Fractal Sequences and Restricted Nim

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Abstract

The *Grundy number* of an impartial game G is the size of the unique Nim heap equal to G. We introduce a new variant of Nim, *Restricted Nim*, which restricts the number of stones a player may remove from a heap in terms of the size of the heap. Certain classes of Restricted Nim are found to produce sequences of Grundy numbers with a self-similar fractal structure. Extending work of C. Kimberling, we obtain new characterizations of these "fractal sequences" and give a bijection between these sequences and certain upper-triangular arrays. As a special case we obtain the game of *Serial Nim*, in which the Nim heaps are ordered from left to right, and players can move only in the leftmost nonempty heap.

1 Introduction

The classic game of Nim, first studied by C. Bouton [4], is played with piles of stones. On her turn, a player can remove any number of stones from any one pile. The winner is the player to take the last stone. Many variants of Nim have been studied; see chapters 14–15 of [3, vol. 3] as well as [1, 2, 5, 9, 14, 15, 17]. In *Restricted Nim*, we place an upper or lower bound on the number of stones that can be removed in terms of the size of the pile. For example, suppose the players are permitted to remove any number of stones strictly smaller than half the size of the pile. Then a pile of size 2^n is a win for the second player: no matter how the first player moves, the second player can respond by reducing the size of the pile to 2^{n-1}; when just two stones remain, the first player is unable to move and loses. Likewise, the first player can win from any pile whose size is *not* a power of two by reducing the size to a power of two.

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In general, we may require that no more than $f(m)$ stones be removed from a pile of size m; here f may be any sequence of nonnegative integers satisfying $f(m) \leq m$. This is the game of *Maximum Nim*. Since the sequence f specifies the rules of the game, we will often refer to f as the *rule sequence*, or simply the *rule*.

Maximum Nim is an example of an *impartial game*. By the *Sprague-Grundy theory of impartial games* [3, 5, 8, 16] any impartial game G is equal to a Nim heap of size g for some g. The integer g is unique and is called the *Grundy number* of G. (For an explanation of impartial games, Sprague-Grundy theory and the notion of equality of games, we refer the reader to the first volume of [3].)

For each n, the game of Maximum Nim with rule f on a pile of size n has a Grundy number g_n. The sequence $(g_n)_{n\geq 0}$ will be called the *Grundy sequence* for Maximum Nim with rule f. By the Sprague-Grundy theory, the sequence g_n satisfies the recurrence

$$g_n = \operatorname{mex}\{g_{n-i}\}_{i=1}^{f(n)}, \tag{1}$$

in which mex S denotes the *minimal excludant* of the set S, the smallest nonnegative integer not in S.

Returning to our example in which the number of stones taken must be strictly smaller than half the size of the pile, the recurrence (1) with $f(n) = \lfloor \frac{n-1}{2} \rfloor$ gives the sequence g_n, starting from $n = 1$, as

$$0, \mathbf{0}, 1, \mathbf{0}, 2, \mathbf{1}, 3, \mathbf{0}, 4, \mathbf{2}, 5, \mathbf{1}, 6, \mathbf{3}, 7, \mathbf{0}, 8, \mathbf{4}, 9, \mathbf{2}, 10, \ldots \tag{2}$$

The odd-indexed terms are just the nonnegative integers in order, while the even-indexed terms, shown in bold, form a copy of the original sequence! This fractal-type property is a consequence of Theorem 2.2. Note that the zeros in the sequence occur at positions indexed by the powers of two; these are precisely the pile sizes resulting in a second player win.

The Grundy sequence (2) is an example of a "divide-and-conquer sequence" [6]. It appeared in [13] in the solution to a card sorting problem. More generally, it is an example of the "fractal sequences" studied by Kimberling [11, 12]. In fact, as shown in Proposition 3.2, all of Kimberling's fractal sequences can be obtained as sequences of Grundy numbers for games of Maximum Nim.

An explicit formula for the Grundy sequence (2) is given by "truncating at the last binary one:" if n is written in binary as

$$n = 2^a + \ldots + 2^y + 2^z$$

with $a > \ldots > y > z \geq 0$, then

$$g_n = 2^{a-z-1} + \ldots + 2^{y-z-1}. \tag{3}$$

By placing a lower bound, rather than an upper bound, on the number of stones that may be taken in a turn, we obtain another variant, *Minimum Nim.* Although the analysis of Minimum Nim is significantly easier than that of Maximum Nim, there is a curious relationship between the two games. For example, if a move consists of taking *at least* half the stones in a pile, any move will reduce by at least one the number of binary digits in the size of the pile. The values of the digits (0 or 1) may change, but the number of digits is always reduced.

