Vertex-Disjoint Short Cycles Containing Specified Edges in a Graph

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Abstract

We call a cycle whose length is at most 5 a *short* cycle. In this paper, we consider the packing of short cycles in a graph with specified edges. A minimum degree condition is obtained, which is slightly weaker than that of the result in [1].

Keywords: vertex-disjoint cycles, packing, specified edges

1 Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. For a vertex x of a graph G, the neighborhood of x is denoted by $N_G(x)$, and $d_G(x) = |N_G(x)|$ is the degree of x in G . For a subgraph H of G and a vertex $x \in V(G)$, we also denote $N_H(x) = N_G(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$. For a subset S of $V(G)$, we write $\langle S \rangle$ for the subgraph induced by S. For a subgraph H of G and a subset S of $V(G)$, $d_H(S) = \sum_{x \in S} d_H(x)$, $N_H(S) = \cup_{x \in S} N_H(x)$ and define $G - H = \langle V(G) - V(H) \rangle$ and $G - S = \langle V(G) - S \rangle$. For a graph G, $|G| = |V(G)|$ is the order of G, $\delta(G)$ is the minimum degree of G, and

$$\sigma_2(G) = \min\{d_G(x) + d_G(y) | xy \notin E(G), x, y \in V(G), x \neq y\}$$

is the minimum degree sum of nonadjacent vertices. (When G is complete, we define $\sigma_2(G) = \infty$.)

For graphs G_1 and G_2, $G_1 + G_2$ is the join of G_1 and G_2. For graphs G_1, G_2 and G_3, $G_1 + G_2 + G_3 = (G_1 \cup G_3) + G_2$. K_n is a complete graph of order n. In this paper, 'disjoint' means 'vertex-disjoint', since we only deal with partitions of the vertex set, and n always denotes the order of a graph. A cycle of length 3 is called a *triangle.*

In [1], Egawa et al. considered the partition of a graph into cycles passing through specified edges and proved the following theorem.

Theorem 1 *Suppose $k \geq 2, n \geq 4k-1$ and $\sigma_2(G) \geq n+2k-2$. Then for any independent edges $e_1, \ldots, e_k \in E(G)$, G can be partitioned into cycles $H_1, \ldots, H_k$ such that $e_i \in E(H_i)$.*

The proof of Theorem 1 consists of two steps, solving packing problems and then extending a packing to a partition. Result of a packing problem is the next theorem.

Theorem 2 *Suppose $k \geq 1$, $n \geq 4k-1$ and $\sigma_2(G) \geq n+2k-2$. Then for any independent edges $e_1, \ldots, e_k \in E(G)$, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|C_i| \leq 4$.*

The following corollary is immediate from Theorem 2.

Corollary 3 *Suppose $k \geq 1$, $n \geq 4k-1$ and $\delta(G) \geq \frac{1}{2}(n+2k-2)$. Then for any independent edges $e_1, \ldots, e_k \in E(G)$, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|C_i| \leq 4$.*

Result of extending a packing to a partition is the following.

Theorem 4 *Suppose $k \geq 1$, $n \geq 3k$, $\sigma_2(G) \geq n+k$, and $e_1, \ldots, e_k \in E(G)$ are independent edges. Moreover, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$. Then G contains k disjoint cycles $H_1, \ldots, H_k$ such that $e_i \in E(H_i)$ and $\bigcup_{i=1}^{k} V(H_i) = V(G)$.*

In [1], the next two examples are shown for Theorem 2 and Corollary 3.

Example 1. Let G be a graph of order n obtained from K_{n-1} by adjoining a new vertex x so that the degree of x is $2k-1$. Take any k independent edges $e_1, \ldots, e_k$ in $\langle \{x\} \cup N_G(x) \rangle$, and let x be an endvertex of e_1. Then there is no cycle through e_1 avoiding any endvertices of $e_2, \ldots, e_k$ and $\sigma_2(G) = n+2k-3$.

