Orientable embedding distributions by genus for certain type of non-planar graphs (I)

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Abstract– We introduce certain type of surfaces M_j^n, for $j = 1, 2, ..., 11$ and determine their genus distributions. At the basis of joint trees introduced by Liu, we develop the *surface sorting method* to calculate the embedding distribution by genus.

Keywords– Embedding distribution; joint tree; surface; genus.

1. Introduction

Graphs considered in this paper are connected graphs with a cycle. Surfaces are closed 2-dimensional manifolds without a boundary. Embeddings of a graph considered are always assumed to be orientable.

Let G be a graph and T be a spanning tree of G. For each non-tree edge e, a *joint tree* $\widetilde{T}$ is obtained by splitting the edge e into two semi-edges with letters e^+ and e^-. For notational convenience, when there is no confusion arises in the context, we shall use e for e^+ throughout this paper. For a rotation σ of G, let G_σ be an

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embedding determined by σ. As shown in [2], there is a joint tree $\widetilde{T}_\sigma$ corresponding to G_σ. Then,

Lemma 1.1 [2] For any $\sigma_1 \neq \sigma_2$, the embeddings G_{σ_1} and G_{σ_2}, as well as $\widetilde{T}_{\sigma_1}$ and $\widetilde{T}_{\sigma_2}$ are not homeomorphic.

Lemma 1.2 [2] Let T and T' be two distinct spanning trees of a graph G, Σ the rotation set of G. There exists a bijection between $(\Sigma, \widetilde{T})$ and $(\Sigma, \widetilde{T'})$ where $(\Sigma, \widetilde{T}) = \{\widetilde{T}_\sigma | \sigma \in \Sigma\}$ and $(\Sigma, \widetilde{T'}) = \{\widetilde{T'}_\sigma | \sigma \in \Sigma\}$.

For a joint tree $\widetilde{T}_\sigma$ of G, $\mathcal{P}^{\sigma}_{\widetilde{T}}$ induced by all semi-edges of $\widetilde{T}_\sigma$ is regarded as the *embedding surface* of $\widetilde{T}_\sigma$. Let $\mathcal{S}$ be a collection of surfaces. For a surface S, let $o(S)$ be the genus of S. In order to determine $o(S)$, an equivalence $\sim$ defined on $\mathcal{S}$ is introduced. It can be determined by the following operations:

$OP1$. $AB \sim (Ae)(e^- B)$ where $e \notin AB$;

$OP2$. $Ae_1e_2Be_2^-e_1^- \sim AeBe^- = Ae^-Be$ where $e \notin AB$;

$OP3$. $Aee^-B \sim AB$ where $AB \neq \emptyset$.

Further, it is seen that each embedding surface is equivalent to one, and only one, of the following canonical forms ([1]) of surfaces:

$$O_i = \begin{cases} a_0a_0^-, & \text{if } i = 0; \\ \prod_{k=1}^{i} a_kb_ka_k^-b_k^-, & \text{if } i \geq 1 \end{cases}$$

which are the sphere ($i = 0$), torus ($i = 1$) and orientable surfaces of genus i ($i \geq 2$). For the detailed information about the background, see [1,2].

For a graph G, let $g_i(G)$ be the number of embeddings with genus i. Then the embedding polynomial of G is as follows:

$$f_G(x) = \sum_{i=0}^{\infty} g_i(G)x^i.$$

Gross and Furst [3] introduced the embedding distributions of graphs by genus. Later, orientable embedding distributions of circular ladders and Möbius ladders [4], closed-end ladders and cobblestone paths [5], bouquets of circles [6] and Ringel ladders [7] were introduced and studied. The total embedding distributions of necklaces, closed-end ladders and cobblestone paths [8] and bouquets of circles [9] were also defined and investigated. In this paper, the genus distributions of M_j^n are calculated for a positive integer n and $j = 1, 2, 3, \cdots, 24$ (See Section 3). In Section 4, the following theorem is proved.

Theorem 4.1 $g_i(G_n)$ is a linear combination $g_{m_j}(n)$'s for $j = 1, 6$, $0 \le m \le i$ and $n \ge 1$.

