TREES WITH PATH-STABLE CENTER

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ABSTRACT. We study the notion of path-congruence $\Phi : T_1 \to T_2$ between two trees T_1 and T_2. We introduce the concept of the trunk of a tree, and prove that, for any tree T, the trunk and the periphery of T are stable. We then give conditions for which the center of T is stable. One such condition is that the central vertices have degree 2. Also, the center is stable when the diameter of T is less than 8.

1. INTRODUCTION

Motivation for the investigations dealt with in this paper comes mainly from Reconstruction Theory. A survey on Reconstruction Theory is given in [2].

Let G_1, G_2 be two (finite, simple) graphs. Let $\mathcal{K}$ be a class of graphs. A *$\mathcal{K}$-congruence* $\Phi : G_1 \to G_2$ is a bijection $V(G_1) \to V(G_2)$ such that, for every $Q \in \mathcal{K}$, and every $v \in V(G_1)$, the number of subgraphs of G_1 containing v and isomorphic to Q equals the number of subgraphs of G_2 containing $\Phi(v)$ and isomorphic to Q. If there is a $\mathcal{K}$-congruence $\Phi : G_1 \to G_2$, we say that G_1 and G_2 are *$\mathcal{K}$-congruent*.

The *$\mathcal{K}$-table* of a graph G is the array whose rows are labelled by the vertices of G, whose columns are labelled by representatives of the isomorphism classes of the graphs of $\mathcal{K}$ such that, for $v \in V(G)$, $Q \in \mathcal{K}$, the entry at position (v, Q) is the number of subgraphs of G containing v isomorphic to Q.

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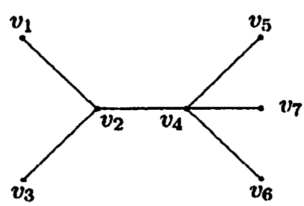
With this definition it follows that two graphs G_1 and G_2 are $\mathcal{K}$-congruent if and only if their $\mathcal{K}$-tables are equal, up to reordering of the rows.

When $\mathcal{K}$ is the class of all paths, the $\mathcal{K}$-table of G is called the *path-table* of G and it is denoted $\mathbb{P}(G)$, and a $\mathcal{K}$-congruence $\Phi : G_1 \to G_2$ is called a *path-congruence.* If there is a path-congruence from G_1 to G_2, then the graphs are said to be *path-congruent.* Since an isomorphism class of paths can simply be described by their (common) length, then the columns of $\mathbb{P}(G)$ can be labelled by the positive integers.

In the special case of a tree T, $\mathbb{P}(T)$ has $|V(T)|$ rows and diam$T$ columns (we omit zero columns) and, for $v \in V(T)$, $l \in \mathbb{N}$, the entry at position (v, l) is the number of paths of length l passing through v (see Fig. 1): we denote this number simply by $p_l(v)$, the tree T being understood.



v \ l	1	2	3
v_1	1	2	3
v_2	3	6	6
v_3	1	2	3
v_4	4	8	6
v_5	1	3	2
v_6	1	3	2
v_7	1	3	2

FIGURE 1. Example of path-table.

In [5] we have shown that path-congruent trees need not be isomorphic. Nonetheless, they have some similarities. In this paper we are concerned with investigating such similarities: we describe some "canonical" subsets of a tree which are preserved by any path-congruence $\Phi : T_1 \to T_2$. In Section 2 we give the necessary definitions and notations. In particular, we introduce the basic notion of *trunk* (following an idea that appeared in [1], but with a different meaning for the word *trunk*), and its generalization, the *p-trunk* . We also define the *stump* of a tree T, which makes an interesting comparison to the center of T. Finally, we define the *trunk-decomposition* of a tree, and the *ramification* of a trunk vertex. In Section 3 we get some results on the stump and the center. In Section 4 we obtain inequalities involving the ramifications of two trunk vertices linked by a path-congruence. In Section 5 we prove that any path-congruence $\Phi : T_1 \to T_2$ must map the trunk (resp. the periphery) of T_1 onto the trunk (resp. the periphery) of T_2, and discuss conditions on a tree T for which any path-congruence $\Phi : T \to T$ must fix the center. In Section 6 we pose the problem of locating the row of $\mathbb{P}(T)$ corresponding to the center, and make some final remarks.

2. Preliminaries.

Let T be a tree. If $x, y \in V(T)$, we denote by $[x, y]$ the unique path whose end-vertices are x and y. If x, y are adjacent vertices, we simply denote by xy the edge whose end-points are x and y. If $x = y$ is allowed we also use the notation $\{x, y\}$ (for example to denote the center of a tree). For other notations we follow [3].

Recall that in a finite graph G with natural metric d, the *eccentricity* of a vertex v is defined by $\mathrm{e}(v) = \max\{d(v, x) | x \in V(G)\}$, and that the *diameter* $\mathrm{diam}G$ of G and *radius* $\mathrm{rad}G$ of G are respectively defined as the maximum and minimum eccentricity of the vertices of G.

Let T be a tree. For any integer p, with $\mathrm{rad}T \leq p \leq \mathrm{diam}T$, the *p-th trunk* of T, denoted $\mathrm{Tr}_p(T)$, is the intersection of all paths of length p, and the *p-th crown*, denoted $\mathrm{Cr}_p(T)$, is the set of all vertices of eccentricity p.

It is immediately seen that the p-th trunk is a path (if nonempty, since in a tree the intersection of two paths is a path), and that, if $p \leq q$, then $\mathrm{Tr}_p(T) \subseteq \mathrm{Tr}_q(T)$. For simplicity, we shall denote by $\mathrm{Tr}(T)$ (without any index) the largest among the $\mathrm{Tr}_p(T)$, and call it the *trunk* of T, whereas we shall denote by $\mathrm{St}(T)$ the smallest *nonempty* among the $\mathrm{Tr}_p(T)$, and call it the *stump* of T.

When $p = \mathrm{rad}T$, $\mathrm{Cr}_p(T)$ is equal to the *center* $Z(T)$. It is well-known that, for a tree T, $|Z(T)| \leq 2$, and if $|Z(T)| = 2$, then the two vertices of $Z(T)$ are adjacent. It is also well-known that $Z(T)$ can be obtained by applying to T the process of iterated pruning. Thus, $Z(T)$ has one or two vertices depending on whether $\mathrm{diam}T$ is even or odd.