We may think of this game as played with piles of red and blue beads: a move consists of removing any number of beads from any one pile, and in addition changing the colors of any number of beads remaining in that pile. Of course, color has no effect on this game, which is just Nim. Playing Minimum Nim with this rule on a pile of size n is thus equivalent to playing ordinary Nim on the binary digits of n. In other words, the Grundy number h_n for a pile of size n is just $\lfloor \log_2 n \rfloor + 1$.

Notice that in our example, n can be uniquely recovered from its pair of Grundy numbers (g_n, h_n) for Maximum and Minimum Nim: by comparing h_n with the number of binary digits in g_n, we can determine how many final zeros were deleted when using (3) to pass from n to g_n. To recover n, simply write g_n in binary and append a final one followed by the appropriate number of zeros. Theorem 4.4 generalizes this observation.

2 Maximum Nim

When the rule sequence f is weakly increasing, the corresponding Grundy sequence g_n for Maximum Nim exhibits a self-similar fractal structure. Sequences f satisfying

$$0 \le f(n) - f(n-1) \le 1 \tag{4}$$

play a special role in the analysis and will be called *regular.* The following lemma converts the recurrence

$$g_n = \text{mex}\{g_{n-i}\}_{i=1}^{f(n)} \tag{5}$$

into a more explicit recurrence (6).

Lemma 2.1. *If f is a regular sequence, the Grundy sequence $(g_n)_{n\ge0}$ for Maximum Nim with rule f satisfies*

$$g_n = \begin{cases} f(n) & \text{if } f(n) > f(n-1); \\ g_{n-f(n)-1}, & \text{if } f(n) = f(n-1). \end{cases} \tag{6}$$

Proof. Fix $0 \le j \le n$. By regularity, $f(n) \le j + f(n-j)$, so $g_{n-j} = \mathrm{mex}\{g_{n-i}\}_{i=j+1}^{j+f(n-j)}$ is distinct from $g_{n-j-1}, \dots, g_{n-f(n)}$. Thus for any n the terms $g_n, g_{n-1}, \dots, g_{n-f(n)}$ are distinct.

If $f(n) > f(n-1)$, then for any $0 < j \le n$, by (5) the term g_{n-j} is the mex of a set of size strictly smaller than $f(n)$, hence $g_{n-j} < f(n)$. Since $g_{n-1}, \dots, g_{n-f(n)}$ are distinct and $< f(n)$, they must be $0, 1, \dots, f(n)-1$ in some order. Thus $g_n = \mathrm{mex}\{0, 1, \dots, f(n)-1\} = f(n)$, completing the proof in the first case.

Now suppose $f(n) = f(n-1)$. Since $g_{n-1}, \dots, g_{n-1-f(n-1)}$ are distinct and $\le f(n)$, they are $0, 1, \dots, f(n)$ in some order, so

$$g_{n-f(n)-1} = g_{n-1-f(n-1)} = \mathrm{mex}\{g_{n-i}\}_{i=1}^{f(n)} = g_n. \quad \square$$

Following [12], we denote by $\Lambda(g)$ the subsequence of g obtained by deleting, for each integer $i \ge 0$, the first term equal to i. As the following theorem shows, the Grundy sequences for Maximum Nim are "self-similar" in the sense that they satisfy $\Lambda(g) = g$.

Theorem 2.2. *Let f be a regular sequence, and let $(g_n)_{n\ge0}$ be the Grundy sequence for Maximum Nim with rule f. Then $\Lambda(g) = g$.*

Proof. By Lemma 2.1, $\Lambda(g)$ consists of precisely those terms g_n for which $f(n) = f(n-1)$. Since f is regular, it follows that all but $f(n)+1$ of the terms $g_0, g_1, \dots, g_n$ lie in the subsequence $\Lambda(g)$. Thus if $f(n) = f(n-1)$, we have by Lemma 2.1

$$\Lambda(g)_{n-f(n)-1} = g_n = g_{n-f(n)-1}. \tag{7}$$

Since f is regular, as n ranges through all positive integers such that $f(n) = f(n-1)$, the quantity $n - f(n) - 1$ ranges through all nonnegative integers, and hence $\Lambda(g) = g$. $\square$