By sorting the embedding surfaces of G_n and using the genus distribution of M_j^n the embedding distribution of G_n is computed, which is the *surface sorting method.*

2. Lemmas

Lemma 2.1 [1] Let A, B, C, D and E be linear sequences ([1]). Then

$$AxByCx^-Dy^-E \sim ADCBExyx^-y^-.$$

Lemma 2.2 [2] Let S and S' be surfaces. If $S \sim S'xyx^-y^-$(of course, $x, y, x^-, y^- \notin S'$), then $o(S) = o(S') + 1$.

Lemma 2.3 Let A, B, C and D be linear sequences. Then

$$xABx^-CD \sim xBAx^-CD \sim xABx^-DC$$

where $x, x^- \notin ABCD$.

Proof. It follows by applying OP1, OP2 and OP3, as well as their inverses. □

Corollary 2.4 Let A, B, C and D be linear sequences. Then,

$$AxBx^-yCy^-zDz^- \sim xBx^-AyCy^-zDz^- \sim xBx^-yCy^-AzDz^-$$

where $x \ne y \ne z$ and $x, y, z, x^-, y^-, z^- \notin ABCD$.

Proof. This follows from Lemma 2.3. □

Lemma 2.5 Let A, B, C and D be linear sequences. Then,

$$AxBx^-yCy^-zDz^- \sim BxAx^-yCy^-zDz^- \sim CxAx^-yBy^-zDz^- \sim DxAx^-yBy^-zCz^-$$

where $x \ne y \ne z$ and $x, y, z, x^-, y^-, z^- \notin ABCD$.

Proof. By applying Lemma 2.3 three times and Corollary 2.4,

$$AxBx^-yCy^-zDz^- \sim Bx^-yCy^-AxzDz^- = yCy^-AxzDz^-Bx^-$$
$$\sim yCy^-x^-AxzDz^-B = ByCy^-x^-AxzDz^-$$
$$\sim x^-AxByCy^-zDz^- \sim Bx^-AxyCy^-zDz^-.$$

By Corollary 2.4, the foregoing conclusion and Lemma 2.3,

$$AxBx^-yCy^-zDz^- \sim xBx^-AyCy^-zDz^- = AyCy^-zDz^-xBx^-$$
$$\sim CyAy^-zDz^-xBx^- = xBx^-CyAy^-zDz^-$$
$$\sim CyAy^-xR_2x^-zDz^-.$$

Similarly, $AxBx^-yCy^-zDz^- \sim DxAx^-yBy^-zCz^-$. □

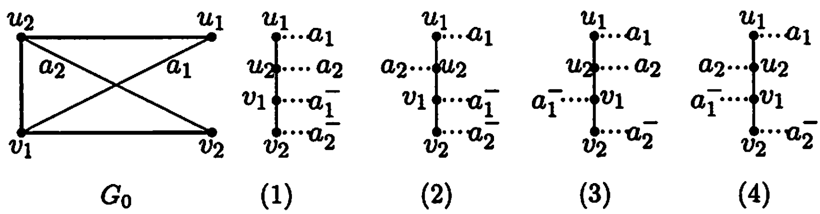

Fig. 1: G_0 and its joint trees

For example, let G_0 be a graph in Fig. 1. Let a_1, a_2 be non-tree edges. Let each vertex have a clockwise rotation in joint trees of G_0. Then, four joint trees of G_0 are shown in Fig. 1 (1), (2), (3) and (4). Their embedding surfaces are $a_1a_2a_1^-a_2^-$, $a_1a_1^-a_2^-a_2$, $a_1a_2a_2^-a_1^-$ and $a_1a_2^-a_1^-a_2$ in correspondence. Then, $f_{G_0}(x) = 2 + 2x$.

3. Genus Distributions of Some Types of Surfaces

For a set of surfaces M, let $g_i(M)$ be the number of surfaces with genus i in M. The *genus distribution* of M is the sequence: $g_0(M), g_1(M), g_2(M), \cdots$. The *genus polynomial* of M is as follows:

$$f_M(x) = \sum_{i=0}^{\infty} g_i(M)x^i.$$

Given a positive integer n, let $y_1, y_2, \cdots, y_n$ denote n distinct letters. Let $Y_1 = y_{k_1}y_{k_2}y_{k_3}\cdots y_{k_r}$, $Y_2 = y_{k_{r+1}}y_{k_{r+2}}y_{k_{r+3}}\cdots y_{k_n}$, $Y_3 =$