Example 2. Let $G = A + K_{2k-2} + B$ with an edge e_1 joining A and B, where A and B are complete graphs with $|A| = \lceil n/2 \rceil - k + 1$ and $|B| = \lfloor n/2 \rfloor - k + 1$. Take any $k-1$ independent edges $e_2, \ldots, e_k$ in K_{2k-2}. Then $e_1, \ldots, e_k$ are k independent edges, but there is no cycle through e_1 avoiding any vertices in K_{2k-2}, while $\delta(G) = \lfloor n/2 \rfloor + k - 2 = \lfloor \frac{n+2k-4}{2} \rfloor$.

Example 2 gives the sharpness of the assumption in Corollary 3 only for even n.

In this paper, we will prove the following theorem.

Theorem 5 *Suppose $n \geq \max\{6k, 4k+6\}$, $k \geq 1$ and $\delta(G) \geq (n+2k-3)/2$. Then for any independent edges $e_1, \ldots, e_k$, G contains k disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ for $1 \leq i \leq k$, and '$|C_i| \leq 4$ for $1 \leq i \leq k$' or '$|C_i| = 5$ for some i, $1 \leq i \leq k$ and the rest are all triangles'.*

By Theorem 5, the degree condition in Theorem 1 can be slightly improved when n is sufficiently large.

Theorem 6 *Suppose $n \geq 6k+2$, $k \geq 2$ and either $\sigma_2(G) \geq n+2k-2$ or $\delta(G) \geq (n+2k-3)/2$. Then for any independent edges $e_1, \ldots, e_k \in E(G)$, G can be partitioned into cycles $H_1, \ldots, H_k$ such that $e_i \in E(H_i)$.*

The following example shows that the conclusion '$|C_i| = 5$ for some i, $1 \leq i \leq k$ and the rest are all triangles' in Theorem 5 is necessary.

Example 3. Suppose n is odd. Let G be a graph obtained from $G' = A + K_{2k-2} + B$, where A and B are complete graphs with $|A| = |B| = (n-2k-1)/2$, by adding new three vertices x, y and z with an edge yz and joining x to A, B and K_{2k-2}, y to A and K_{2k-2}, and z to B and K_{2k-2}. Take any $k-1$ independent edges $e_2, \ldots, e_k$ in K_{2k-2} and let $e_1 = yz$. Then $e_1, \ldots, e_k$ are k independent edges, but e_1 can not be contained in a cycle of length 3 or 4 avoiding the vertices of K_{2k-2}, while $\delta(G) = (n+2k-3)/2$.

For k independent edges $e_1 = x_1y_1, \ldots, e_k = x_ky_k$, a cycle C is called *admissible* if $|E(C) \cap \{e_1, \ldots, e_k\}| = 1$ and $|V(C) \cap \{x_1, \ldots, x_k, y_1, \ldots, y_k\}| = 2$. For $1 \leq r \leq k$, a set of cycles $\{C_1, \ldots, C_r\}$ is *admissible* if each C_i is admissible, mutually disjoint, and $|C_i| \leq 4$ for $1 \leq i \leq r$ or $|C_i| = 5$ for some i, $1 \leq i \leq r$ and the rest are all triangles. If we say 'r admissible cycles', it means that a set of these r cycles is admissible.

2 Proof of Theorem 5

We distinguish two cases according to the value of k.

Case 1 $k \geq 2$.

Let G be an edge-maximal counterexample and $e_i = x_iy_i$ for $1 \leq i \leq k$. Since if G is a complete graph, G contains k admissible cycles, G is not complete. Let x and y be nonadjacent vertices of G and define $G' = G+xy$, the graph obtained from G by adding the edge xy. Then G' is not a counterexample by the maximality of G, and so G' has k admissible cycles $C_1, \dots, C_k$. Without loss of generality, we may assume that $xy \in E(C_k)$. Then G has $k-1$ admissible cycles $C_1, \dots, C_{k-1}$. We take these cycles such that $|\bigcup_{i=1}^{k-1} V(C_i)|$ is as small as possible. We may assume that $e_i \in E(C_i)$. Let $L = \langle \bigcup_{i=1}^{k-1} V(C_i) \rangle$, $M = G - L$, $D = M - \{x_k, y_k\}$.