Lemma 2.1 and Theorem 2.2 provide an easy algorithm for writing down the first n terms of the Grundy sequence g in time $O(n)$. (This is a significant improvement over the recurrence (5), which requires time on the order of $\sum_{i=1}^{n} f(i)$.) First, make a table of the values $f(0), \dots, f(n)$, marking those indices $n_1 < \dots < n_k$ for which $f(n_i) > f(n_i - 1)$. Next, write the integers $0, 1, \dots, k$ in positions $0, n_1, \dots, n_k$; this takes care of the first case in (6). Finally, fill in the gaps between the n_i in the unique way possible so that the gapped sequence forms a copy of the original; this is done by copying earlier terms according to the second case of (6). The example below illustrates this algorithm for the rule sequence $f(n) = \lfloor \sqrt{n} \rfloor$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(n)$	0	1	1	1	2	2	2	2	2	3	3	3	3	3	3	3	4
case 1	0	1			2					3							4
case 2			0	1		0	1	2	0		1	2	0	3	1	2	

Since $\lfloor\sqrt{n}\rfloor$ exceeds $\lfloor\sqrt{n-1}\rfloor$ precisely when n is a perfect square, we have $n_i = i^2$. The first case of (6) gives $g_{i^2} = i$, and the second case is used to compute the remaining terms.

Our next result reduces the problem of computing Grundy numbers for a general weakly increasing rule sequence f to the case of regular f, so that Theorem 2.2 applies.

Proposition 2.3. *If f is any weakly increasing sequence, the Grundy sequence for Maximum Nim with rule f is the same as that with rule f', where the regular sequence f' is defined inductively by*

$$f'(n) = \min\{f(n), 1 + f'(n-1)\}.$$

Proof. Let g_n and g'_n be the Grundy sequences corresponding to rules f and f', and induct on n to show $g_n = g'_n$. If $f'(n) = f(n)$, then the inductive hypothesis, together with (5), implies $g_n = g'_n$. Otherwise, $f'(n) = 1 + f'(n-1) < f(n)$. Since f' is regular, by Lemma 2.1 we have $g'_n = f'(n) > g'_{n-j} = g_{n-j}$ for all $0 < j \le n$, hence

$$g'_n \ge \operatorname{mex}\{g_{n-j}\}_{j=1}^{f(n)} \ge \operatorname{mex}\{g_{n-j}\}_{j=1}^{f'(n)} = \operatorname{mex}\{g'_{n-j}\}_{j=1}^{f'(n)} = g'_n,$$

hence $g'_n = \operatorname{mex}\{g_{n-j}\}_{j=1}^{f(n)} = g_n$. □

By way of example, consider the rule sequence $f(n) = \max\{2^k \le n\} - 1$: players may remove any number of stones less than the greatest power of two not exceeding the size of the pile. Since f is not regular, we use Proposition 2.3 to pass to the regular sequence f' before applying Theorem 2.2. The following chart gives values for f, f' and g.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(n)$	0	0	1	1	3	3	3	3	7	7	7	7	7	7	7	7	15
$f'(n)$	0	0	1	1	2	3	3	3	4	5	6	7	7	7	7	7	8
g_n	0	0	1	0	2	3	1	0	4	5	6	7	2	3	1	0	8

If n is one less than a power of two, then $g_n = 0$. Otherwise, writing n in binary, after the inital 1 there will be a string of ones, possibly empty, followed by a zero: $n = (11^k0b_1 \dots b_j)_2$. Now g_n is obtained by deleting this string of ones and the zero that follows it: $g_n = (1b_1 \dots b_j)_2$.

3 Fractal Sequences

We now show that the Grundy sequences for Maximum Nim with a weakly increasing rule f are precisely the "fractal sequences" studied by Kimberling [11, 12]. Following [12], we call a sequence $(g_n)_{n\ge 0}$ *infinitive* if for every integer $k \geq 0$ infinitely many terms g_n are equal to k. A *fractal sequence* $(g_n)_{n\geq 0}$ is defined in [12] as an infinitive sequence satisfying two additional properties:

(F2) If $j < k$, the first instance of j in g precedes the first instance of k;

(F3) The subsequence $\Lambda(g)$ of g obtained by deleting the first instance of each integer k is g itself.

