Stability Number and Fractional F-factors in Graphs *

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Abstract

Let G be a graph with vertex set V(G) and let f be a nonnegative integer-valued function defined on V(G). A spanning subgraph F of G is called a fractional f-factor if $d^h_G(x) = f(x)$ for every $x \in V(F)$. In this paper we prove that if $\delta(G) \geq b$, and $\alpha(G) \leq \frac{4a(\delta-b)}{(b+1)^2}$, then G has a fractional f-factor. Where a and b are integers such that $0 \leq a \leq f(x) \leq b$ for every $x \in V(G)$. Therefore we prove that the fractional analogue of Conjecture in [2] is true.

Key words: stability number; fractional f-factor; graph
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1 Introduction

The graphs considered in this paper will be simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $d_G(x)$ the degree of a vertex x in G. Let g and f be two integer-valued functions defined on V(G) such that $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. Denote by $\alpha(G)$ the stability number of a graph G, by $\delta(G)$ the minimum degree of vertices in G. If A and B are disjoint subsets of V(G), then $e_G(A, B)$ denotes the number of edges that join a vertex in A and a vertex in B. If $A = \{x\}$, then $e_G(x, B)$ denotes the number of edges that join x and a vertex in B. For a subset S of V(G), we denote by $G - S$ the subgraph obtained from G by deleting the vertices in S together with edges incident with vertices in S. In the following we

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write $f(W) = \sum_{x\in W} f(x)$ and $f(\emptyset) = 0$ for any $W \subseteq V(G)$. In particular, we set $d_{G-S}(T) = \sum_{x\in T} d_{G-S}(x)$ for $S, T \subseteq V(G)$ and $S \bigcap T = \emptyset$. We also set $N_G[A] = N_G(A) \bigcup A$. A fractional (g,f)-indicator function is a function h that assigns to each edge of a graph G a number $h(e)$ in [0,1] so that for each vertex x we have $g(x) \le d_G^h(x) \le f(x)$, where $d_G^h(x) = \sum_{x\in E_x} h(e)$ is the fractional degree of x in G with $E_x = \{e : e = xy \in E(G)\}$. When $g(x) = f(x)$ for every $x \in V(G)$, a fractional (g,f)-indicator function is called a fractional f-indicator function. Let h be a fractional f-indicator function of a graph G. Set $E_h = \{e : e \in E(G) \;\; and \;\; h(e) \neq 0\}$. If G_h is a spanning subgraph of G such that $E(G_h) = E_h$, then G_h is called a fractional f-factor of G. In this paper, we prove the fractional analogue of Conjecture in [2]. Notations and definitions not given here can be found in [1] and [6].

Many authors have investigated graph factors and factorization [5,8], connected factors [3], and fractional *(g, f)-factors* [4,7]. There is a necessary and sufficient condition for a graph G to have an fractional *f-factor* which was given by Guizhen Liu and Lanju Zhang.

Theorem A. [4] *A graph G has an fractional f-factor if and only if for every subset S of V(G)*

$$\gamma(S,T) = f(S) + \sum_{x\in T} d_{G-S}(x) - f(T) \ge 0$$

where $T = \{x : x \in V(G) - S \text{ and } d_{G-S}(x) \le f(x)\}$.

In [2], we gave a sufficient condition for the existence of an *f-factor* in a $K_{1,n}$-free graph in terms of its stability number and minimum degree δ, where $a \le f(x) \le b$ for every vertex $x \in V(G)$. The following theorem is the main result in [2].

Theorem B.[2] *Let G be an $K_{1,n}$-free graph and let f be a nonnegative integer-valued function defined on $V(G)$ such that $1 \le n-1 \le a \le f(x) \le b$ for every $x \in V(G)$. If $f(V(G))$ is even, $\delta(G) \ge b+n-2$, and $\alpha(G) \le \frac{4a(\delta-b-n+2)}{(b+1)^2(n-1)}$, then G has an f-factor.*

Furthermore, we made the following conjecture in [2].

Conjecture.[2] *Let G be a connected graph and let f be an nonnegative integer-valued function defined on $V(G)$ such that $0 \le a \le f(x) \le b$ for every $x \in V(G)$. If $\delta(G) \ge b$, $f(V(G))$ is even and $\alpha(G) \le \frac{4a(\delta-b)}{(b+1)^2}$, then G has an f-factor.*

In this paper, we prove the following theorems which show that the fractional analogue of the conjecture is true.

Theorem 1. *Let G be a connected graph and let f be a nonnegative integer-valued function defined on $V(G)$ such that $0 \le a \le f(x) \le b$ for* *every $x \in V(G)$. If $\delta(G) \geq b$, and $\alpha(G) \leq \frac{4a(\delta-b)}{(b+1)^2}$, then G has an fractional f-factor.*

2 Proof of Theorem 1

If $\alpha(G) = 0$, then G is an empty graph. Theorem 1 is trivial. Now we assume that $\alpha(G) \geq 1$, If $\alpha(G) = 1$, the graph G is a complete graph. Then obviously G has an fractional f-factor. So we may assume that $2 \leq \alpha(G) \leq \frac{4a(\delta-b)}{(b+1)^2}$.

Let G be a graph satisfying the hypothesis of Theorem 1, we prove the theorem by contradiction. Suppose that G has no fractional f-factors. Then $\gamma(S,T) < 0$ for some subsets S of V(G) by Theorem A. We choose such a subset S of V(G) which satisfies

$$\gamma(S,T) = f(S) + \sum_{x \in T} d_{G-S}(x) - f(T) < 0. \tag{1}$$

Where $T = \{x : x \in V(G) - S \text{ and } d_{G-S}(x) \leq f(x)\}$.

We first prove the following claims.

Claim 1. $|T| \geq a+1$.

