ISOMORPHISMS OF CONNECTED CYCLIC ABELIAN COVERS OF SYMMETRIC DIGRAPHS

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Abstract

Let D be a connected symmetric digraph, A a finite abelian group, $g \in A$ and Γ a group of automorphisms of D. We consider the number of Γ-isomorphism classes of connected g-cycic A-covers of D for an element g of odd order. Specially, we enumerate the number of I-isomorphism classes of connected g-cycic A-covers of D for an element g of odd order and the trivial automorphism group I of D, when A is the cyclic group $\mathbf{Z}_{p^n}$ and the direct sum of m copies of $\mathbf{Z}_p$ for any prime number $p(> 2)$.

Key words: digraph covering; enumeration

1 Introduction

Graphs and digraphs treated here are finite and simple.

A graph H is called a covering of a graph G with projection $\pi : H \longrightarrow G$ if there is a surjection $\pi : V(H) \longrightarrow V(G)$ such that $\pi|_{N(v')} : N(v') \longrightarrow N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$, where $N(v) = N_G(v) = \{w \in V(G) \mid (v,w) \in D(G)\}$, etc. The projection $\pi : H \longrightarrow G$ is an n-fold covering of G if π is n-to-one. A covering $\pi : H \longrightarrow G$ is said to be regular if there is a subgroup B of the automorphism group $Aut\ H$ of H acting freely on H such that the quotient graph H/B is isomorphic to G.

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Let G be a graph and A a finite group. Let $D(G)$ be the arc set of the symmetric digraph corresponding to G. Then a mapping $\alpha : D(G) \longrightarrow A$ is called an ordinary voltage assignment if $\alpha(v,u) = \alpha(u,v)^{-1}$ for each $(u,v) \in D(G)$. The (ordinary) derived graph G^α derived from an ordinary voltage assignment α is defined as follows:

$V(G^\alpha) = V(G) \times A$, and $((u,h),(v,k)) \in D(G^\alpha)$ if and only if $(u,v) \in D(G)$ and $k = h\alpha(u,v)$.

The graph G^α is called an A-covering of G. The A-covering G^α is an $\mid A \mid$-fold regular covering of G. Every regular covering of G is an A-covering of G for some group A (see [3]). Furthermore the 1-cyclic A-cover $D_1(\alpha)$ of a symmetric digraph D can be considered as the A-covering $\tilde{D}^\alpha$ of the underlying graph $\tilde{D}$ of D.

A general theory of graph coverings is developed in [4]. $\mathbf{Z}_2$-coverings (double coverings) of graphs were dealed in [5] and [17]. Hofmeister [6] and, independently, Kwak and Lee [11] enumerated the I-isomorphism classes of n-fold coverings of a graph, for any $n \in \mathrm{N}$. Dresbach [2] obtained a formula for the number of strong isomorphism classes of regular coverings of graphs with voltages in finite fields. The I-isomorphism classes of regular coverings of graphs with voltages in finite dimensional vector spaces over finite fields were enumerated by Hofmeister [7]. Hong, Kwak and Lee [9] gave the number of I-isomorphism classes of $\mathbf{Z}_n$-coverings, $\mathbf{Z}_p \bigoplus \mathbf{Z}_p$-coverings and D_n-coverings, n:odd, of graphs, respectively.

In the case of connected coverings, Kwak and Lee [13] enumerated the I-isomorphism classes of connected n-fold coverings of a graph G. Furthermore, Kwak, Chun and Lee [12] gave some formulas for the number of I-isomorphism classes of connected A-coverings of a graph G when A is a finite abelian group or D_n.

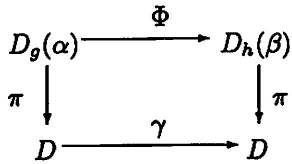
Let D be a symmetric digraph with arc set $A(D)$, A a finite group, and a function $\alpha : A(D) \longrightarrow A$ an ordinary voltage assignment. For $g \in A$, a g-cyclic A-cover (or g-cyclic cover) $D_g(\alpha)$ of D is the digraph as follows:

$V(D_g(\alpha)) = V(D) \times A$, and $((u,h),(v,k)) \in A(D_g(\alpha))$ if and only if $(u,v) \in A(D)$ and $k^{-1}h\alpha(u,v) = g$.

The natural projection $\pi : D_g(\alpha) \longrightarrow D$ is a function from $V(D_g(\alpha))$ onto $V(D)$ which erases the second coordinates. A digraph D' is called a cyclic A-cover of D if D' is a g-cyclic A-cover of D for some $g \in A$. In the case that A is abelian, then $D_g(\alpha)$ is called simply a cyclic abelian cover.