$y_{m_1}^{-}y_{m_2}^{-}y_{m_3}^{-}\cdots y_{m_s}^{-}$ and $Y_4 = y_{m_{s+1}}^{-}y_{m_{s+2}}^{-}y_{m_{s+3}}^{-}\cdots y_{m_n}^{-}$ where $n \geq k_1 > k_2 > k_3 > \cdots > k_r \geq 1$, $1 \leq k_{r+1} < k_{r+2} < k_{r+3} < \cdots < k_n \leq n$, $1 \leq m_1 < m_2 < m_3 < \cdots < m_s \leq n$, $n \geq m_{s+1} > m_{s+2} > m_{s+3} > \cdots > m_n \geq 1$ and $0 \leq r, s \leq n$, $k_p \neq k_q$, $m_p \neq m_q$ for $p \neq q$. Let $M_1^n = \{Y_1Y_2Y_3Y_4\}$, $M_2^n = \{Y_1Y_2Y_4Y_3\}$, $M_3^n = \{Y_1Y_3Y_2Y_4\}$, $M_4^n = \{aY_1Y_2a^-Y_3Y_4\}$, $M_5^n = \{aY_1Y_3a^-Y_2Y_4\}$, $M_6^n = \{aY_1Y_4a^-Y_2Y_3\}$, $M_7^n = \{aY_1a^-Y_2Y_4Y_3\}$, $M_8^n = \{Y_1Y_2aY_3a^-bY_4b^-\}$, $M_9^n = \{Y_1Y_3aY_2a^-bY_4b^-\}$, $M_{10}^n = \{Y_1Y_4aY_2a^-bY_3b^-\}$, $M_{11}^n = \{Y_1aY_2a^-bY_3b^-cY_4c^-\}$, $M_{12}^n = \{aY_2a^-Y_3Y_1Y_4\}$, $M_{13}^n = \{Y_1Y_4Y_3Y_2\}$, $M_{14}^n = \{Y_1Y_3Y_4Y_2\}$, $M_{15}^n = \{Y_1Y_4Y_2Y_3\}$, $M_{16}^n = \{aY_2a^-Y_1Y_3Y_4\}$, $M_{17}^n = \{aY_3a^-Y_1Y_4Y_2\}$, $M_{18}^n = \{aY_4a^-Y_1Y_2Y_3\}$, $M_{19}^n = \{Y_3Y_4aY_1a^-bY_2b^-\}$, $M_{20}^n = \{Y_2Y_4aY_1a^-bY_3b^-\}$, $M_{21}^n = \{Y_2Y_3aY_1a^-bY_4b^-\}$, $M_{22}^n = \{aY_1a^-Y_2Y_3Y_4\}$, $M_{23}^n = \{aY_3a^-Y_1Y_2Y_4\}$ and $M_{24}^n = \{aY_4a^-Y_1Y_3Y_2\}$.

Lemma 3.1 $f_{M_1^n}(x) = f_{M_{13}^n}(x)$, $f_{M_2^n}(x) = f_{M_{14}^n}(x)$, $f_{M_3^n}(x) = f_{M_{15}^n}(x)$, $f_{M_7^n}(x) = f_{M_{16}^n}(x) = f_{M_{17}^n}(x) = f_{M_{18}^n}(x)$, $f_{M_8^n}(x) = f_{M_{19}^n}(x)$, $f_{M_9^n}(x) = f_{M_{20}^n}(x)$, $f_{M_{10}^n}(x) = f_{M_{21}^n}(x)$ and $f_{M_{12}^n}(x) = f_{M_{22}^n}(x) = f_{M_{23}^n}(x) = f_{M_{24}^n}(x)$.

Proof. For Y_1, Y_2, Y_3 and Y_4 defined above, denote $Y_1' = y_{n+1-k_{r+1}} y_{n+1-k_{r+2}}y_{n+1-k_{r+3}}\cdots y_{n+1-k_n}$, $Y_2' = y_{n+1-k_1}y_{n+1-k_2}y_{n+1-k_3}\cdots y_{n+1-k_r}$, $Y_3' = y_{n+1-m_{s+1}}^{-}y_{n+1-m_{s+2}}^{-}y_{n+1-m_{s+3}}^{-}\cdots y_{n+1-m_n}^{-}$ and $Y_4' = y_{n+1-m_1}^{-}y_{n+1-m_2}^{-}y_{n+1-m_3}^{-}\cdots y_{n+1-m_s}^{-}$. Define a function ϕ from M_7^n to M_{16}^n as follows:

$$\phi(aY_1a^-Y_2Y_4Y_3) = aY_2'a^-Y_1'Y_3'Y_4'.$$

Obviously, ϕ is a bijection. Since $o(aY_1a^-Y_2Y_4Y_3) = o(aY_2'a^-Y_1'Y_3'Y_4')$, it follows that $f_{M_7^n}(x) = f_{M_{16}^n}(x)$.