When $p = \mathrm{diam}T$, $\mathrm{Cr}_p(T)$ is also called the *periphery* of T, denoted by $\mathcal{P}(T)$, and its vertices are called *peripheral* vertices. Clearly, a vertex $v \in V(T)$ is peripheral if and only if there exists $y \in V(T)$ such that $d(v,y) = \mathrm{diam}T$. In this case v and y are said to be *antipodal* vertices. When $|\mathrm{Tr}(T)| > 1$, the set $\mathcal{P}(T)$ can be partitioned into two subsets $\mathcal{P}_R(T), \mathcal{P}_L(T)$ (conventionally) called *right-peripheral* and *left-peripheral*, defined by stating that two peripheral vertices belong to the same element of the partition if they are *not* antipodal.

A quite general type of "decomposition" for a graph G can be described as follows. Let G be a graph and S an induced connected subgraph. Given any $v \in V(S)$, we define the *S-branch* from v, denoted $\mathrm{Br}_S(v)$, to be the maximal connected subgraph H of G containing v and such that $H \cap S = \{v\}$. If $S = \{v_1, ..., v_t\}$, the sequence $(S, \mathrm{Br}_S(v_1), ..., \mathrm{Br}_S(v_t))$ is the *S-decomposition* of G. The *S-ramification* $\mathrm{ram}_S\, v_i$ of $v_i \in V(S)$ is defined to be the eccentricity of v_i within $\mathrm{Br}_S(v_i)$. Although the "decomposition" just described may be unnatural for a general graph G (for instance, when G is a cycle and S a path), it is useful in the case of a tree, where we take S to be the trunk $\mathrm{Tr}(T)$. In this particular case we simply write $\mathrm{Br}(v)$ and $\mathrm{ram}\, v$ for the $\mathrm{Tr}(T)$-*branch* and $\mathrm{Tr}(T)$-*ramification* of v, calling them *branch* from v and *ramification* of v. The trunk-decomposition of a tree becomes a decomposition in the usual sense, since the trunk and the branches make a partition of the edge-set of T.

To avoid complex notation, in the sequel of the paper, when x is a vertex of a graph G, we will often write $x \in G$ instead of $x \in V(G)$.

3. Some properties of the trunk.

In this section we prove a few results regarding $\mathrm{Tr}_p(T)$. Recall that when $p = \mathrm{diam}T$, we simply write $\mathrm{Tr}(T)$ instead of $\mathrm{Tr}_p(T)$.

Proposition 3.1. *Let T be a tree. Then the end-vertices of* $\mathrm{Tr}(T)$ *have degree in T different from* 2.

Proof. To begin with, note that an end-vertex v of $\mathrm{Tr}(T)$ has $\deg_T v = 1$ if and only if $v \in \mathcal{P}(T)$. Let x be an end-vertex of $\mathrm{Tr}(T)$ with $\deg_T x \neq 1$. If $\deg_T x = 2$, then all paths of length $\mathrm{diam}T$ (which, by definition, contain x) necessarily also contain the vertices adjacent to x. This contradicts the fact that x is an end-vertex of $\mathrm{Tr}(T)$. $\square$

Proposition 3.2. *Let T be a tree with more than one vertex. Then $Z(T) \subseteq \mathrm{Tr}(T)$, and either $Z(T) = \mathrm{Tr}(T)$ or at least one vertex of $Z(T)$ is not an end-vertex of* $\mathrm{Tr}(T)$. *Moreover, if* $\mathrm{diam}T$ *is even and $|\mathrm{Tr}(T)| > 1$, then the center c of T is not an end-vertex of* $\mathrm{Tr}(T)$.

Proof. It is well-known that any diametral path of a tree contains $Z(T)$. Thus $Z(T) \subseteq \mathrm{Tr}(T)$. Assume $Z(T) \subsetneq \mathrm{Tr}(T)$. If $|Z(T)| = 2$, then, by assumption, $|\mathrm{Tr}(T)| \geq 3$. Since the two central vertices are adjacent, they obviously cannot both be the end-vertices of $\mathrm{Tr}(T)$. If $|Z(T)| = 1$, then, by assumption, $|\mathrm{Tr}(T)| \geq 2$. Let c be the center, and x be one end-vertex of the trunk. From Proposition 3.1, $\deg_T x = 1$ or $\deg_T x > 3$. If $\deg_T x = 1$ then clearly $x \neq c$, since $\deg_T c \neq 1$ (T is a nontrivial tree). If $\deg_T x \geq 3$, there exist w_1, w_2 neighbours of x such that both lie on a diametral path (see Fig. 2).

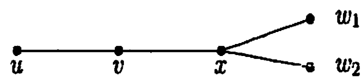


FIGURE 2

If x were the center (in addition to being an end-vertex of the trunk), then, by using the two paths of maximum length $(..., u, v, x, w_1, ...)$ and $(..., u, v, x, w_2, ...)$, one could also construct another path of maximum length $(..., w_1, x, w_2, ...)$, not containing the trunk vertex v, a contradiction. Thus, again $x \neq c$. ☐

In general, there is no relationship between $\mathrm{St}(T)$ and $Z(T)$, as the examples in Figure 3 show ($\mathrm{St}(T) \subset Z(T)$ is impossible when $|Z(T)| = 1$, since $\mathrm{St}(T)$ is nonempty by definition).

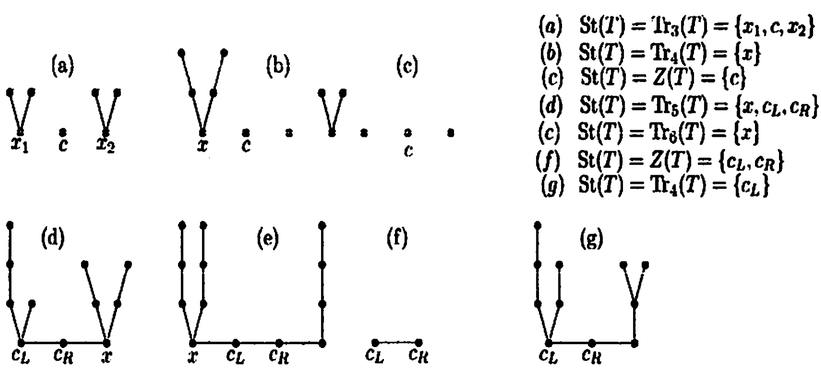


FIGURE 3. In (a), (b), (c) $Z(T) = \{c\}$, in (d), (e), (f), (g) $Z(T) = \{c_L, c_R\}$

However, the following proposition shows that, if $|\mathrm{St}(T)| \geq 2$, then $\mathrm{St}(T) \supseteq Z(T)$.