By an *instance* of an integer k in g we mean a term $g_n = k$. If g is an infinitive squence, denote by $\hat{g}(k)$ the position of the first instance of k in g. If g is fractal, the sequence $\hat{g}$ is increasing by property (F2).

Lemma 3.1. *Let g and h be fractal sequences. If $\hat{g} = \hat{h}$, then $g = h$.*

Proof. Induct on n to show $g_n = h_n$. If $n = \hat{g}(k)$ for some k, then $g_n = h_n = k$. Otherwise, let k be such that $\hat{g}(k) < n < \hat{g}(k+1)$. By property (F3) and the inductive hypothesis,

$$g_n = \Lambda(g)_{n-k-1} = g_{n-k-1} = h_{n-k-1} = \Lambda(h)_{n-k-1} = h_n. \quad \square$$

Proposition 3.2. *Let $(g_n)_{n\geq 0}$ be an infinitive sequence. The following are equivalent.*

(i) *g is a fractal sequence;*

(ii) *g is the Grundy sequence for Maximum Nim for some weakly increasing rule sequence f;*

(iii) *g is the Grundy sequence for Maximum Nim for some regular rule sequence f.*

Remark. Equivalently, conditions, (ii) and (iii) may be replaced by the condition that g satisfies the recurrence

$$g_n = \operatorname{mex}\{g_{n-i}\}_{i=1}^{f(n)}$$

for a weakly increasing or regular sequence f, respectively.

Proof. We'll show (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii). The first implication is trivial. If g is the Grundy sequence for Maximum Nim with rule f, with f weakly increasing, by Proposition 2.3 it follows that g is also the Grundy sequence for rule f', which is regular. By Theorem 2.2, it follows that g is a fractal sequence.

For the final implication, let $f(n) = \max\{g_m\}_{m\leq n}$. By property (F2), we have $0 \leq f(n) - f(n-1) \leq 1$, i.e. f is regular. Let h_n be the Grundy sequence for Maximum Nim with rule f. We will show $g = h$. By Theorem

2.2, h is a fractal sequence, and by Lemma 2.1

$$\begin{aligned}\hat{h}(k) &= \min\{n|f(n) = k\} \\ &= \min\{n|\max\{g_m\}_{m\leq n} = k\} \\ &= \min\{n|g_n = k\} \\ &= \hat{g}(k).\end{aligned}$$

By Lemma 3.1 it follows that $g = h$. □

Kimberling [10, 12] has given characterizations of fractal sequences—the notions of *interspersion* and *dispersion*—which on the surface have nothing to do with self-similarity. These are defined in terms of an *associated array* [12] $A = A(g) = (a_{ij})_{i,j\geq 0}$ whose i-th row consists of the instances of i in g listed in increasing order. The array $A(g)$ contains every positive integer exactly once, and its rows are increasing. An array having these properties is called an *interspersion* if, in addition, its columns are increasing and

(I4) $a_{ij} < a_{kl} < a_{i,j+1}$ implies $a_{i,j+1} < a_{k,l+1} < a_{i,j+2}$.

In [12] it is shown that

Theorem 3.3. *g is a fractal sequence if and only if $A(g)$ is an interspersion.*

We find it illuminating to recast the definition of an interspersion in terms of the sequence itself, rather than its associated array. If M is a set of nonnegative integers and g an infinitive sequence, the *restriction of g to M*, denoted $g|M$, is the subsequence of g formed by deleting all terms g_n for which $g_n \notin M$. In these terms, an *interspersion* is an infinitive sequence g such that for any $i < j$ the restriction $g|\{i, j\}$ has the form

$$i, i, i, \ldots, i, j, i, j, i, j, \ldots;$$

after an initial segment of i's, instances of i and j must alternate.

When M is infinite, it is often useful to relabel the sequence $g|M$ so as to make it infinitive. If $M = \{m_0, m_1, \ldots\}$ with $m_0 < m_1 < \ldots$, the *relabeling* of $g|M$ is the sequence obtained by replacing each instance of m_i with i.

Our next result characterizes the restrictions of an interspersion. Taking M to be the set of positive integers, we obtain as a special case Theorem 5 of [12].