Claim 1 $d_{C_i}(z) \le 3$ *for any* $z \in V(D)$ *and* $1 \le i \le k-1$.

(*Proof.*) Let $z \in V(D)$. If $d_{C_i}(z) \ge 4$ for some i, $1 \le i \le k-1$, $\langle V(C_i) \cup \{z\} \rangle$ contains a cycle passing through e_i which is shorter than C_i. □

Claim 2 $d_D(x_k) \ge 2$ *and* $d_D(y_k) \ge 2$.

(*Proof.*) Suppose $d_D(x_k) \le 1$. Then

$$\frac{n+2k-3}{2} \le d_G(x_k) \le |L| + 2 \le \max\{4k-4, 3k-1\} + 2.$$

Then $n \le \max\{6k-1, 4k+5\}$. This is a contradiction. □

Take any $z \in N_D(x_k)$ and $z' \in N_D(y_k)$, and let $S = \{x_k, y_k, z, z'\}$. Since M does not contain an admissible cycle passing through e_k length at most 4 (if such cycle exists, it contradicts G does not contain k admissible cycles or the minimality of $|L|$), $zz', x_k z', y_k z \notin E(G)$, and $d_S(w) \le 2$ for any $w \in V(M) - S$. Then

$$d_M(S) \le 2(|M| - 4) + 6 = 2|M| - 2.$$

Therefore,

$$\begin{aligned} d_L(S) &\ge 4\delta(G) - (2|M| - 2) = 2n + 4k - 6 - 2(n - |L|) + 2 \\ &= 2|L| + 4k - 4 = \sum_{i=1}^{k-1} (2|C_i| + 4). \end{aligned} \tag{1}$$

Claim 3 $d_{C_i}(S) \le 2|C_i| + 4$ *for* $1 \le i \le k-1$.

(*Proof.*) Suppose $|C_i| \ge 4$. By Claim 1, $d_{C_i}(\{z, z'\}) \le 6$. If $d_{C_i}(\{x_k, y_k\}) \ge |C_i| + 3$, there is a triangle $x_k y_k a x_k$ for some $a \in V(C_i) - \{x_i, y_i\}$. Hence $d_{C_i}(\{x_k, y_k\}) \le |C_i| + 2$, and we get $d_{C_i}(S) \le 2|C_i| + 4$ if $|C_i| = 4$ and $d_{C_i}(S) \le 2|C_i| + 3$ if $|C_i| = 5$.

Suppose $|C_i| = 3$, $C_i = x_i y_i a x_i$ and $d_{C_i}(S) \geq 2|C_i| + 5 = 11$. If $\{zx_i, zy_i, x_k a, z'a\} \subseteq E(G)$, then $x_i y_i z x_i$ and $x_k y_k z' a x_k$ are two admissible cycles. Then, since $d_{C_i}(S) \geq 11$, we may assume that $\{za, y_k a, z' x_i, z' y_i\} \subseteq E(G)$. But this means that there are two admissible cycles $x_i y_i z' x_i$ and $x_k y_k a z x_k$. □

By Claim 3, the equality holds for (1), that is, $d_{C_i}(S) = 2|C_i| + 4$ for all i, $1 \leq i \leq k-1$.

Claim 4 $|C_i| = 3$ *for* $1 \leq i \leq k-1$.