Let $Y_1'' = y_{m_s}\cdots y_{m_3}y_{m_2}y_{m_1}$, $Y_2'' = y_{m_n}\cdots y_{m_{s+3}}y_{m_{s+2}}y_{m_{s+1}}$, $Y_3'' = y_{k_r}^{-}\cdots y_{k_3}^{-}y_{k_2}^{-}y_{k_1}^{-}$ and $Y_4'' = y_{k_n}^{-}\cdots y_{k_{r+3}}^{-}y_{k_{r+2}}^{-}y_{k_{r+1}}^{-}$. Define a function ψ from M_7^n to M_{17}^n such that $\psi(aY_1a^-Y_2Y_4Y_3) = aY_3''a^-Y_1''Y_4''Y_2''$.

Obviously, ψ is a bijection. By Lemma 2.3,

$$aY_3''a^-Y_1''Y_4''Y_2'' \sim aY_3''a^-Y_4''Y_2''Y_1''.$$

Since $o(aY_1a^-Y_2Y_4Y_3) = o(aY_3''a^-Y_4''Y_2''Y_1'')$,

$$o(aY_1a^-Y_2Y_4Y_3) = o(aY_3''a^-Y_1''Y_4''Y_2'').$$

Thus, $f_{M_7^n}(x) = f_{M_{17}^n}(x)$.

All other equalities can be verified in a similar way. □

Theorem 3.2 Let $g_{i_j}(n)$ be the number of surfaces with genus i in M_j^n for $n \geq 0$, $i \geq 0$ and $1 \leq j \leq 11$. Let $f_{M_j^0}(x) = 1$. Then, for $n \geq 1$, $g_{i_j}(n) =$

$$4g_{i_7}(n-1), \text{ if } j = 1 \text{ and } 0 \leq i \leq \left[\frac{n}{2}\right];$$
$$2g_{i_2}(n-1) + 2g_{i_4}(n-1), \text{ if } j = 2 \text{ and } 0 \leq i \leq \left[\frac{n}{2}\right];$$
$$g_{i_3}(n-1) + g_{i_5}(n-1) + 2g_{i_7}(n-1), \text{ if } j = 3 \text{ and } 0 \leq i \leq \left[\frac{n}{2}\right];$$
$$4g_{(i-1)_2}(n-1), \text{ if } j = 4 \text{ and } 1 \leq i \leq \left[\frac{n+1}{2}\right];$$
$$2g_{(i-1)_2}(n-1) + 2g_{i_5}(n-1), \text{ if } j = 5 \text{ and } 0 \leq i \leq \left[\frac{n+1}{2}\right];$$
$$2g_{(i-1)_3}(n-1) + 2g_{i_{10}}(n-1), \text{ if } j = 6 \text{ and } 1 \leq i \leq \left[\frac{n+1}{2}\right];$$
$$g_{(i-1)_1}(n-1) + g_{(i-1)_2}(n-1) + g_{i_7}(n-1) + g_{i_8}(n-1), \quad \text{if } j = 7, 12 \text{ and } 0 \leq i \leq \left[\frac{n+1}{2}\right];$$
$$4g_{(i-1)_7}(n-1), \text{ if } j = 8 \text{ and } 1 \leq i \leq \left[\frac{n}{2}\right] + 1;$$
$$g_{(i-1)_5}(n-1) + 2g_{(i-1)_7}(n-1) + g_{i_9}(n-1), \quad \text{if } j = 9 \text{ and } 0 \leq i \leq \left[\frac{n}{2}\right] + 1;$$
$$g_{(i-1)_6}(n-1) + 2g_{(i-1)_7}(n-1) + g_{i_{11}}(n-1), \quad \text{if } j = 10 \text{ and } 1 \leq i \leq \left[\frac{n}{2}\right] + 1;$$
$$2g_{(i-1)_9}(n-1) + 2g_{(i-1)_{10}}(n-1), \quad \text{if } j = 11 \text{ and } 1 \leq i \leq \left[\frac{n+1}{2}\right] + 1.$$

