, and they induce a cycle along with d, e, a contradiction. If $d = 4q+2, e = 4q+3$, then $4q$ vertices $3, 4, \cdots, 4q+1, 4n$ receive q colors where any of which colors four vertices. If $3 < h_1 < h_2 < 4n$ receive the same color β, then vertices h_1, h_2 are all adjacent to $4n$ and any other color colors four vertices $v, v+2, v+4, v+6$ for some integer $v \geq 4$, so vertex $4n+3$ receives color α, and it induces a cycle along with d, e, thus a contradiction. If $f(3) \neq f(4n)$, then any color colors four vertices $v, v+2, v+4, v+6$ for some $3 \leq v \leq 4n-6$ in $\{3, 4, \cdots, 4q+1, 4n\}$, but it is impossible for any v with $3 \leq v \leq 4n-6$ such that $4q+1 = v+4$ and $4n = v+6$. Similarly, it is not difficult to get a contradiction for $d = 4q+2, e = 4n$.

Therefore, $vla(G(D_{4q,2})) \geq \lceil \frac{4q+1}{4} \rceil + 1$. The case $m \neq 4q$ can be settled similarly.

Since $vla(G(D_{4q,2})) \leq \lceil \frac{4q}{4} \rceil + 2 = \lceil \frac{4q+1}{4} \rceil + 1$, $vla(G(D_{4q,2})) = q+2 = \lceil \frac{4q}{4} \rceil + 2$.

In addition, suppose $m = 8l+1$ for some integer l and $0 \leq h < 8(l+1)$. Let

$$f'(h) = \begin{cases} 2u+1, & \text{for } h = 8u, 8u+2, 8u+4, 8u+6, \\ 2u+2, & \text{for } h = 8u+1, 8u+3, 8u+5, 8u+7. \end{cases}$$

Then f' can be extended to a path coloring of $G(D_{8l+1,2})$, so we have $vla(G(D_{8l+1,2})) \leq 2\lfloor\frac{8l+1}{8}\rfloor + 2 = \lceil\frac{8l+1}{4}\rceil + 1 = 2l + 2$. But $vla(G(D_{m,2})) \geq \lceil\frac{m+1}{4}\rceil + 1 = 2l + 2$, so $vla(G(D_{m,2})) = 2l + 2 = \lceil\frac{m+1}{4}\rceil + 1$. The proof is completed. □

This theorem means that $vla(G(D_{m,2})) = \lceil\frac{m+1}{4}\rceil + 1$ or $\lceil\frac{m}{4}\rceil + 2$ for $m \geq 6$ and $m \neq 4q$. For some small m, we can determine $vla(G(D_{m,2}))$. For example, by Theorem 2.3, $vla(G(D_{6,2})) \geq 3$. But for $m = 7$, let $f(n) = i$ if $n \equiv 3i, 3i+1, 3i+2 (\mod 9)$ for $i = 0, 1, 2$. Then f is a path coloring of $G(D_{7,2})$. Therefore $vla(G(D_{6,2})) = vla(G(D_{7,2})) = 3$.

Let $f(n) = i$ if $n \equiv 3i, 3i+1, 3i+2 (\mod 12)$ for $i = 0, 1, 2, 3$. Then f is a path coloring of $G(D_{10,2})$. Therefore $vla(G(D_{10,2})) = vla(G(D_{8,2})) = vla(G(D_{9,2})) = 4$ by Theorem 2.3.

Suppose that $m = 13$. Let $f(n) = i$ if $n \equiv 3i, 3i+1, 3i+2 (\mod 15)$ for $i = 0, 1, 2, 3, 4$. Then f is a path coloring of $G(D_{13,2})$, so $vla(G(D_{13,2})) = vla(G(D_{12,2})) = 5$ by Theorem 2.3. For $m = 11$, Theorem 2.3 implies $4 \leq vla(G(D_{11,2})) \leq 5$ (in fact, it is not difficult to prove that $vla(G(D_{11,2})) = 5$).

Now we consider the case $k \geq 3$.