(*Proof.*) By the proof of Claim 3, we only consider the case $|C_i| = 4$. Let $C_i = x_i y_i a b x_i$. Since $d_{C_i}(\{z, z'\}) = 6$, $d_{C_i}(z) = d_{C_i}(z') = 3$ and each of $N_{C_i}(z)$ and $N_{C_i}(z')$ is $\{a, b, x_i\}$ or $\{a, b, y_i\}$. Hence we may assume that $\{za, z'a, zb, z'b, zy_i\} \subseteq E(G)$ by symmetry. Then $x_k a \notin E(G)$ and since $d_{C_i}(\{x_k, y_k\}) = 6$, we may assume that $y_k a \in E(G)$. (Otherwise, we get an admissible triangle $x_k y_k b x_k$.) By Claim 2, we can take $z'' \in N_D(x_k) - \{z\}$. Since also $d_{C_i}(\{z'', z'\}) = 6$, $z''a \in E(G)$. Then $x_i y_i z b x_i$ and $x_k y_k a z'' x_k$ are admissible cycles. □

Claim 5 $d_{C_i}(\{z, z'\}) = 6$ *for some* i, $1 \leq i \leq k-1$.

(*Proof.*) Suppose $d_{C_i}(\{z, z'\}) \leq 5$ for $1 \leq i \leq k-1$. Then $d_L(\{z, z'\}) \leq 5k - 5$. Since $N_D(z) \cap N_D(z') = \phi$,

$$d_M(\{z, z'\}) \leq |M| - 2 = n - 3(k-1) - 2 = n - 3k + 1.$$

Hence we get

$$d_G(\{z, z'\}) \leq (5k-5) + (n - 3k + 1) = n + 2k - 4 < 2\delta(G).$$

This is a contradiction. □

Without loss of generality, we may assume that $d_{C_1}(\{z, z'\}) = 6$. This means that $N_{C_1}(z) = N_{C_1}(z') = V(C_1)$. Let $C_1 = x_1 y_1 a x_1$ and take any $z'' \in N_D(x_k) - \{z\}$. Let $S' = \{x_k, y_k, z', z''\}$. Then, since $N_{C_1}(S') = 2|C_1| + 4 = 10$ also holds, $d_{C_1}(z'') \geq 2$. Hence $x_1 y_1 z x_1$ and $x_k y_k z' a z'' x_k$ or $x_1 y_1 z'' x_1$ and $x_k y_k z' a z x_k$ are two admissible cycles, and this gives k admissible cycles which consist of $k-1$ admissible triangles and an admissible cycle of length 5. This completes the proof of Case 1.

Case 2 $k = 1$.

In this case, the assumption is $\delta(G) \geq (n-1)/2$. Let $e_1 = xy$, $x, y \in V(G)$ and $M = V(G) - \{x, y\}$. We may assume that $N(x) \cap N(y) = \phi$, since otherwise there is an admissible triangle. If there are $z \in N_M(x)$ and $z' \in N_M(y)$ such that $N(z) \cap N(z') \neq \phi$, there is an admissible cycle. Hence we may assume that $N(z) \cap N(z') = \phi$ for any $z \in N_M(x)$ and $z' \in N_M(y)$. Let $D = V(G) - (N(x) \cup N(y))$ and take any $z \in N_M(x)$ and $z' \in N_M(y)$. Then

$$
\begin{aligned}
n &\geq 2 + |N_M(x)| + |N_M(y)| + |N_D(z)| + |N_D(z')| \\
&\geq 2 + |N_M(x)| + |N_M(y)| \\
&\quad + \left(\frac{n-1}{2} - (|N_M(x)| - 1) - 1\right) + \left(\frac{n-1}{2} - (|N_M(y)| - 1) - 1\right) \\
&= n + 1.
\end{aligned}
$$

This is a contradiction. This completes the proofs of Case 2 and Theorem 5.

Acknowledgement

The author would like to thank Professor Hikoe Enomoto and Professor Katsuhiro Ota for their helpful suggestions. Also, he would like to thank the refree for valuable suggestions.

References

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