Let α and β be two ordinary voltage assignments from $A(D)$ into A, and let Γ be a subgroup of the automorphism group $Aut\ D$ of D, denoted $\Gamma \leq Aut\ D$. Let $g, h \in A$. Then two cyclic A-covers $D_g(\alpha)$ and $D_h(\beta)$ are called Γ-isomorphic, denoted $D_g(\alpha) \cong_\Gamma D_h(\beta)$, if there exist an isomorphism Φ :

$D_g(\alpha) \longrightarrow D_h(\beta)$ and a $\gamma \in \Gamma$ such that $\pi\Phi = \gamma\pi$, i.e., the diagram commutes. Let $I = \{1\}$ be the trivial group of automorphisms.



Cheng and Wells [1] discussed isomorphism classes of cyclic triple covers (1-cyclic $\mathbf{Z}_3$-covers) of a complete symmetric digraph. Moreover, Mizuno and Sato [15] gave a formula for the characteristic polynomial of a cyclic A-cover of a symmetric digraph, for any finite group A. Mizuno and Sato [14] enumerated the number of I-isomorphism classes of g-cyclic $\mathbf{Z}_p^n$-covers of a connected symmetric digraph D for an element g of odd order and a prime number $p(>2)$. Furthermore, Mizuno and Sato [16] gave a formula for the number of I-isomorphism classes of g-cyclic $\mathbf{Z}_{p^n}$-covers of D for any prime $p(>2)$.

In Section 2, we discuss the number of Γ-isomorphism classes of connected g-cycic A-covers of D for a finite abelian group A and $g \in A$ of odd order. In Section 3, we enumerate the number of I-isomorphism classes of connected g-cycic $\mathbf{Z}_p^n$-covers and connected h-cycic $\mathbf{Z}_{p^n}$-covers of D, where $p(>2)$ is prime.

2 Isomorphisms of cyclic abelian covers

Let D be a symmetric digraph and A a finite group. A group Γ of automorphisms of D acts on the set $C(D)$ of ordinary voltage assignments from $A(D)$ into A as follows:

$$\alpha^{\gamma}(x, y) = \alpha(\gamma(x), \gamma(y)) \textit{ for all } (x, y) \in A(D),$$

where $\alpha \in C(D)$ and $\gamma \in \Gamma$. Any voltage $g \in A$ determines a permutation $\rho(g)$ of the symmetric group S_A on A which is given by $\rho(g)(h) = hg$, $h \in A$.

From now on, assume that D is connected and A is abelian. Let G be the underlying graph of D, T be a spanning tree of G and w a root of T. For any $\alpha \in C(D)$ and any walk W in G, the net α-voltage of W, denoted $\alpha(W)$, is the sum of the voltages of the edges of W. Then the T-voltages α_T of α is defined as follows:

$$\alpha_T(u, v) = \alpha(P_u) + \alpha(u, v) - \alpha(P_v) \textit{ for each } (u, v) \in D(G) = A(D),$$

where P_u and P_v denote the unique walk from w to u and v in T, respectively. For a function $f : A(D) \longrightarrow A$, the net f-values $f(W)$ of any walk

W is defined as the net f-voltage of W. For a function $f : A(D) \longrightarrow A$, let the pseudolocal voltage group $A_f(v)$ of f at v denote the subgroup of A consisting of all net f-values of the closed walk based at $v \in V(D)$. By the hypothesis that A is abelian, we have $A_f(v) = A_f(w)$ for a function $f : A(D) \longrightarrow A$ and any two vertices $v, w \in V(D)$. Thus, let $A_f = A_f(v)$ for any $v \in V(D)$. Notice that if $\alpha \in C(D)$ and $g \in A$, then the pseudolocal voltage groups $A_{\alpha-g}$ and A_{α_T-g} are equal to the group generated by g and $A_\alpha = A_{\alpha_T}$, where $(\alpha_T - g)(u,v) = \alpha_T(u,v) - g$, $(u,v) \in A(D)$. Moreover, $D_g(\alpha)$ is connected if and only if $A_{\alpha-g}$ is the full group A. Let $ord(g)$ be the order of $g \in A$.

Let D be a connected symmetric digraph, G its underlying graph and A a finite abelian group. The set of ordinary voltage assignments of G with voltages in A is denoted by $C^1(G; A)$. Note that $C(D) = C^1(G; A)$. Furthremore, let $C^0(G; A)$ be the set of functions from $V(G)$ into A. We consider $C^0(G; A)$ and $C^1(G; A)$ as additive groups.