Proposition 3.4. *Let g be an interspersion, and let M be a set of nonnegative integers.*

(i) *If M is finite, then $g|M$ is eventually periodic with period $\#M$.*

(ii) *If M is infinite, the relabeling of $g|M$ is an interspersion.*

Proof. (i) Let $m = \#M$, and fix $i \in M$. With finitely many exceptions, between consecutive instances of i in $g|M$ there is exactly one instance of each $j \in M - \{i\}$. Thus for sufficiently large n the m terms

$$(g|M)_n, (g|M)_{n+1}, \ldots, (g|M)_{n+m-1}$$

are a permutation of M. In particular, both $(g|M)_n$ and $(g|M)_{n+m}$ are equal to the unique element $j \in M$ not contained in $\{(g|M)_{n+i}\}_{i=1}^{m-1}$, so $g|M$ is eventually periodic mod m.

(ii) Write $M = \{m_0, m_1, \ldots\}$ with $0 \le m_0 < m_1 < \ldots$. For $i < j$, since the restriction $g|\{m_i, m_j\}$ has the form

$$m_i, m_i, \ldots, m_i, m_j, m_i, m_j, \ldots,$$

the restriction of the relabeling of $g|M$ to $\{i, j\}$ has the form

$$i, i, \ldots, i, j, i, j, \ldots,$$

so the relabeling of $g|M$ is an interspersion. □

If g is an interspersion, the restriction $g|\{i,j\}$ is determined by the number s_{ij} of instances of i in g preceding the first instance of j. (If $i = 0$, we do not count the instance $g_0 = 0$.) The array $S(g) := (s_{ij})_{i,j \ge 0}$ is strictly upper-triangular and satisfies

$$s_{ij} + s_{jk} - 1 \le s_{ik} \le s_{ij} + s_{jk}. \tag{8}$$

Equality holds on the left or the right side of (8) accordingly as the restriction $g|\{i,j,k\}$ has the form,

$$i, i, \ldots, i, j, i, j, \ldots, i, j, k, i, j, k, i, \ldots$$

or

$$i, i, \ldots, i, j, i, j, \ldots, i, j, i, k, j, i, k, \ldots.$$

An upper-triangular array satisfying (8) will be called a *subadditive triangle*.

For example, the array

$$\begin{array}{ccccccccccc}
2 & 3 & 3 & 4 & 4 & 4 & 4 & 5 & 5 & 5 & \ldots \\
 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & \ldots \\
 & & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & \ldots \\
 & & & 1 & 1 & 1 & 2 & 2 & 2 & 2 & \ldots \\
 & & & & 1 & 1 & 1 & 1 & 2 & 2 & \ldots \\
 & & & & & 1 & 1 & 1 & 1 & 1 & \ldots \\
 & & & & & & 1 & 1 & 1 & 1 & \ldots \\
 & & & & & & & 1 & 1 & 1 & \ldots \\
 & & & & & & & & 1 & 1 & \ldots \\
 & & & & & & & & & 1 & \ldots
\end{array}$$

is the subadditive triangle associated to the Grundy sequence

$$0,0,1,0,2,1,3,0,4,2,5,1,6,3,7,0,8,4,9,2,10,\ldots$$

for Maximum Nim with rule $f(n) = \lfloor \frac{n-1}{2} \rfloor$.

In Theorem 3.6 we show that the correspondence between fractal sequences and subadditive triangles is a bijection.

Lemma 3.5. *A subadditive triangle $(s_{ij})_{i,j\geq 0}$ is determined by its column sums $c_j = \sum_{i=0}^{j-1} s_{ij}$.*

Proof. For $i < j < k$ write

$$\varepsilon_{ijk} = s_{ij} + s_{jk} - s_{ik}.$$

By subadditivity (8), each ε_{ijk} is either 0 or 1. For fixed i, we induct on j to show that the column sums $c_i, c_{i+1}, \ldots, c_j$ determine the entry s_{ij}. We have

$$\begin{aligned} c_j &= \sum_{p=0}^{j-1} s_{pj} \\ &= \sum_{p=0}^{i-1}(s_{pi} + s_{ij} - \varepsilon_{pij}) + s_{ij} + \sum_{p=i+1}^{j-1} (s_{ij} - s_{ip} + \varepsilon_{ipj}) \\ &= c_i + j s_{ij} - \sum_{p=i+1}^{j-1} s_{ip} + \varepsilon, \end{aligned} \qquad (9)$$

where the error term

$$\varepsilon = \sum_{p=i+1}^{j-1} \varepsilon_{ipj} - \sum_{p=0}^{i-1} \varepsilon_{pij}$$