Proposition 3.3. *For any p such that $|\mathrm{Tr}_p(T)| > 1$, $\mathrm{Tr}_p(T) \supseteq Z(T)$.*

Proof. Let $Z(T) = \{c_L, c_R\}$ (possibly $c_L = c_R$). We have already noted that $\mathrm{Tr}(T) \supseteq Z(T)$. We now show that $\mathrm{Tr}(T)$ can be thought of as three consecutive paths. Let t_L and t_R denote the end-vertices of $\mathrm{Tr}(T)$. If $c_L \neq c_R$, let t_L and t_R be in the components of $T \setminus c_Lc_R$ containing c_L and c_R respectively. Then the paths are $P_L = [t_L, c_L]$, c_Lc_R, and $P_R = [c_R, t_R]$. We shall prove that if there is a path C of length p not containing $Z(T)$, then there is another path D of length p such that $|C \cap D \cap \mathrm{Tr}(T)| \leq 1$. Since $\mathrm{Tr}_p(T) \subseteq C \cap D \cap \mathrm{Tr}(T)$, it will follow that $|\mathrm{Tr}_p(T)| \leq 1$, contradicting the assumption. Assume that there is a path C as above. Then either $C \cap \mathrm{Tr}(T) \subseteq P_R$ or $C \cap \mathrm{Tr}(T) \subseteq P_L$. We can assume w.l.o.g. that $C \cap \mathrm{Tr}(T) \subseteq P_R$ (Fig. 4).

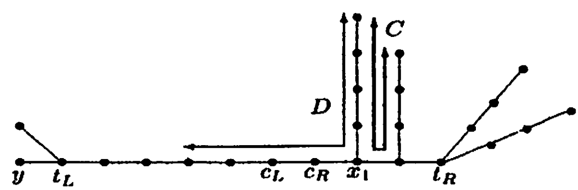


FIGURE 4

Let $x_1 \in \mathrm{Tr}(T)$ be the vertex of C closest to c_R (possibly $x_1 = c_R$). The length p of C satisfies $p \leq \mathrm{rad}T - d(c_L, x_1) + \mathrm{ram}\, x_1$. This implies $p \leq \mathrm{rad}T + \mathrm{ram}\, x_1$. Consequently, there certainly exists a path D of length p contained in $P_L \cup [c_L, c_R, x_1] \cup \mathrm{Br}(x_1)$, hence $C \cap D \cap \mathrm{Tr}(T) \subseteq \{x_1\}$. □

4. THE RAMIFICATIONS OF PATH-CONGRUENT VERTICES

Recall from Section 2 that if $v \in \mathrm{Tr}(T)$, then $\mathrm{ram}\, v$ is defined to be the eccentricity of v within the branch $\mathrm{Br}(v)$. Thus, $\mathrm{ram}\, v$ is a nonnegative integer, and $\mathrm{ram}\, v = 0$ if and only if $\deg v \leq 2$.

In this section we study the relation between $\mathrm{ram}\, x$ and $\mathrm{ram}\, y$ when $x, y \in \mathrm{Tr}(T)$, are *path-congruent* vertices, i.e. there is a path-congruence $\Phi : T \to T$ taking x to y. This is equivalent to saying that x and y have the same row in $\mathbb{P}(T)$ (see also Section 6). Note, however, that if $x \in V(T_1), y \in V(T_2)$ with $T_1 \neq T_2$, then the fact that x and y have the same path-row does not generally imply the existence of a path-congruence $\Phi : T_1 \to T_2$ taking x to y.

Lemma 4.1. *Let T be a tree, $\Phi : T \to T$ be a path-congruence and $x \in V(T)$. For any $l \geq 1$, let $\mathcal{A}_l$ (resp. $\mathcal{B}_l$) be the set of the paths in T of length l containing x but not $\Phi(x)$ (resp. $\Phi(x)$ but not x). Then $|\mathcal{A}_l| = |\mathcal{B}_l|$. If a (resp. b) denotes the maximum length of a path containing x but not $\Phi(x)$ (resp. $\Phi(x)$ but not x), then $a = b$.*

Proof. Let $p_l(x, \Phi(x))$ be the number of paths in T of length l passing through both x and $\Phi(x)$. Then $|\mathcal{A}_l| = p_l(x) - p_l(x, \Phi(x))$ and $|\mathcal{B}_l| = p_l(\Phi(x)) - p_l(x, \Phi(x))$. Since Φ is a path-congruence, $p_l(x) = p_l(\Phi(x))$, hence $|\mathcal{A}_l| = |\mathcal{B}_l|$.
If $a < b$, since $\mathcal{B}_b \neq \emptyset$, the equality $|\mathcal{A}_l| = |\mathcal{B}_l|$ would imply $\mathcal{A}_b \neq \emptyset$, contradicting the definition of a. Analogously, it cannot be $a > b$. Consequently, $a = b$. □

In order to use the equality $a = b$, we seek explicit formulae for a and b involving ramifications.

Lemma 4.2. *Let T be a tree with $|\mathrm{Tr}(T)| > 1$. Then, for every $x \in \mathrm{Tr}(T)$*

$$\text{ram}\, x + d(x, Z(T)) \leq \text{rad}T - |Z(T)| + 1,$$

with equality holding if and only if x is a trunk end-vertex.

Proof. If x is an end-vertex of $\text{Tr}(T)$, then $\text{ram}\, x + d(x, Z(T)) = \text{rad}T - |Z(T)| + 1$.

If x is not an end-vertex of $\text{Tr}(T)$, then there exists a path P (Fig. 5) of length $l = \text{ram}\, x + d(x, Z(T)) + \text{rad}T$ containing one end-vertex of $\text{Tr}(T)$ but not the other.