The automorphism group $Aut\ A$ acts on $C^0(G; A)$ and $C^1(G; A)$ as follows:

$$(\sigma s)(x) = \sigma(s(x)) \ for \ x \in V(D),$$

$$(\sigma\alpha)(x,y) = \sigma(\alpha(x,y)) \ for \ (x,y) \in A(D),$$

where $s \in C^0(G; A)$, $\alpha \in C^1(G; A)$ and $\sigma \in Aut\ A$.

We proceed according to an analogue of a method in [12].

Theorem 1 *Let D be a connected symmetric digraph, G the underlying graph of D, T a spanning tree of G and $\Gamma \leq Aut\ G$. Let A, B be two finite abelian groups, $g \in A$ and $h \in B$. Let $\alpha \in C^1(G; A)$ and $\beta \in C^1(G; B)$. Assume that the orders of g and h are odd. Then the following are equivalent:*

1. $D_g(\alpha) \cong {}_\Gamma D_h(\beta)$.
2. *There exist $\gamma \in \Gamma$ and an isomorphism $\sigma : A_{\alpha_T-g}(w) \longrightarrow B_{\beta_{\gamma T}-h}(\gamma(w))$ such that*

   $$\beta^\gamma_{\gamma T}(u,v) = \sigma(\alpha_T(u,v)) \ for \ each \ (u,v) \in A(D),$$

   and

   $$\sigma(g) = h.$$

   where $w \in V(D)$.

Furthermore, if both α and β derive connected cyclic abelian covers, then the above statement 1 is also equivalent to:

There exist $\gamma \in \Gamma$ and an group isomorphism $\sigma : A \longrightarrow B$ such that

$$\beta^\gamma_{\gamma T}(u,v) = \sigma(\alpha_T(u,v)) \ for \ each \ (u,v) \in A(D),$$

and

$$\sigma(g) = h.$$

Proof. By an analogue of the proof of Theorem 2 of [16], $D_g(\alpha) \cong {}_\Gamma D_h(\beta)$ if and only if there exist $\gamma \in \Gamma$ and an isomorphism $\sigma : A_{\alpha_T - g}(w) \longrightarrow B_{\beta_{\gamma T} - h}(\gamma(w))$ such that

$$\beta^{\gamma}_{\gamma T}(u,v) - h = \sigma(\alpha_T(u,v) - g) \ for\ each\ (u,v) \in A(D).$$

Note that $\alpha_T(u,v) = \beta^{\gamma}_{\gamma T}(u,v) = 0$ for all $(u,v) \in A(T)$.

But we have $\beta^{\gamma}_{\gamma T}(v,u) - h = \sigma(\alpha_T(v,u) - g)$. Thus we have $-2h = -2\sigma(g)$. Since $ord(g) = ord(h)$ are odd, it follows that $\sigma(g) = h$.
Q.E.D.

Corollary 1 *Let D be a connected symmetric digraph, G the underlying graph of D, T a spanning tree of G and A a finite abelian group.*

1. *Let $g, h \in A$, $\alpha, \beta \in C(D)$ and $\Gamma \leq Aut\ G$. Assume that the orders of g and h are equal and odd. Then the following are equivalent:*

   (a) $D_g(\alpha) \cong {}_\Gamma D_h(\beta)$.

   (b) There exist $\gamma \in \Gamma$ and an isomorphism $\sigma : A_{\alpha_T - g}(w) \longrightarrow A_{\beta_{\gamma T} - h}(\gamma(w))$ such that

   $$\beta^{\gamma}_{\gamma T}(u,v) = \sigma(\alpha_T(u,v)) \ for\ each\ (u,v) \in A(D),$$

   and

   $$\sigma(g) = h.$$

   If both α and β derive connected cyclic A-covers, then the condition (a) is also equivalent to say that there exist $\gamma \in \Gamma$ and an automorphism $\sigma \in Aut A$ such that

   $$\beta^{\gamma}_{\gamma T}(u,v) = \sigma(\alpha_T(u,v)) \ for\ each\ (u,v) \in A(D),$$

   and

   $$\sigma(g) = h.$$

2. *Let $g \in A$ and $\alpha, \beta \in C(D)$. Assume that the order of g is odd. If both $D_g(\alpha)$ and $D_g(\beta)$ are connected, then the following are equivalent:*

   (a) $D_g(\alpha) \cong {}_I D_g(\beta)$.