is bounded by

$$-i \leq \varepsilon \leq j - 1 - i. \qquad (10)$$

By the induction hypothesis, the sum

$$\Sigma := \sum_{p=i+1}^{j-1} s_{ip}$$

appearing in (9) is determined by the column sums $c_i, c_{i+1}, \ldots, c_{j-1}$. Solving (9) for s_{ij} we obtain

$$s_{ij} = \frac{1}{j}\left[c_j - c_i + \Sigma - \varepsilon\right] \qquad (11)$$

in which every term on the right hand side, except the error term ε, is determined by the column sums. By the bounds (10), there is a unique value of ε making the right hand side an integer, and hence s_{ij} is determined by the column sums $c_i, c_{i+1}, \dots, c_j$. □

Theorem 3.6. *The map $g \mapsto S(g)$ is a bijection between fractal sequences and subadditive triangles.*

Proof. Given a fractal sequence g, write s_{ij} for the typical entry of $S(g)$. For fixed j, the column sum

$$c_j = \sum_{i=0}^{j-1} s_{ij}$$

counts each term preceding the first instance of j in g exactly once. Thus $\hat{g}(j) = 1 + c_j$. By Lemma 3.1, the sequence $\hat{g}$ determines g, so the map $g \mapsto S(g)$ is 1–1.

To show that the map is onto, given a subadditive triangle $S = \{s_{ij}\}$, let g be the unique fractal sequence satisfying

$$\hat{g}(j) = 1 + \sum_{i=0}^{j-1} s_{ij}.$$

Then $S(g)$ and S have the same column sums c_j. By Lemma 3.5, it follows that $S = S(g)$. □

4 Minimum Nim

In the game of Minimum Nim with rule f, a move consists of removing *strictly* more than $f(m)$ stones from a pile of size m. In Maximum Nim, on the other hand, taking exactly $f(m)$ stones is permitted. The effect of this convention is to simplify the statements of Proposition 4.3 and Theorem 4.4, which describe the relationship between Minimum and Maximum Nim. The Grundy sequence $(h_n)_{n\geq 0}$ for Minimum Nim obeys the recurrence

$$h_n = \text{mex}\{h_i\}_{i=0}^{n-f(n)-1}. \tag{12}$$

If f is a regular sequence, the sequence $(n - f(n))_{n\geq 0}$ is also regular. To avoid trivialities that arise when this sequence is eventually constant, we require that

$$n - f(n) \to \infty \tag{13}$$

as $n \to \infty$. Proposition 4.1 solves the game of Minimum Nim with rule f in the case that f is a regular sequence satisfying (13).

Recall the notation $\hat{h}(n) = \min\{k : h_k = n\}$. If h is regular, the sequence $\hat{h}$ determines h.

Proposition 4.1. *Let f be a regular sequence satisfying* (13), *and let $(h_n)_{n\geq 0}$ be the Grundy sequence for Minimum Nim with rule f. Then h is a regular sequence, $\hat{h}(0) = 0$ and*

$$\hat{h}(n) = q(\hat{h}(n-1)),$$

where

$$q(k) = \min\{j : j - f(j) > k\}. \tag{14}$$

Proof. Let $S_n = \{h_0, h_1, \ldots, h_{n-f(n)-1}\}$. Since f is regular, $S_{n-1} \subset S_n$ and S_n contains at most one element not in S_{n-1}. By (12), $h_n = \text{mex } S_n$ and hence

$$h_{n-1} \leq h_n \leq 1 + h_{n-1},$$

i.e. h is regular. Since $h_0 = 0$ we have $\hat{h}(0) = 0$ and

$$\begin{aligned}
\hat{h}(n) &= \min\{k : \text{mex}\{h_i\}_{i=0}^{k-f(k)-1} = n\} \\
&= \min\{k : h_{k-f(k)-1} = n-1\} \\
&= \min\{k : k - f(k) - 1 \geq \hat{h}(n-1)\} \\
&= q(\hat{h}(n-1)). \quad \square
\end{aligned}$$

For example, if $f(n) = \lfloor \frac{n-1}{2} \rfloor$ then $q(k) = 2k$, and Proposition 4.1 gives the corresponding Grundy sequence h for Minimum Nim as

$$0, 1, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 5, \ldots;$$

as we remarked in the introduction, its n-th term is $\lfloor \log_2 n \rfloor + 1$.