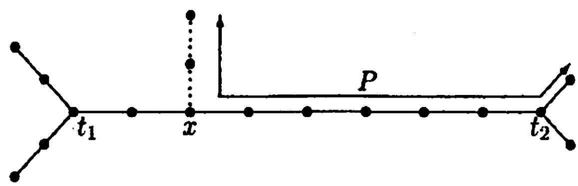


FIGURE 5

If $\text{ram}\, x \geq \text{rad}T - d(x, Z(T)) - |Z(T)| + 1$, one has

$$l \geq \text{rad}T - d(x, Z(T)) - |Z(T)| + 1 + d(x, Z(T)) + \text{rad}T = \text{diam}T.$$

The case $l > \text{diam}T$ is impossible, while the case $l = \text{diam}T$ contradicts the fact that x is not an end-vertex of the trunk. □

Lemma 4.3. *Let T be a tree with $|\text{Tr}(T)| > 1$. Let $v_1, v_2 \in \text{Tr}(T)$, with $v_1 \neq v_2$. Let $k = d(v_1, v_2)$, and denote by $w_0, ..., w_k$ the vertices of $[v_1, v_2]$ (set $w_0 = v_1$). Let a (resp. b) be the length of a longest path containing v_2 but not v_1 (resp. containing v_1 but not v_2). Let t_1, t_2 be the end-vertices of $\text{Tr}(T)$ (let t_1 be the one closer to v_1). Then*

$$a = k + \max\{\mathrm{ram}\, p + d(p, v_2) \mid p \in [v_2, t_2]\}$$
$$+ \max\{\mathrm{ram}\, w_i - i \mid i = 1, \ldots, k\},$$

$$b = \max\{\mathrm{ram}\, q + d(q, v_1) \mid q \in [t_1, v_1]\}$$
$$+ \max\{\mathrm{ram}\, w_i + i \mid i = 0, \ldots, k-1\}.$$

Proof. Let P be a path containing v_2 but not v_1 (see Fig. 6). We can think of P as the splicing of four paths: a first path within the branch of a vertex $w_i \in [v_1, v_2]$ (with $w_i \neq v_1$, but possibly $w_i = v_2$), a second path $[w_i, v_2]$, a third path $[v_2, p]$ with $p \in [v_2, t_2]$, and a fourth path within the branch of p.

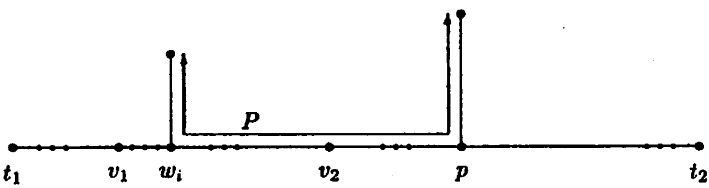


FIGURE 6

Therefore, $a = \max\{\mathrm{ram}\, w_i + d(w_i, v_2) + d(v_2, p) + \mathrm{ram}\, p \mid i = 1, \ldots, k,\ p \in [v_2, t_2]\}$. Since we can vary w_i and p independently, clearly

$$a = \max\{\mathrm{ram}\, w_i + d(w_i, v_2) \mid i = 1, \ldots, k\}$$
$$+ \max\{d(v_2, p) + \mathrm{ram}\, p \mid p \in [v_2, t_2]\}.$$

The formula for a given in the statement then follows from the fact that $d(w_i, v_2) = k - i$.

Let Q be a path containing v_1 but not v_2 (see Fig. 7).

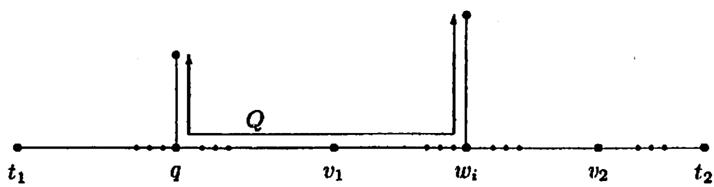


FIGURE 7

By an argument analogous to the previous one, we obtain

$$b = \max\{\mathrm{ram} w_i + i | i = 0, ..., k-1\} + \max\{\mathrm{ram} q + d(q, v_1) | q \in [t_1, v_1]\}.$$

□

Lemma 4.4. *Let T be a tree, and $\Phi : T \to T$ be a path-congruence. Let $Z(T) = \{c_L, c_R\}$ (possibly $c_L = c_R$), and assume, w.l.o.g., $\Phi(c_L) \notin Z(T)$ (therefore $c_L \neq \Phi(c_L) \neq c_R$). Let a (resp. b) be the length of a longest path containing c_L but not $\Phi(c_L)$ (resp. containing $\Phi(c_L)$ but not c_L). Let $k = d(c_L, \Phi(c_L))$, and denote by $w_0, ..., w_k$ the vertices of $[\Phi(c_L), c_L]$ (set $w_0 = \Phi(c_L)$).*

(1). *If $c_L = c_R$, then*

$$a = \mathrm{rad}T + k + \max\{\mathrm{ram}\, w_i - i \mid i = 1, ..., k\},$$
$$b = \mathrm{rad}T - k + \max\{\mathrm{ram}\, w_i + i \mid i = 0, ..., k-1\}.$$

(2). *If $c_L \neq c_R$ and $c_R \notin [\Phi(c_L), c_L]$, then*

$$a = \mathrm{rad}T + k + \max\{\mathrm{ram}\, w_i - i \mid i = 1, ..., k\},$$
$$b = \mathrm{rad}T - k - 1 + \max\{\mathrm{ram}\, w_i + i \mid i = 0, ..., k-1\}.$$

(3). *If $c_L \neq c_R$ and $c_R \in [\Phi(c_L), c_L]$, then*

$$a = \mathrm{rad}T + k - 1 + \max\{\mathrm{ram}\, w_i - i \mid i = 1, ..., k\},$$
$$b = \mathrm{rad}T - k + \max\{\mathrm{ram}\, w_i + i \mid i = 0, ..., k-1\}.$$

Proof. (1). Let c be the center of T, that is $c = c_L = c_R$, and let t_1 and t_2 be defined as in Lemma 4.3. Letting $\Phi(c) = v_1$ and $c = v_2$, from Lemma 4.3 we obtain

$$a = \max\{\operatorname{ram} w_i - i \mid i = 1, ..., k\} + k + \max\{\operatorname{ram} p + d(p, c) \mid p \in [c, t_2]\}.$$

By Lemma 4.2, we have

$$\max\{\operatorname{ram} p + d(p, c) \mid p \in [c, t_2]\} = \operatorname{ram} t_2 + d(t_2, c) = \operatorname{rad} T.$$

Also, from Lemma 4.3,

$$b = \max\{\operatorname{ram} w_i + i \mid i = 0, ..., k-1\} + \max\{\operatorname{ram} q + d(q, \Phi(c)) \mid q \in [t_1, \Phi(c)]\}.$$

Now, $d(q, \Phi(c)) = d(q, c) - d(c, \Phi(c)) = d(q, c) - k$, so that, again by Lemma 4.2, we have

$$\max\{\operatorname{ram} q + d(q, c) \mid q \in [c, \Phi(c)]\} = \operatorname{rad} T.$$

Claims (2) and (3) are obtained by using Lemma 4.2 and Lemma 4.3 as above, once we assume $\Phi(c_L) = v_1$ and $c_L = v_2$. □

We now establish arithmetic relationships among the ramifications of a vertex $z \in Z(T)$, its image $\Phi(z)$ under a path-congruence Φ, and the interior vertices on the $[z, \Phi(z)]$ path.