   (b) There exists $\sigma \in Aut A$ such that

   $$\beta_T(u,v) = \sigma(\alpha_T(u,v)) \ for\ each\ (u,v) \in A(D) \setminus A(T)$$

   and

   $$\sigma(g) = g.$$

Let D be a connected symmetric digraph and A a finite abelian group, $g \in A$. Furthermore, let G be the underlying graph of D and T a spanning tree of G. Then set, $C_T(D) = \{\alpha_T \mid \alpha \in C(D) = C^1(G; A)\}$. By Corollary 1 of [16], we have $D_g(\alpha) \cong {}_I D_g(\alpha_T)$ for any $\alpha \in C(G; A)$ and any $g \in A$. Thus we consider only an element α in $C_T(D)$.

Let $\alpha \in C_T(D)$ and $v \in V(D)$. The component of g-cyclic A-cover $D_g(\alpha)$ containing $(v, 0)$ is called the identity component of $D_g(\alpha)$. By the definition of cyclic A-covers, each component of $D_g(\alpha)$ is isomorphic to the identity component. Furthermore, the identity component of g-cyclic A-cover $D_g(\alpha)$ is just a g-cyclic $A_{\alpha-g}$-cover if g is of odd order.

Lemma 1 *Let D be a connected symmetric digraph, G be the underlying graph of D, T a spanning tree of G and A a finite abelian group, $g \in A$, $\alpha, \beta \in C_T(D)$ and $v \in V(D)$. Then the following are equivalent:*

1. $D_g(\alpha) \cong {}_I D_g(\beta)$.

2. *The identity component of $D_g(\alpha)$ is I-isomorphic to that of $D_g(\beta)$.*

For a finite abelian group A and $g \in A$, let $Iso(D, A, g, I)$ denote the number of I-isomorphism classes of g-cyclic A-covers of D. Furthermore, let $Isoc(D, A, g, I)$ be the number of I-isomorphism classes of connected g-cyclic A-covers of D.

Theorem 2 *Let D be a connected symmetric digraph, A a finite abelian group and $g \in A$. Assume that the order of g is odd. Then*

$$Iso(D, A, g, I) = \sum_{B} Isoc(D, B, g, I),$$

where B runs over all representatives of isomorphism classes of subgroups of A which contain g and have an isomorphism fixing g.

Proof. We use Lemma 1. Let B and C be any two isomorphic subgroups of A which contain g and have an isomorphism fixing g. Then by Theorem 1, we have $Isoc(D, B, g, I) = Isoc(D, C, g, I)$ Furhthermore, $Isoc(D, B, g, I)$ is equal to the number of I-isomorphism classes of connected g-cyclic A-covers of D whose pseudolocal voltage groups are isomorphic to B.
Q.E.D.

3 Isomorphisms of connected cyclic abelian covers

Let D be a connected symmetric digraph and A a finite abelian group. For A and a natural number n, let

$$F_g(A;n) = \{(g_1, \cdots, g_n) \in A^n \mid \{g, g_1, \cdots, g_n\} \text{ generates } A\}.$$

We give a formula for the number of I-isomorphism classes of connected g-cyclic A-covers of D for an element g of odd order.

Theorem 3 *Let D be a connected symmetric digraph, A a finite abelian group and $g \in A$. Furthermore, assume that the order of g is odd. Then*

$$Isoc(D, A, g, I) = \mid F_g(A; B(D)) \mid / \mid (Aut\ A)_g \mid,$$

where $B(D) = \mid A(D) \mid /2 - \mid V(D) \mid +1$ is the Betti number of D and $(Aut\ A)_g = \{\sigma \in Aut\ A \mid \sigma(g) = g\}$.

Proof. Let G be the underlying graph of D and T a spanning tree of G. Then the pseudolocal voltage group $A_{\alpha_T - g}(w)$ for any connected g-cyclic A-covers $D_g(\alpha_T)$ of D is isomorphic to A. By Corollary 1.2, the number of I-isomorphism classes of connected g-cyclic A-covers of D is equal to that of $(Aut\ A)_g$-orbits on $F_g(A; B(D))$. By Burnside's Lemma, we have

$$Isoc(D, A, g, I) = \frac{1}{\mid (Aut\ A)_g \mid} \sum_{\sigma \in (Aut\ A)_g} \mid F_g(A; B(D))^\sigma \mid,$$

where U^σ is the set consisting of the elements of U fixed by σ.