Theorem 2.4. *For any positive integers m, k with $m > k \geq 3$,*

$$\begin{array}{ll} vla(G(D_{m,k})) = \lceil\frac{k}{2}\rceil, & \text{for } m \leq \lfloor\frac{3k}{2}\rfloor - 1, \\ \lceil\frac{m+1}{3}\rceil \leq vla(G(D_{m,k})) \leq k, & \text{for } \lfloor\frac{3k}{2}\rfloor \leq m < 3k. \end{array}$$

Let $m = 4kl + j \geq 3k$ for some l and j, $0 \leq j < 4k$, then

$$\lceil\frac{m+k+1}{4}\rceil \leq vla(G(D_{m,k})) \leq \begin{cases} k(\lfloor\frac{m}{4k}\rfloor + 1), & \text{for } 0 \leq j < k, \\ k\lceil\frac{m}{4k}\rceil + \lceil\frac{j-k+1}{2}\rceil, & \text{for } k \leq j < 2k, \\ k\lceil\frac{m}{4k}\rceil + \lceil\frac{k}{2}\rceil, & \text{for } 2k \leq j < \lfloor\frac{5k-2}{2}\rfloor, \\ k(\lceil\frac{m}{4k}\rceil + 1), & \text{for } \lfloor\frac{5k-2}{2}\rfloor \leq j < 4k. \end{cases}$$

Proof. Suppose $m \leq k + \lfloor\frac{k}{2}\rfloor - 1$. Since vertex set $\{0, 1, \cdots, k-1\}$ induces a complete subgraph of order k, $vla(G(D_{m,k})) \geq \lceil\frac{k}{2}\rceil$. Let $f_1(kl + i) \equiv i (\mod \lceil\frac{k}{2}\rceil)$ for $l \in Z$ and $0 \leq i < k$, that is, all vertices in $V_i = \{\cdots, i, \lceil\frac{k}{2}\rceil + i, k+i, k + \lceil\frac{k}{2}\rceil + i, 2k+i, \cdots\}$ receive color i for every $0 \leq i \leq \lfloor\frac{k}{2}\rfloor - 1$ and all vertices in $V_{(k-1)/2} = \{\cdots, (k-1)/2, k + (k-1)/2, 2k + (k-1)/2, \cdots\}$ receive color $\frac{k-1}{2}$ if k is odd. It is clear that V_i induces a path since $2k + i - (\lceil\frac{k}{2}\rceil + i) = k + \lfloor\frac{k}{2}\rfloor > m$, and $V_{(k-1)/2}$ is an independent set if k is odd. So f_1 is a path $\lceil\frac{k}{2}\rceil$-coloring and it follows that $vla(G(D_{m,k})) \leq \lceil\frac{k}{2}\rceil$. Therefore, $vla(G(D_{m,k})) = \lceil\frac{k}{2}\rceil$.

Suppose that $k + \lfloor\frac{k}{2}\rfloor \leq m \leq 2k - 1$. By Lemma 2.1, $vla(G(D_{m,k})) \leq k$. Let H be the subgraph of $G(D_{m,k})$ induced by vertices $0, 1, 2, \cdots, m$.

Then H is a complete k-partite graph $K(2,\cdots,2,1,\cdots,1\}$ with partition sets $X_0=\{0,k\}$, $X_1=\{1,k+1\}$, $\cdots$, $X_{m-k}=\{m-k,m\}$, $X_{m-k+1}=\{m-k+1\}$, $\cdots$, $X_{k-1}=\{k-1\}$. It is obvious that any four vertices of H induce a subgraph which contains a cycle, and any three vertices of H that are exactly contained in three different partite induce a cycle. So $vla(H)=2k-m-1+\lceil 2\frac{m-k+1-(2k-1-m)}{3}\rceil=\lceil\frac{m+1}{3}\rceil$. Therefore, $vla(G(D_{m,k}))\geq vla(H)\geq\lceil\frac{m+1}{3}\rceil$ for $2k-1\geq m\geq k+\lfloor\frac{k}{2}\rfloor$.