The following lemma, which explains the importance of the function q, is closely related to the fact that fractal sequences are also *dispersions* [10].

Lemma 4.2. *Let f be a regular sequence satisfying* (13), *and let $(g_n)_{n\geq 0}$ be the Grundy sequence for the corresponding game of Maximum Nim. With q as in* (14), *we have $g_{q(n)} = g_n$.*

Proof. Since $(n - f(n))_{n\geq 0}$ is a regular sequence, by (14) we have $q(n) - f(q(n)) = n+1$ and $q(n) - 1 - f(q(n)-1) = n$, hence $f(q(n)) = f(q(n)-1)$. By Lemma 2.1, it follows that

$$g_{q(n)} = g_{q(n)-f(q(n))-1} = g_n. \quad \square$$

Our next proposition relates the Grundy sequences for Minimum and Maximum Nim. We write $q^0(n) = n$, $q^i(n) = q(q^{i-1}(n))$.

Proposition 4.3. *Let f be a regular sequence satisfying* (13), *and let $(g_n)_{n\geq 0}$ and $(h_n)_{n\geq 0}$ be the Grundy sequences for Maximum and Minimum Nim with rule f. Then*

$$h_n = \#\{0 < k \leq n : g_k = 0\}.$$

Proof. Let $z_0 = 0$, and let z_i be the first instance of zero in g following z_{i-1}. We'll show $z_i = q^i(0)$, where q is given by (14). By Lemma 4.2, we have $g_{q^i(0)} = 0$ for all i. Conversely, suppose $g_m = 0$ for some $m > 0$. By Lemma 2.1, $f(m) = f(m-1)$, hence $m - f(m) > m - 1 - f(m-1)$ and $m = q(m - f(m) - 1)$ by (14). Then $g_{m-f(m)-1} = g_m = 0$ by Lemma 4.2, and by induction it follows that $m = q^i(0)$ for some i. Proposition 4.1 now implies that $z_i = \hat{h}(i)$. Since h is regular,

$$h_n = \max\{i : \hat{h}(i) \leq n\} = \max\{i : z_i \leq n\} = \#\{0 < k \leq n : g_k = 0\}. \quad \square$$

Our next result shows that n can be uniquely recovered from the pair (g_n, h_n).

Theorem 4.4. *Let f be a regular sequence satisfying* (13), *and let $(g_n)_{n\geq 0}$ and $(h_n)_{n\geq 0}$ be the Grundy sequences for Maximum and Minimum Nim with rule f. Let $\{s_{ij}\}_{i,j\geq 0}$ be the subadditive triangle associated to the sequence g. The map $n \mapsto (g_n, h_n)$ is a bijection between nonnegative integers and pairs (i, j) of nonnegative integers satisfying $j \geq s_{0i}$.*

Proof. g is a fractal sequence by Proposition 3.2, and hence an interspersion by Theorem 3.3. Thus if $g_m = g_n$ for some $m < n$, there is some term $g_i = 0$ with $m < i \leq n$. By Proposition 4.3 it follows that $h_m < h_n$, hence the map $n \mapsto (g_n, h_n)$ is 1–1.

Since g is an interspersion, instances of 0 and i in g alternate after the first instance of i, so by Proposition 4.3, for every $j \geq s_{0i}(g)$ there is an index n such that $g_n = i$ and $h_n = j$. $\square$

Corollary 4.5. *The array $A = (a_{ij})_{i,j\geq 0}$ whose entry a_{ij} is the unique integer n such that $i = g_n$, $j = h_n - s_{0i}$ is an interspersion.*

Proof. The entry a_{ij} of A is the position of the j-th instance of i in g; i.e. A is the associated array of g. By Theorem 3.3, since g is a fractal sequence, A is an interspersion. $\square$

The array $A' = \{a'_{ij}\}_{j\geq s_{0i}}$ shown below is the inverse to the map $n \mapsto (g_n, h_n)$ for the rule sequence $f(n) = \lfloor \frac{n-1}{2} \rfloor$. The entry a'_{ij} is the unique integer n for which $i = g_n$, $j = h_n$. The blank spaces in the lower left correspond to pairs (i,j) satisfying $j < s_{0i}$, for which no such n exists.

0	1	2	4	8	16	32	...
		3	6	12	24	48	...
			5	10	20	40	...
			7	14	28	56	...
				9	18	36	...
				11	22	44	...
				13	26	52	...
				15	30	60	...
					17	34	...
					19	38	...
					⋮	⋮	

If the rows of A' are left-justified, by Corollary 4.5 the resulting array A is an interspersion.