Theorem 4.5. *Let T be a tree, and $\Phi : T \to T$ be a path-congruence. Let $Z(T) = \{c_L, c_R\}$ (possibly $c_L = c_R$), and assume $\Phi(c_L) \notin Z(T)$ (therefore $c_L \neq \Phi(c_L) \neq c_R$). Let $k = d(c_L, \Phi(c_L))$, and denote by $w_0, ..., w_k$ the vertices of $[\Phi(c_L), c_L]$ (set $w_0 = \Phi(c_L)$). The following hold*

(1) *If $c_L = c_R = c$, then*

$$\operatorname{ram} \Phi(c) \geq \operatorname{ram} w_i + 2k - i, \quad (i = 1, ..., k).$$

Moreover, $\operatorname{ram} \Phi(c) = \operatorname{ram} c + k$ *if and only if* $\operatorname{ram} c \geq \operatorname{ram} w_i + k - i$ *for all* $i = 1, ..., k$.

(2) *If* $c_L \neq c_R$ *and* $c_R \notin [\Phi(c_L), c_L]$, *then*

$$\operatorname{ram} \Phi(c_L) \geq \operatorname{ram} w_i + 2k - i + 1, \quad (i = 1, ..., k).$$

Moreover, $\operatorname{ram} \Phi(c_L) = \operatorname{ram} c_L + k + 1$ *if and only if* $\operatorname{ram} c_L \geq \operatorname{ram} w_i + k - i$ *for all* $i = 1, ..., k$.

(3) *If* $c_L \neq c_R$ *and* $c_R \in [\Phi(c_L), c_L]$, *then*

$$\operatorname{ram} \Phi(c_L) \geq \operatorname{ram} w_i + 2k - i - 1, \quad (i = 1, ..., k).$$

Moreover, $\operatorname{ram} \Phi(c_L) = \operatorname{ram} c_L + k - 1$ *if and only if* $\operatorname{ram} c_L \geq \operatorname{ram} w_i + k - i$ *for all* $i = 1, ..., k$.

Proof. From $a = b$ of Lemma 4.1 and part (1) of Lemma 4.4, we obtain

$$\begin{aligned} (*) \quad & \max\{\operatorname{ram} c, \operatorname{ram} w_i + k - i \mid i = 1, ..., k-1\} \\ & = \max\{\operatorname{ram} \Phi(c) - k, \operatorname{ram} w_i - k + i \mid i = 1, ..., k-1\}. \end{aligned}$$

Observe that $\operatorname{ram} w_i + k - i > \operatorname{ram} w_i - k + i$ for all $i = 1, ..., k-1$. Therefore, from $(*)$,

$$\max\{\operatorname{ram} \Phi(c) - k, \operatorname{ram} w_i - k + i \mid i = 1, ..., k-1\} = \operatorname{ram} \Phi(c) - k,$$

and consequently, again by (*),

$$\operatorname{ram}\Phi(c) \geq \operatorname{ram} w_i + 2k - i, \quad (i = 1, ..., k).$$

In particular, if $i = k$ we get $\operatorname{ram}\Phi(c) \geq \operatorname{ram} c + k$, where equality holds if and only if

$$\max\{\operatorname{ram} c, \operatorname{ram} w_i + k - i \mid i = 1, ..., k-1\} = \operatorname{ram} c,$$

that is $\operatorname{ram} c \geq \operatorname{ram} w_i + k - i$ for all $i = 1, ..., k$.

Claims (2) and (3) are similarly obtained by Lemma 4.1, and, respectively, parts (2) and (3) of Lemma 4.4. □

Remark 4.6. With regards to (1) of Theorem 4.5, notice that both possibilities $\operatorname{ram}\Phi(c) = \operatorname{ram} c + k$ and $\operatorname{ram}\Phi(c) > \operatorname{ram} c + k$ can actually occur, as one can easily see by examining the trees in Figure 8 (where $k = 2$).

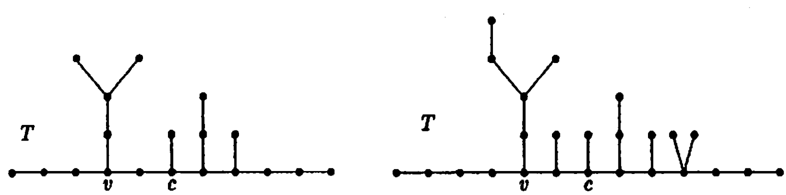


FIGURE 8. The cases $\operatorname{ram}\Phi(c) = \operatorname{ram} c + 2$ and $\operatorname{ram}\Phi(c) > \operatorname{ram} c + 2$.

For both trees, the path congruence $\Phi : T \to T$ is defined as follows. For $x \in V(T)$, put

$$\Phi(x) = \begin{cases} x & \text{if } x \neq v, c \\ v & \text{if } x = c \\ c & \text{if } x = v. \end{cases}$$

5. Some stable subsets. Stability conditions for $Z(T)$

We start this section by pointing out some general properties of a path-congruence $\Phi : T_1 \to T_2$ between trees T_1 and T_2. This will allow us to determine some stable sets.

Recall that, for $v \in V(T)$, $p_l(v)$ denotes the number of paths of length l passing through v. If $\Phi : T_1 \to T_2$ is a path-congruence, then, by definition, $p_l(\Phi(v)) = p_l(v)$ for any positive integer l. It is also clear that $\mathrm{diam} T_1 = \mathrm{diam} T_2$, and thus $|Z(T_1)| = 1$ if and only if $|Z(T_2)| = 1$. The next result gives other quantities that are preserved under a path-congruence.