Suppose that $2k\leq m<3k$. We also have $vla(G(D_{m,k}))\leq k$ by Lemma 2.1. Let $X_0'=\{0,k,2k\}$, $X_1'=\{1,k+1,2k+1\}$, $\cdots$, $X_{m-2k}'=\{m-2k,m-k,m\}$, $X_{m-2k+1}'=\{m-2k+1,m-k+1\}$, $\cdots$, $X_{k-1}'=\{k-1,2k-1\}$, then $X_0'\cup X_1'\cup\cdots\cup X_{k-1}'=\{0,1,2,\cdots,m\}$ induces a supergraph H' of a complete k-partite graph $K(3,3,\cdots,3,2,\cdots,2)$. It is clear that any four vertices of H' induce a cycle. Therefore, $vla(H')=m-2k+1+\lceil 2\frac{k-1-(m-2k)}{3}\rceil=\lceil\frac{m+1}{3}\rceil$ and then $vla(G(D_{m,k}))\geq\lceil\frac{m+1}{3}\rceil$.

Suppose that $m=4kl+j\geq 3k$. If $0\leq j<k$ and $0\leq n<4k(l+1)$, let $f_3(n)=i+kt$ for $n-(i+4kt)=0,k,2k,3k,0\leq i<k,0\leq t\leq l$, and $f_3(n+4ks(l+1))=f_3(n)$ for every $s\in Z$. It is obvious that f_3 is a periodic path coloring of $G(D_{m,k})$. Therefore $vla(G(D_{m,k}))\leq(l+1)k=k(\lfloor\frac{m}{4k}\rfloor+1)$.

If $k\leq j<2k$, let

$$f_4(n)=\begin{cases} i+kt, & \text{for } n-(4kt+i)=0,k,2k,3k,\\ & 0\leq i<k,0\leq t\leq l,\\ k(l+1)+\lfloor\frac{n-4k(l+1)}{2}\rfloor, & \text{for } 4k(l+1)\leq n\leq m+3k,\end{cases}$$

and other vertices be colored periodically, that is, $f_4(n+(m+3k)t)=f_4(n)$ for any $t\in Z$. Then f_4 is a path coloring, and so $vla(G(D_{m,k}))\leq k\lceil\frac{m}{4k}\rceil+\lceil\frac{m+3k-4k(l+1)+1}{2}\rceil=k\lceil\frac{m}{4k}\rceil+\lceil\frac{j-k+1}{2}\rceil$.

If $2k\leq j<2k+\lfloor\frac{k}{2}\rfloor-1$, for $0\leq n\leq m+3k$, let

$$f_5(n)=\begin{cases} i+kt, & \text{for } n-(4kt+i)=0,k,2k,3k,\\ & 0\leq i<k,0\leq t\leq l,\\ k(l+1)+i, & \text{for } n-i-4k(l+1)=0,\lceil\frac{k}{2}\rceil,k,\ 0\leq i<\lceil\frac{k}{2}\rceil,\end{cases}$$

and other vertices be colored periodically, that is, $f_5(n+(m+3k)t)=f_5(n)$ for any $t\in Z$. Then f_5 is a path coloring, so $vla(G(D_{m,k}))\leq k\lceil\frac{m}{4k}\rceil+\lceil\frac{k}{2}\rceil$.

If $2k+\lfloor\frac{k}{2}\rfloor-1\leq j<4k$, for $0\leq n<4k(l+2)$, let $f_6(n)=i+kt$ for $n-(i+4kt)=0,k,2k,3k$, $0\leq i<k$, $0\leq t\leq l+1$, and $f_6(n+4ks(l+2))=f_6(n)$ for every $s\in Z$. Then f_6 is a periodical path coloring and so $vla(G(D_{m,k}))\leq(l+2)k=k(\lceil\frac{m}{4k}\rceil+1)$.