Proof. If $|T| \leq a$, then by (1) and since $|S| + d_{G-S}(x) \geq d_G(x) \geq \delta(G) \geq b$ for $x \in T$ we obtain

$$\begin{aligned} 0 > \gamma(S,T) &= f(S) + \sum_{x \in T} d_{G-S}(x) - f(T) \\ &\geq a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| \\ &\geq \sum_{x \in T} (|S| + d_{G-S}(x) - b) \geq 0, \end{aligned}$$

which is a contradiction. So $|T| \geq a+1$.

Since $T \neq \emptyset$, let $h = min\{d_{G-S}(x) \mid x \in T\}$.

Claim 2. $h \leq b$.

Proof. According to the definition of T, we know that any $x \in T$ satisfies $d_{G-S}(x) \leq f(x) \leq b$. Hence $h \leq b$.

Now we proceed to prove Theorem 1.

We take $x_1 \in T$ such that x_1 is the vertex with the least degree in $G[T]$. Let $N_1 = N_G[x_1] \bigcap T$ and $T_1 = T$. For $i \geq 2$, if $T - \bigcup_{j<i} N_j \neq \emptyset$, let $T_i = T - \bigcup_{j<i} N_j$. Then take $x_i \in T_i$ such that x_i is the vertex with the least degree in $G[T_i]$, and set $N_i = N_G[x_i] \bigcap T_i$. We continue this

procedures until we reach the situation in which $T_i = \emptyset$ for some i, say for $i = k+1$. Then from the above definition we know that $x_1, x_2, \cdots, x_k$ is an independent set of G. Since $T \neq \emptyset$, we have $k \geq 1$.

Let $| N_i |= n_i$. From the definition of N_i, we can get the following properties.

$$\alpha(G[T]) \geq k, \tag{2}$$

$$| T |= \sum_{1\leq i\leq k} n_i, \tag{3}$$

$$\sum_{1\leq i\leq k} (\sum_{x\in N_i} d_{T_i}(x)) \geq \sum_{1\leq i\leq k} (n_i^2 - n_i). \tag{4}$$

It is easy to see that

$$d_{G-S}(T) \geq \sum_{1\leq i\leq k} (n_i^2 - n_i) + \sum_{1\leq i<j\leq k} e_G(N_i, N_j) \geq \sum_{1\leq i\leq k} (n_i^2 - n_i). \tag{5}$$

Let $f(n_i) = n_i(n_i - b - 1)$. By differentiation we know that the minimum value of $n_i(n_i - b - 1)$ is $-(b+1)^2/4$, i.e. $n_i(n_i - b - 1) \geq -(b+1)^2/4$. From (1),(3),(4) we get

$$\begin{aligned} 0 > \gamma(S,T) &= f(S) + \sum_{x\in T} d_{G-S}(x) - f(T) \\ &\geq a | S | + d_{G-S}(T) - b | T | \\ &\geq a | S | + \sum_{1\leq i\leq k} (n_i(n_i - b - 1)) \\ &\geq a | S | -(b+1)^2 k/4. \end{aligned}$$

Since $\delta(G) \leq| S | + h$, we know that

$$| S |\geq \delta(G) - h \tag{6}$$

Since $\alpha(G) \geq \alpha(G[T]) \geq k$, from (2), (6), and the assumption $\alpha(G) \leq \frac{4a(\delta-b)}{(b+1)^2}$, we get

$$\begin{aligned} 0 > \gamma(S,T) &\geq a | S | -(b+1)^2 k/4 \\ &\geq a(\delta(G) - h) - \frac{(b+1)^2}{4} \cdot \frac{4a(\delta - b)}{(b+1)^2} \\ &= a(\delta(G) - h) - a(\delta(G) - b). \end{aligned}$$

Since $h \leq b$ by Claim 2, we have

$$0 > \gamma(S,T) \geq 0$$

Thus we get a contradiction and complete the proof. ∎

Remark. The condition of Theorem 1 is sharp. The upper bound on the stability number condition ($\alpha(G) \leq \frac{4a(\delta-b)}{(b+1)^2}$) is best possible in the following sense. We cannot replace $\alpha(G) \leq \frac{4a(\delta-b)}{(b+1)^2}$ by $\alpha(G) \leq \frac{4a(\delta-b)}{(b+1)^2} + 1$ in Theorem 1, which is shown by the following example.

Let $k = \lfloor\frac{4a(\delta-b)}{(b+1)^2}\rfloor$. We set $G_1 = K_2$ and $G_2 = \bigcup_{i=1}^{k+1} K_b^i$, where K_b^i is a complete graph with b vertices$(1 \leq i \leq k+1)$. Then let $G = G_1 + G_2$ be the join of G_1 and G_2(that is, $V(G) = V(G_1)\bigcup V(G_2)$ and $E(G) = E(G_1)\bigcup E(G_2)\bigcup\{uv : u \in V(G_1), v \in V(G_2)\}$). Then we have $\alpha(G) = \frac{4a(\delta-b)}{(b+1)^2} + 1$ and $\delta(G) \geq b$. Thus, if we take $S = V(G_1)$, and $f(x) = a$, $x \in V(G_1)$; $f(x) = b$, $x \in V(G_2)$, then obviously $T = V(G_2)$. Then

$$\begin{aligned}\gamma(S,T) &= f(S) + \sum_{x\in T} d_{G-S}(x) - f(T)\\ &= 2a + b\cdot(b-1)(k+1) - bb(k+1)\\ &= 2a - b(k+1).\end{aligned}$$

Note that $\alpha(G) \geq 2$, $k \geq 2$. So

$$2a - b(k+1) \leq 2a - 3b < 0.$$

Therefore, according to Theorem A, G has no any fractional f-factors.

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