Proposition 5.1. *Let T_1 and T_2 be trees and $\Phi : T_1 \to T_2$ be a path-congruence. Then*

- (*i*) $\Phi(\mathrm{Tr}_k(T_1)) = \mathrm{Tr}_k(T_2)$ *for all* k.
- (*ii*) $\Phi(\mathcal{P}(T_1)) = \mathcal{P}(T_2)$.
- (*iii*) *If* $|\mathcal{P}_L(T_1)| \neq |\mathcal{P}_R(T_1)|$, *then* Φ *preserves the partition into right-peripheral and left-peripheral vertices.*
- (*iv*) $\{|\mathcal{P}_L(T_1)|, |\mathcal{P}_R(T_1)|\} = \{|\mathcal{P}_L(T_2)|, |\mathcal{P}_R(T_2)|\}$.

Proof. We argue in terms of the path-tables, because it is more immediate.

(*i*) For $i = 1, 2$, a vertex u belongs to $\mathrm{Tr}_k(T_i)$ if and only if the k-th entry $p_k(u)$ of its path-row has value $N_k^{(i)} = \frac{1}{k+1} \sum_v p_k(v)$, that is the total number of paths of length $k \geq 1$ in T_i. Since $\Phi : T_1 \to T_2$ is a path-congruence, then the tables $\mathbb{P}(T_1)$ and $\mathbb{P}(T_2)$ are equal up to reordering of the rows. Thus, in particular, $\mathrm{diam} T_1 = \mathrm{diam} T_2$, $N_k^{(1)} = N_k^{(2)}$, and $|\mathrm{Tr}_k(T_1)| = |\mathrm{Tr}_k(T_2)|$.
From the fact that Φ permutes the path-rows and is injective, it follows that $\Phi(\mathrm{Tr}_k(T_1)) = \mathrm{Tr}_k(T_2)$.

(ii) It is enough to note that a vertex is peripheral if and only if its path-row has its first entry 1, and a nonzero last entry. Reasoning as above, we get $\Phi(\mathcal{P}(T_1)) = \mathcal{P}(T_2)$.

(iii) Assume $|\mathcal{P}_L(T_1)| \neq |\mathcal{P}_R(T_1)|$. Then a peripheral vertex v will belong to $\mathcal{P}_R(T_1)$ if and only if its path-row has last entry equal to $|\mathcal{P}_L(T_1)|$. Since $\mathbb{P}(T_1)$ and $\mathbb{P}(T_2)$ are equal up to reordering of the rows, it follows $\{|\mathcal{P}_L(T_1)|, |\mathcal{P}_R(T_1)|\} = \{|\mathcal{P}_L(T_2)|, |\mathcal{P}_R(T_2)|\}$. From the fact that Φ permutes path-rows and is injective, it then follows that $\Phi(\mathcal{P}_L(T_1)) = \mathcal{P}_L(T_2)$ or else $\Phi(\mathcal{P}_L(T_1)) = \mathcal{P}_R(T_2)$.

(iv) Since Φ is a path-congruence, so is Φ^{-1}. The result follows from (iii) in case $|\mathcal{P}_L(T_1)| \neq |\mathcal{P}_R(T_1)|$ or (using Φ^{-1}) in case $|\mathcal{P}_L(T_2)| \neq |\mathcal{P}_R(T_2)|$. It remains to show that if $|\mathcal{P}_L(T_1)| = |\mathcal{P}_R(T_1)|$ and $|\mathcal{P}_L(T_2)| = |\mathcal{P}_R(T_2)|$ then these two numbers are equal. Indeed, this follows from the equality $|\mathcal{P}_L(T_i)| + |\mathcal{P}_R(T_i)| = |\mathcal{P}(T_i)|$ and the statement (ii). □

Let G be a finite graph and Ω a group of permutations of $V(G)$. A subset S of $V(G)$ is said to be *Ω-stable* if, for any $\Phi \in \Omega$, $\Phi(S) = S$.

According to this definition we have, for example, that for a tree T, $Z(T)$ is $\mathrm{Aut}(T)$-stable. Here we only deal with the group Ω of all path-congruences on a tree T. Thus, we shall call *path-stable*, or simply *stable*, an Ω-stable subset. Note that, as a path-congruence Φ is not generally an automorphism, the graph induced by a path-stable set S may differ from the graph induced by $\Phi(S)$. It seems an interesting problem to describe the stable subsets of T. Perhaps a bit surprisingly, $Z(T)$ is not always stable, so we are led to formulate sufficient conditions for the stability of $Z(T)$.

Proposition 5.2. *Let T be a tree. For every p, $\mathrm{Tr}_p(T)$ is stable. The set $\mathcal{P}(T)$ is also stable, and if $|\mathcal{P}_L(T)| \neq |\mathcal{P}_R(T)|$ then both $\mathcal{P}_L(T)$ and $\mathcal{P}_R(T)$ are stable.*

Proof. The results follow immediately from Proposition 5.1, upon setting $T_1 = T_2 = T$. □

Theorem 5.3. *Let* T *be a tree and* $Z(T) = \{c_L, c_R\}$ *(possibly* $c_L = c_R$ *) its center. Then* $Z(T)$ *is stable if at least one of the following holds*

(i) $|\mathrm{Tr}(T)| = 1$.

(ii) $\min\{\mathrm{ram}\, c_L, \mathrm{ram}\, c_R\} \geq \mathrm{ram}\, x$ *for all* $x \in \mathrm{Tr}(T)$.

(iii) $\mathrm{ram}\, c_L = \mathrm{rad}T - |Z(T)|$ *and* $\mathrm{ram}\, c_R = \mathrm{rad}T - |Z(T)|$

(iv) $\deg c_L = 2$ *and* $\deg c_R = 2$.