Now we begin to prove $vla(G(D_{m,k}))\geq\lceil\frac{m+k+1}{4}\rceil$. Assume, on the contrary, that $vla(G(D_{m,k}))<\lceil\frac{m+k+1}{4}\rceil$. Let $n=\lceil\frac{m+k+1}{4}\rceil-1$, then $G(D_{m,k})$ has a path n-coloring f. Let G_m be a subgraph of $G(D_{m,k})$

induced by vertices $0, 1, 2, \cdots, m+k$, then f is also a path n-coloring of G_m. Note that $|V(G_m)| = m+k+1$. There must be 5 vertices a, b, c, d and e receiving the same color α, where $0 \le a < b < c < d < e \le m+k$.

Claim 4.1. $\min\{c-a, d-b, e-c\} \ge k$ and $\max\{c-a, e-c\} \le m$.

Assume, for example, that $c-a < k$, then $ab, ac, bc \in E(G_m)$ and a, b, c induce a cycle, a contradiction. So the claim holds.

Claim 4.2. $\min\{d-a, e-b\} > m$ and $\max\{b-a, e-d\} < k$.

Suppose that $d-a \le m$, we will prove that $d = c+k = b+2k = a+3k$. If $d-c \ne k$, then $cd \in E(G_m)$, and $ac \notin E(G_m)$ or $bc \notin E(G_m)$ (for otherwise, a, b, c and d induce a $K_{1,3}$, a contradiction). In the former case, $c-a = k$ and $ab, bc, ad \in E(G_m)$, a, b, c and d induce a cycle of length 4, a contradiction. In the latter case, $c-b = k$ and $cd, bd, ad \in E(G_m)$, a, b, c and d induce a $K_{1,3}$, a contradiction. Hence $d-c = k$. It is similar to prove that $c-b = b-a = k$. Thus $ac, bd, ad \in E(G_m)$ and $de \notin E(G_m)$, so $e-d = k$ and $ce, be \in E(G_m)$, a, b, c, d and e induce a cycle of length 5, a contradiction, too. Therefore $d-a > m$. Similarly, we can obtain that $e-b > m$. Hence, $b-a < k$ and $e-d < k$.

We will come to contradictions, whatever the relative position of a and c may be.

Case 1. $c-a = k$.

Since $ab, bc \in E(G_m)$, bd is not in $E(G_m)$, so $d-b > m$ by Claim 4.2. Thus $e-d < k, b < k$, and $k < d-c < m+k-k = m$, so $cd \in E(G_m)$. But $e-c \le m$ and $e-c > d-(b+k) = d-b-k > m-k > k$, therefore $ce \in E(G_m)$, and then b, c, d and e induce a $K_{1,3}$, a contradiction.

Case 2. $c-a > k$.

Since $b-a < k$ and $c-a \le m$, $ab, ac \in E(G_m)$ and $bc \notin E(G_m)$, so $c-b = k$ by Claim 4.1. If $bd \in E(G_m)$, then $cd \notin E(G_m)$, so $d-c = k$ since $d-c \le m$. Thus $ad \in E(G_m)$, however, it is impossible by Claim 4.2. If $bd \notin E(G_m)$, then $d-b > m$, $e-d < k < d-c < e-c$, so $cd, de, ce \in E(G_m)$ by Claim 4.1. Thus c, d and e induce a cycle of length 3, a contradiction.

By all above arguments, we have $vla(G(D_{m,k})) \ge \lceil \frac{m+k+1}{4} \rceil$. Thus the theorem is proved. □

By this theorem, it is obtained that $vla(G(D_{3k-3,k})) = vla(G(D_{3k-2,k})) = vla(G(D_{3k-1,k})) = k$ for any positive integer k. For $m = 2k-1$, we will get $vla(G(D_{m,k})) \ge \lceil \frac{m+2}{3} \rceil$. Let $\bar{X}_0 = \{0, k, 2k\}$, $\bar{X}_1 = \{1, k+1\}, \cdots, \bar{X}_{k-1} = \{k-1, 2k-1\}$. Then vertex subset $\{0, 1, \cdots, 2k\}$ induces a supergraph of the complete k-partite graph $K(3, 2, \cdots, 2) = \bar{H}$ and $vla(\bar{H}) = 1 + \lceil \frac{2(k-1)}{3} \rceil = \lceil \frac{2k+1}{3} \rceil$, so $vla(G(D_{2k-1,k})) \ge \lceil \frac{2k+1}{3} \rceil = \lceil \frac{m+2}{3} \rceil$. Therefore, $vla(G(D_{m,3})) = 3$ for $5 \le m \le 8$. So the upper bound is sharp.