If $\mid F_g(A; B(D))^\sigma \mid \neq 0$, then there exists a $(g_1, \cdots, g_n) \in F_g(A; B(D))$ such that $\sigma(g_i) = g_i$ for all $i = 1, \cdots, r$. Since $\{g\} \cup \{g_1, \cdots, g_n\}$ generates A, we have $\sigma(h) = h$ for each $h \in A$, i.e., $\sigma = 1$. Therefore it follows that

$$Isoc(D, A, g, I) = \mid F_g(A; B(D)) \mid / \mid (Aut\ A)_g \mid .$$

Q.E.D.

Let $p (> 2)$ be a prime number and $\mathbf{Z}_p^n$ the direct sum of n copies of the cyclic group $\mathbf{Z}_p$.

Theorem 4 *Let D be a connected symmetric digraph and $g \in \mathbf{Z}_p^n \setminus \{0\}$. Then the number of I-isomorphism classes of connected g-cyclic $\mathbf{Z}_p^n$-covers of D is*

$$Isoc(D, \mathbf{Z}_p^n, g, I) = \frac{p^{B-n+1}(p^B - 1) \cdots (p^{B-n+2} - 1)}{(p^{n-1} - 1)(p^{n-2} - 1) \cdots (p - 1)},$$

where $B = B(D)$.

Proof. Since $\mathbf{Z}_p^n$ is the n-dimensional vector space over $\mathbf{Z}_p$, the general linear group $GL_n(\mathbf{Z}_p)$ is the automorphism group of $\mathbf{Z}_p^n$. By Lemma 2 of [7], we have

$$\mid (Aut\ \mathbf{Z}_p^n)_g \mid = p^{\frac{n(n-1)}{2}}(p^{n-1}-1)(p^{n-2}-1)\cdots(p-1).$$

Let K be the orthogonal complement of $< g >$ in the vector space $\mathbf{Z}_p^n$. Note that $K \cong \mathbf{Z}_p^{n-1}$. Then we have

$$F_g(\mathbf{Z}_p^n; B) = F_0(\mathbf{Z}_p^n; B) \cup \{(g_1, \cdots, g_B) \in (\mathbf{Z}_p^n)^B \mid \{g_1, \cdots, g_B\}\ generates\ K\}.$$

By Lemma 1 of [12], we have

$$\mid F_0(\mathbf{Z}_p^n; B) \mid = p^{\frac{n(n-1)}{2}}(p^B-1)(p^{B-1}-1)\cdots(p^{B-n+1}-1).$$

Now, set $X = \{(g_1, \cdots, g_B) \in (\mathbf{Z}_p^n)^B \mid \{g_1, \cdots, g_B\}\ generates\ \mathbf{Z}_p^{n-1}\}$. Let $(g_1, \cdots, g_B) \in X$. Furthermore, let $\mathbf{A}$ be the $n \times B$ matrix having $g_1, g_2, \cdots, g_B$ as column vectors. By the definition of X, the rank of $\mathbf{A}$ is $n-1$. Let $u_1, u_2, \cdots, u_n$ be the row vectors of $\mathbf{A}$. Then $(g_1, \cdots, g_B) \in X$ if and only if $dim\ < u_1, u_2, \cdots, u_n >= n-1$. Thus we have

$$\begin{aligned} \mid X \mid &= \mid \{(u_1, u_2, \cdots, u_n) \in (\mathbf{Z}_p^n)^B \mid dim\ < u_1, u_2, \cdots, u_n >= n-1\} \mid \\ &= (p^B-1)(p^B-p)\cdots(p^B-p^{n-2})p^{n-1} \\ &= p^{\frac{n(n-1)}{2}}(p^B-1)(p^{B-1}-1)\cdots(p^{B-n+2}-1). \end{aligned}$$

Therefore it follows that

$$\mid F_g(\mathbf{Z}_p^n; B) \mid = p^{\frac{n(n-1)}{2}+B-n+1}(p^B-1)(p^{B-1}-1)\cdots(p^{B-n+2}-1).$$

By Theorem 3, the result follows.
Q.E.D.

For $n = 2$ and $g \in \mathbf{Z}_p^2 \setminus \{0\}$, we have

$$Isoc(D, \mathbf{Z}_p, g, I) = p^B$$

and

$$Isoc(D, \mathbf{Z}_p^2, g, I) = \frac{p^{B-1}(p^B-1)}{p-1},$$

where $B = B(D)$. By Theorem 2, these imply that

$$Iso(D, \mathbf{Z}_p^2, g, I) = p^B + \frac{p^{B-1}(p^B-1)}{p-1}.$$

This is given in Corollary 4.6 of [14].