Proof. From *(i)* stability immediately follows, since $Z(T) \subseteq \mathrm{Tr}(T)$, and $\mathrm{Tr}(T)$ is stable. We can next assume $|\mathrm{Tr}(T)| > 1$. Let $\Phi : T \to T$ be a path-congruence, and suppose $\Phi(Z(T)) \neq Z(T)$. Without loss of generality we can assume $\Phi(c_L) \notin Z(T)$ (possibly $c_L = c_R$). Since $\Phi(c_L) \in \mathrm{Tr}(T)$, condition *(ii)* implies $\mathrm{ram}\, c_L \geq \mathrm{ram}\, \Phi(c_L)$. By Theorem 4.5 this cannot happen unless $\Phi(c_L) = c_R$; but this contradicts the assumption that $\Phi(c_L) \notin Z(T)$. Condition *(iii)* and Lemma 4.2 imply *(ii)*, since at least one vertex of $Z(T)$ is not a trunk end-vertex. Assume now *(iv)*. Since Φ preserves the degree, we have $2 = \deg c_L = \deg \Phi(c_L)$, hence $\mathrm{ram}\Phi(c_L) = 0$. Therefore $\mathrm{ram}\Phi(c_L) = \mathrm{ram}\, c_L$. By Theorem 4.5 this cannot happen unless $\Phi(c_L) = c_R$; but this contradicts the assumption that $\Phi(c_L) \notin Z(T)$. □

Theorem 5.4. *The center of any tree* T *with* $\mathrm{diam}T < 8$ *is stable.*

Proof. Let $Z(T) = \{c_L, c_R\}$ be the center of T, where, if $\mathrm{diam}T$ is even, we let $c_L = c_R = c$.
Assume, by contradiction, that $\Phi(Z(T)) \neq Z(T)$ for some path-congruence $\Phi : T \to T$, and, without loss of generality, that $\Phi(c_L) \notin Z(T)$, so that $c_L \neq \Phi(c_L) \neq c_R$. By Theorem 4.5, $\mathrm{ram}\Phi(c_L) > \mathrm{ram} c_L$ (note that, if $c_R \in [\Phi(c_L), c_L]$, it must be $k \geq 2$).

If $\mathrm{ram}\, c_L = 0$, then $\deg c_L = 2$, and, since Φ preserves the degree, $\deg \Phi(c_L) = 2$, so that $\mathrm{ram}\, \Phi(c_L) = 0 = \mathrm{ram}\, c_L$, which is impossible. We shall next assume $\mathrm{ram}\, c_L \geq 1$, and consequently $\mathrm{ram}\, \Phi(c_L) \geq 2$. It is easy to see that the statement is true if $\mathrm{diam} T \leq 5$, so we discuss only the cases $\mathrm{diam} T = 6$ and $\mathrm{diam} T = 7$.

If $\mathrm{diam} T = 6$ and $\mathrm{ram}\, c = 2$, one would have $\mathrm{ram}\, \Phi(c) \geq 3$, which is impossible. Hence $\mathrm{ram}\, c = 1$, and T must contain, as a subtree, the graph T_1 in Fig. 9

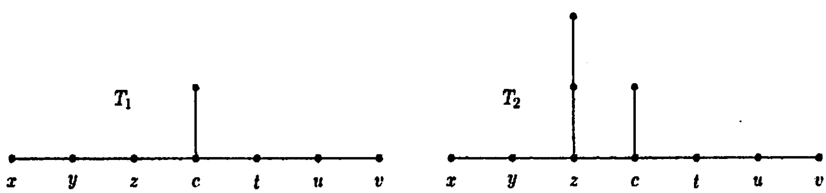


FIGURE 9

Note that $\mathrm{Tr}(T) \subseteq \{x, y, z, c, t, u, v\}$. Let $\Phi : T \to T$ be a path-congruence such that $\Phi(c) \neq c$. Since $\mathrm{diam} T = 6$, the only possibility is that $\mathrm{ram}\, \Phi(c) = 2$, and consequently $\Phi(c)$ is adjacent to c. Consequently, $\Phi(c)$ is an end-vertex of $\mathrm{Tr}(T)$, and, w.l.o.g. we can suppose that $\Phi(c) = z$, so that T must contain the tree T_2 in Fig. 9 Let D be the set of vertices of T which are at distance 2 from z and are not adjacent to c. Since Φ is a path-congruence, we have $p_2(z) = p_2(c)$, that is $\binom{\deg z}{2} + |D| + \deg c - 1 = \binom{\deg c}{2} + \deg t - 1 + \deg z - 1$, which implies $\deg t = |D| + 1$.

Now, denote by $p_3(z, c)$ the number of paths in T, of length 3, passing through both z and c. We have $p_3(z) = p_3(z, c) + (\deg z - 2)|D|$ and $p_3(c) = p_3(z, c) + (\deg c - 2)(\deg t - 1) + r$, where r is the number of vertices of T at distance 2 from t and not adjacent to c. Since $\deg t - 1 = |D|$ and $r \geq 1$ (being $\mathrm{diam} T = 6$), we get $p_3(c) > p_3(z)$, contradicting the assumption that Φ is a path-congruence.

Let, now, $\mathrm{diam}T = 7$. Then we can find a path in T of length

$$\begin{aligned} l &= \mathrm{rad}T + d(\Phi(c_L), Z(T)) + \mathrm{ram}\,\Phi(c_L) \\ &\geq 4 + d(\Phi(c_L), Z(T)) + 2 = 6 + d(\Phi(c_L), Z(T)). \end{aligned}$$

Since $\mathrm{diam}T = 7$, then $d(\Phi(c_L), Z(T)) = 1$ and $\mathrm{ram}\,\Phi(c_L) = 2$. This implies that $\Phi(c_L)$ is adjacent to c_R, since otherwise c_L would be between $\Phi(c_L)$ and c_R, and we would have $\mathrm{ram}\,\Phi(c_L) > \mathrm{ram}\,c_L + 1 \geq 2$. Consequently, T must contain, as a subtree, the tree T_3 in Fig. 10, where $z = \Phi(c_L)$.

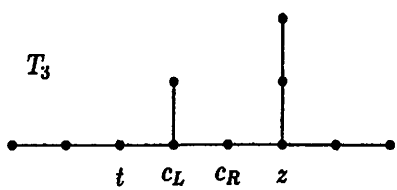


FIGURE 10

Let D be the set of vertices of T which have distance 2 from z and are not adjacent to c_R. From now on, we can argue in analogy with the case $\mathrm{diam}T = 6$, ending again with a contradiction to the assumption that Φ is a path-congruence. □

Remark 5.5. The assumption $\mathrm{diam}T < 8$ in Theorem 5.4 is essential. In fact it is possible to prove that for any pair (D, m), $D, m \in \mathbb{N}$, $D \geq 8$, $1 \leq m \leq \lfloor \frac{D+1}{2} \rfloor - |Z(T)| - 1$, there exists at least a tree T such that

(*i*) $\mathrm{diam}T = D$;

(*ii*) there exists $c \in Z(T)$ with $\mathrm{ram}\,c = m$;

(*iii*) there exists a path-congruence $\Phi : T \to T$ such that $\Phi(c) \notin Z(T)$.