By Theorem 2.4 and Lemma 2.1, $vla(G(D_{5,4})) = 2$.

Suppose $m = 9$, let $f(n) = 0$ if $n \equiv 0,1,3(\mod 12)$, $f(n) = 1$ if $n \equiv 2,4,5(\mod 12)$, $f(n) = 2$ if $n \equiv 6,7,9(\mod 12)$ and $f(n) = 3$ if $n \equiv 8,10,11(\mod 12)$, then f is a path coloring of $G(D_{9,3})$. So $vla(G(D_{9,3})) = 4$ and the lower bound in Theorem 2.4 is sharp.

It is easy to verify that $4 \leq vla(G(D_{10,3})) \leq 5$ (if we define $f(n) = 1$ for $n \equiv 0,1,3,14,16,17(\mod 30)$, $f(n) = 2$ for $n \equiv 2,4,5,18,19,21(\mod 30)$, $f(n) = 3$ for $n \equiv 6,7,9,20,22,23(\mod 30)$, $f(n) = 4$ for $n \equiv 8,10,11,24,25,27(\mod 30)$ and $f(n) = 5$ for $n \equiv 12,13,15,26,28,29(\mod 30)$, then f is a path coloring).

We will give another upper bound of $vla(G(D_{m,k}))$ for $m \geq 3k \geq 9$.

Theorem 2.5. *For any $m \geq 3k \geq 9$, $vla(G(D_{m,k})) \leq d\lceil\frac{m+3k+1}{4d}\rceil$ where $d = \gcd(k, m+3k+1)$ is the greatest common divisor of k and $m+3k+1$.*

Proof. Define a circulant graph H on the set $\{0, 1, \cdots, m+3k\}$ with generating set $D_{m,k}$, that is, ij is an edge of H if and only if $(j-i) \mod (m+3k+1) \in D_{m,k}$ or $(i-j) \mod (m+3k+1) \in D_{m,k}$. It is enough to find a path n-coloring f of H, where $n = d\lceil\frac{m+3k+1}{4d}\rceil$. Let $w = \frac{m+3k+1}{d}$. Divide the vertex set of H into d subsets such that each subset has w vertices and is of the form $\{i, i+k, \cdots, i+(w-1)k\} \mod (m+3k+1)$. Any consecutive four vertices in a subset constitute a linear forest, so each subset can be partitioned into $\lceil\frac{w}{4}\rceil = \lceil\frac{m+3k+1}{4d}\rceil$ linear forests of size 4, except the last one, whose size might be smaller than 4. Therefore, the vertex set of H can be partitioned into $d\lceil\frac{m+3k+1}{4d}\rceil$ linear forests and so H has a path n-coloring f. It is easy to verify that the coloring f can be extended to a path n- coloring f' of $G(D_{m,k})$ by letting $f'(y) = f(x)$, where $x \equiv y \mod (m+3k+1)$. Hence, $vla(G(D_{m,k})) \leq d\lceil\frac{m+3k+1}{4d}\rceil$. □

$vla(G(D_{112,16})) \leq \lceil\frac{161}{4}\rceil = 41$ by Theorem 2.5, and $vla(G(D_{112,16})) \leq 16(\lceil\frac{112}{64}\rceil + 1) = 48$ by Theorem 2.4, so the upper bound of $vla(G(D_{112,16}))$ in Theorem 2.5 is smaller than that in Theorem 2.4. Since $3\lceil\frac{29}{12}\rceil + \lceil\frac{2+1}{2}\rceil = 11$ and $3\lceil\frac{39}{12}\rceil = 12$, the upper bound of $vla(G(D_{29,3}))$ in Theorem 2.4 is smaller than that in Theorem 2.5.

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