Proof. Since the proof arguments for these equalities are similar, we shall only prove the typical cases when $j = 7, 12$ and $0 \leq i \leq [(n+1)/2]$, and leave the verifications of the other equalities to the readers. For $n \geq 1$, let $Z_1 = y_{k_1} y_{k_2} y_{k_3} \cdots y_{k_r}$, $Z_2 = y_{k_{r+1}} y_{k_{r+2}} y_{k_{r+3}} \cdots y_{k_{n-1}}$, $Z_3 = y_{\bar{m}_1} y_{\bar{m}_2} y_{\bar{m}_3} \cdots y_{\bar{m}_s}$ and $Z_4 = y_{\bar{m}_{s+1}} y_{\bar{m}_{s+2}} y_{\bar{m}_{s+3}} \cdots y_{\bar{m}_{n-1}}$ where

$n-1 \geq k_1 > k_2 > k_3 > \cdots > k_r \geq 1$, $1 \leq k_{r+1} < k_{r+2} < k_{r+3} < \cdots < k_{n-1} \leq n-1$, $1 \leq m_1 < m_2 < m_3 < \cdots < m_s \leq n-1$, $n-1 \geq m_{s+1} > m_{s+2} > m_{s+3} > \cdots > m_n \geq 1$ and $0 \leq r, s \leq n-1$, $k_p \neq k_q$, $m_p \neq m_q$ for $p \neq q$. Then, $M_7^{n-1} = \{aZ_1a^- Z_2Z_4Z_3\}$ and $M_7^n = \{aa_nZ_1a^-Z_2Z_4Z_3a_n^-, aa_nZ_1a^-Z_2a_n^-Z_4Z_3, aZ_1a^-Z_2a_na_n^-Z_4Z_3, aZ_1a^- Z_2a_nZ_4Z_3a_n^-\}$.

By Lemma 2.1, $aa_nZ_1a^-Z_2Z_4Z_3a_n^- \sim Z_1Z_2Z_4Z_3aa_na^-a_n^-$ and $aa_nZ_1a^-Z_2a_n^-Z_4Z_3 \sim Z_2Z_1Z_4Z_3aa_na^-a_n^-$.

By OP3, $aZ_1a^-Z_2a_na_n^-Z_4Z_3 \sim aZ_1a^-Z_2Z_4Z_3$.

By Lemma 2.3 and OP2,

$$\begin{aligned} aZ_1a^-Z_2a_nZ_4Z_3a_n^- &\sim aZ_1a^-Z_2a_nZ_3Z_4a_n^- \sim aZ_1a^-a_n^-Z_2a_nZ_3Z_4 \\ &= Z_3Z_4aZ_1a^-a_n^-Z_2a_n = Z_3Z_4aZ_1a^-bZ_2b^-. \end{aligned}$$

Thus, for $0 \leq i \leq \left[\frac{n+1}{2}\right]$,

$g_{i_7}(n) = g_{(i-1)_1}(n-1) + g_{(i-1)_2}(n-1) + g_{i_7}(n-1) + g_{i_8}(n-1)$.(1)

Similarly, for $0 \leq i \leq \left[\frac{n+1}{2}\right]$,

$g_{i_{12}}(n) = g_{(i-1)_1}(n-1) + g_{(i-1)_2}(n-1) + g_{i_{12}}(n-1) + g_{i_8}(n-1)$.(2)

By using induction on n for (1) and (2), $g_{i_7}(n) = g_{i_{12}}(n)$. □

4. Genus Distributions of G_n

Given a graph G. Let u_0v_0 be an edge on a cycle of G. For a positive integer n, add $2n$ vertices $u_1, u_2, u_3, \cdots, u_n$, $v_1, v_2, v_3, \cdots, v_n$ on e in sequence. Connect u_lv_l $(1 \leq l \leq n)$ by a new edge a_l to obtain a new graph G_n. Note that when $n \geq 3$, G_n is non-planar.

Theorem 4.1 $g_i(G_n)$ is a linear combination $g_{m_j}(n)$'s for $j = 1, 6$, $0 \leq m \leq i$ and $n \geq 1$.