A detailed proof of this fact is given in [6]. In Figure 11 we only show two examples with diam$T = 8$, where $p_l(v) = p_l(c)$ for all $l \in \mathbb{N}$ and center ramification 1 and 2 (that, by Theorem 5.3 case (*iii*), are the only ramification values for which we can have unstable center). In both cases, a path-congruence Φ such that $\Phi(c) = v$ is defined as in Remark 4.6.

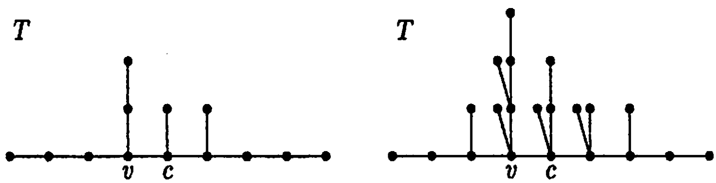


FIGURE 11. diam$T = 8$ and $p_l(v) = p_l(c)$ for all $l \in \mathbb{N}$.

A concept related to Ω-stability is that of Ω-closure. Let S be a subset of $V(T)$. Let Ω be a group of permutations of $V(T)$. The *Ω-closure* of S, denoted $\overline{S}$ (Ω being understood), is the set

$$\overline{S} := \bigcup_{\Phi \in \Omega} \Phi(S).$$

Remark 5.6. Since Ω is a group, $\overline{S}$ is an Ω-stable set, in fact the smallest Ω-stable set containing S.

congruences $T \to T$) shows that, although $Z(T)$ may not be stable, the union $\overline{Z(T)}$ of all images of $Z(T)$ under all path-congruences is somewhat confined.

Proposition 5.7. *$\overline{Z(T)}$ is contained in the smallest* $\mathrm{Tr}_p(T)$ *with* $|\mathrm{Tr}_p(T)| > 1$.

Proof. By Proposition 3.3, $Z(T)$ is contained in the smallest $\mathrm{Tr}_p(T)$ of order greater than 1, but since $\mathrm{Tr}_p(T)$ is stable by Proposition 5.2, then also $\overline{Z(T)}$ is contained in $\mathrm{Tr}_p(T)$. □

6. Algorithms on $\mathbb{P}(T)$ and final remarks

The question is how to locate in $\mathbb{P}(T)$ the rows corresponding to the vertices of previously defined subsets of $V(T)$. For example, the "largest" crown $\mathcal{P}(T)$ is represented by those rows which have first entry (degree-column) equal to 1, and last entry (diameter-column) different from zero. As another example, to locate the rows of $\mathrm{Tr}_p(T)$, one can first calculate the total number N_p of the paths of length p by summing up the entries in column p and dividing by $p+1$. Then, the rows sought are those whose p-th entry is equal to N_p. Note that whenever two rows of $\mathbb{P}(T)$ are equal, then the function $V(T) \to V(T)$ which exchanges the two vertices corresponding to these equal rows and fixes all other vertices is a path-congruence (see also the end of Section 4). Thus a minimal (w.r.t. inclusion) stable subset of $V(T)$ corresponds to a maximal set of identical rows. For example, if $\mathrm{diam}T$ is even, then the center c is stable if and only if its row is different from all the other rows of $\mathbb{P}(T)$. It would be interesting to develop an algorithm that selects the center row (under uniqueness assumption). Although the algorithm of iterated pruning can easily be applied to the adjacency matrix, it is not clear how the pruning operation affects a path-table.

We have seen that the "largest" crown $\mathcal{P}(T)$ is stable, and the "smallest" crown $Z(T)$ is not (in general). It would then be interesting to find out which of the other sets $\mathrm{Cr}_p(T)$ are stable.

Another problem would be the search in T for pairs of path-congruent non-similar vertices. For example, in Figure 8 are depicted trees in which the two vertices c, v are path-congruent (take for example the path-congruence Φ of $V(T)$ which exchanges c and v, and fixes all other vertices), but they are not similar since any automorphism of T fixes c.

The problem of searching "large" subsets of mutually $\mathcal{K}$-congruent (not just path-congruent) non-similar vertices of a graph G (not just a tree) seems also interesting.

As a further remark we wish to point out that the notion of path-congruence we have discussed in this paper, is similar to a notion introduced by Randič in [7]. A *Randič-relation* between two trees T_1, T_2 is a bijection $\sigma : V(T_1) \to V(T_2)$ such that for every vertex v of T_1 and any integer $l \geq 1$, the number of paths contained in T_1 of length l and *starting* at v, equals the number of paths contained in T_2 of length l and *starting* at $\sigma(v)$. T_1, T_2 will then be said *Randič-related*. The *Randič-table* $\mathbb{S}(T)$ of a tree T (called *path layer matrix* in [4]) is the rectangular array having n rows and $\mathrm{diam}(T)$ columns such that the (i,j)-entry is the number of paths in T of length j starting at the vertex v_i (see [5]). It is clear that two trees T_1, T_2

one can renumber the vertices of T_2 such that $\mathbb{P}(T_1) = \mathbb{P}(T_2)$ (resp. $\mathbb{S}(T_1) = \mathbb{S}(T_2)$). Randič conjectured that Randič-related trees are isomorphic ([7]). Slater has shown that it is not so. In ([8]) he has described an infinite set of example-pairs. In Fig. 12 it is depicted one of these examples.

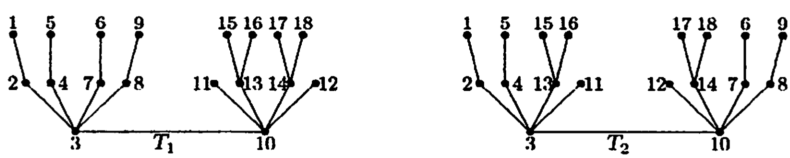


FIGURE 12. A pair of non-isomorphic trees with the same Randič-table

In addition to the path-table and the Randič-table, of interest in the present context could be a table in which the row corresponding to a given vertex v contains, for each $l \in \mathbb{N}$, the number of paths of length l starting at v and ending at a leaf.

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