5 Serial Nim

In general, it seems difficult to describe the behavior of the Grundy sequences for Maximum and Minimum Nim when the rule sequence f is not weakly increasing. Certain special cases are of interest, however. In the game of *Serial Nim*, heaps are arranged in a row from left to right, and players can remove stones only from the leftmost nonempty heap. If the heaps have sizes $a_1, \ldots, a_k$, we denote the Grundy number of the resulting game by $[a_1, \ldots, a_k]$. This bracket is "right-associative" in the sense that $[a_1, \ldots, a_k] = [a_1, [a_2, \ldots, a_k]]$. (However, it is *not* left-associative!) If f is a rule sequence of the form

$$1, 2, \ldots, a_1, 1, 2, \ldots, a_2, \ldots,$$

then a single heap of size n in the corresponding game of Maximum Nim is equivalent to a row of heaps of sizes $n - \sum_{i=1}^{k} a_i, a_k, a_{k-1}, \ldots, a_1$ in Serial Nim, where k is such that $\sum_{i=1}^{k} a_i < n \leq \sum_{i=1}^{k+1} a_i$.

Consider the case of two heaps of sizes a, b. Since $[0, b] = b$ and

$$[a, b] = \operatorname{mex}\{[i, b]\}_{0 \leq i < a},$$

by induction on a the sequence $([a, b])_{a \geq 0}$ has the form

$$b, 0, 1, \ldots, b-1, b+1, b+2, \ldots;$$

in other words, for $a > 0$ the bracket $[a, b]$ is $a-1$ or a accordingly as $a \leq b$ or $a > b$.

Our next result treats the general case of k heaps. As with two heaps, the Grundy number of the game is always equal either to the size a_1 of the first heap or to $a_1 - 1$. Moreover if the heap in position m is the leftmost heap whose size differs from the first, then the Grundy number depends only on the parity of m and the relative size of a_m and a_1. In this respect, Serial Nim behaves like a simplified version of the game "End-nim" studied by Albert and Nowakowski [1], in which players may remove stones from either the leftmost or the rightmost nonempty heap. Although the End-nim positions of Grundy number zero were classified in [1], in general its Grundy numbers seem to behave erratically. By contrast, the following result completely characterizes the Grundy numbers for Serial Nim.

Proposition 5.1. *Let* $a_1, \ldots, a_k$ *be positive integers, and set* $a_{k+1} = 0$. *Let* $m = \min\{j | a_j \neq a_1\}$. *If* m *is odd and* $a_m < a_1$, *or* m *is even and* $a_m > a_1$, *then* $[a_1, \ldots, a_k] = a_1 - 1$; *otherwise* $[a_1, \ldots, a_k] = a_1$.

Proof. Induct on k. The base case $k = 2$ is discussed above. Write $a = [a_1, \ldots, a_k] = [a_1, b]$, where $b = [a_2, \ldots, a_k]$. By the inductive hypothesis, $a = [a_1, b] = a_1 - 1$ or a_1 accordingly as $b \geq a_1$ or $b \leq a_1 - 1$. If m is odd, then $a_2 = a_1$ and by the inductive hypothesis $b = a_2 - 1$ or a_2 accordingly as $a_m > a_2$ or $a_m < a_2$, i.e. $a = a_1 - 1$ or a_1 accordingly as $a_m < a_1$ or $a_m > a_1$.

Suppose now that m is even. If $m = 2$, then either $a_2 < a_1$, in which case $b \leq a_2 \leq a_1 - 1$, so $a = a_1$; or $a_2 > a_1$, in which case $b \geq a_2 - 1 \geq a_1$, hence $a = a_1 - 1$. If $m > 2$, then $a_2 = a_1$ and $b = a_2 - 1$ or a_2 accordingly as $a_m < a_2$ or $a_m > a_2$, i.e. $a = a_1 - 1$ or a_1 accordingly as $a_m > a_1$ or $a_m < a_1$. □

A closely related game is "Smallest Nim," [3, v. 3] in which players may take stones only from the heap (or one of the heaps) of smallest size. Smallest Nim is the special case of Serial Nim in which the piles are arranged in nondecreasing order of size. Further Nim variants in which moves are permitted to occur in only one pile are studied in [2].

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