Proof. Let T_n be a spanning tree of G_n such that u_0u_1, u_lu_{l+1}, u_nv_1, v_lv_{l+1} and v_nv_0 are tree edges for $1 \leq l \leq n-1$. Let each vertex of joint trees have a clockwise rotation at u_l and v_l for $1 \leq l \leq n$. Let S be an embedding surface of G. Then, $S = Y_4Y_1A_0Y_2Y_3B_0$ such that A_0B_0 is an embedding surface of G_0 where Y_1, Y_2, Y_3 and Y_4 are referred above.

Suppose that $o(A_0B_0) = t$. Then,

$$o(Y_4Y_1A_0Y_2Y_3B_0) = o(Y_4Y_1A_1Y_2Y_3A_1^-) + t.$$

If A_1 is an empty sequence, then $o(Y_4Y_1A_0Y_2Y_3B_0) = o(Y_4Y_1Y_2Y_3)+t$, otherwise we have $o(Y_4Y_1A_0Y_2Y_3B_0) = o(Y_4Y_1aY_2Y_3a^-) + t$.

Thus, the theorem is found. □

In fact, let G be a connected graph and e be an edge of G, the Theorem 4.1 still holds.

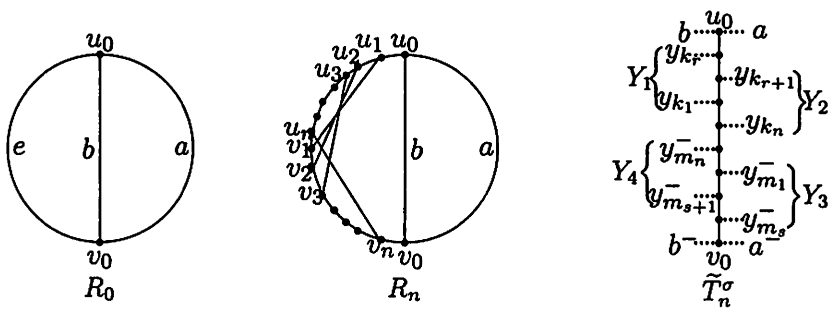

Fig.2: R_0, R_n and $\widetilde{T}_n^\sigma$

For example, let a, b and a_l $(1 \le l \le n)$ be non-tree edges for the graph R_n(Fig.2). Let joint trees of R_n have a clockwise rotation at each vertex. Then, embedding surfaces of R_n have four types $Y_4Y_1baY_2Y_3a^-b^-$, $Y_4Y_1abY_2Y_3b^-a^-$, $Y_4Y_1abY_2Y_3a^-b^-$ and $Y_4Y_1baY_2Y_3$ b^-a^- where Y_1, Y_2, Y_3 and Y_4 are referred above. By OP2, OP3 and Lemma 2.3,

$$\begin{aligned} Y_4Y_1baY_2Y_3a^-b^- \sim Y_4Y_1aY_2Y_3a^- &= a^-Y_4Y_1aY_2Y_3 \\ &= aY_4Y_1a^-Y_2Y_3 \sim aY_1Y_4a^-Y_2Y_3 \end{aligned}$$

and $Y_4Y_1abY_2Y_3b^-a^- \sim aY_1Y_4a^-Y_2Y_3$.

By Lemma 2.1, $Y_4Y_1abY_2Y_3a^-b^- \sim Y_4Y_1Y_2Y_3aba^-b^-$ and

$$Y_4Y_1baY_2Y_3b^-a^- \sim Y_4Y_1Y_2Y_3bab^-a^-.$$

Thus, $g_i(R_n) = 2g_{i_6}(n) + 2g_{(i-1)_1}(n)$.

The distribution polynomials of R_n for $n = 1, 2, 3, \cdots, 7$ are as follows:

$$f_{R_1}(x) = 2^2(1 + 3x);$$

$$f_{R_2}(x) = 2^2(1 + 11x + 4x^2);$$
$$f_{R_3}(x) = 2^5(3x + 5x^2);$$
$$f_{R_4}(x) = 2^6(2x + 11x^2 + 3x^3);$$
$$f_{R_5}(x) = 2^5(5x + 57x^2 + 66x^3);$$
$$f_{R_6}(x) = 2^4(15x + 225x^2 + 656x^3 + 128x^4);$$
$$f_{R_7}(x) = 2^6(6x + 107x^2 + 519x^3 + 392x^4).$$

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