In general, the following result holds. This formula is an explicit form of the formula in [14].

Corollary 2 (14, Theorem 4.4) *Let D be a connected symmetric digraph and $g \in \mathbf{Z}_p^n \setminus \{0\}$. Then the number of I-isomorphism classes of g-cyclic $\mathbf{Z}_p^n$-covers of D is*

$$Iso(D, \mathbf{Z}_p^n, g, I) = \sum_{k=1}^{n} \frac{p^{B-k+1}(p^B-1)\cdots(p^{B-k+2}-1)}{(p^{k-1}-1)(p^{k-2}-1)\cdots(p-1)},$$

where $B = B(D)$.

Let $p(>2)$ be a prime number and $\mathbf{Z}_{p^n}$ the cyclic group of order p^n.

Theorem 5 *Let D be a connected symmetric digraph, $\mathbf{Z}_{p^n}$ the cyclic group of order $p^n(p(>2)$: prime) and $g \in \mathbf{Z}_{p^n} \setminus \{0\}$. Furthermore, let $ord(g) = p^{n-\mu}(\mu < n)$ be the order of g. Then the number of I-isomorphism classes of connected g-cyclic $\mathbf{Z}_{p^n}$-covers of D is*

$$Isoc(D, \mathbf{Z}_{p^n}, g, I) = \begin{cases} p^{(n-1)B-\mu}(p^B-1) & \text{if } \mu \geq 1, \\ p^{nB} & \text{otherwise,} \end{cases}$$

where $B = B(D)$.

Proof. By Theorem 6 of [16], we have

$$|\ (Aut\ \mathbf{Z}_{p^n})_g\ | = p^{\mu}.$$

At first, assume that $\mu \geq 1$. Let $(g_1, \cdots, g_B) \in F_g(\mathbf{Z}_{p^n}; B)$. Then the set $\{g_1, \cdots, g_B, g\}$ generates the group $\mathbf{Z}_{p^n}$. Since g is not a generator of $\mathbf{Z}_{p^n}$, the set $\{g_1, \cdots, g_B\}$ generates the group $\mathbf{Z}_{p^n}$. Thus we have

$$F_g(\mathbf{Z}_{p^n}; B) = F_0(\mathbf{Z}_{p^n}; B).$$

By Lemma 1 of [12], we have

$$|\ F_g(\mathbf{Z}_{p^n}; B)\ | = p^{(n-1)B}(p^B-1).$$

Next, let $\mu = 0$. Then g is a generator of $\mathbf{Z}_{p^n}$. Thus it follows that

$$F_g(\mathbf{Z}_{p^n}; B) = (\mathbf{Z}_{p^n})^B,$$

i.e.,

$$|\ F_g(\mathbf{Z}_{p^n}; B)\ | = p^{nB}.$$

By Theorem 3, the result follows.
Q.E.D.

For $n = 2$ and $g \in \mathbf{Z}_{p^2} \setminus \{0\}$, we have

$$Isoc(D, \mathbf{Z}_{p^2}, g, I) = p^{B-1}(p^B-1)$$

and

$$Isoc(D, \mathbb{Z}_p, g, I) = p^B.$$

By Theorem 2, these imply that

$$Iso(D, \mathbb{Z}_{p^2}, g, I) = p^B + p^{B-1}(p^B - 1).$$

In general, the following result is obtained. This formula is an alternative form of the formula in [16].

Corollary 3 (16, Theorem 6) *Let D be a connected symmetric digraph and $g \in \mathbb{Z}_{p^n} \setminus \{0\}$. Furthermore, let $ord(g) = p^{n-\mu} (0 < \mu < n)$ be the order of g. Then the number of I-isomorphism classes of g-cyclic $\mathbb{Z}_{p^n}$-covers of D is*

$$Iso(D, \mathbb{Z}_{p^n}, g, I) = p^{(n-\mu-1)B} + p^{(n-\mu-1)B}(p^B - 1)\frac{p^{(\mu+1)(B-1)} - 1}{(p^{B-1} - 1)},$$

where $B = B(D)$.

We state a few problems on enumeration of isomorphism classes of cyclic A-covers.

Problem 1 *When g is even order, can we obtain a characterization when two g-cyclic A-covers are Γ-isomorphic ?*

We have no information on enumeration of cyclic A-covers for any non-abelian group A.

Problem 2 *Can we drive enumeration formula for any other abelian groups or non-abelian groups ?*

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