

Semisymmetric Cubic Graphs Constructed from Bi-Cayley Graphs of A_n *

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Abstract

For a group T and a subset S of T , the bi-Cayley graph $\text{BCay}(T, S)$ of T with respect to S is the bipartite graph with vertex set $T \times \{0, 1\}$ and edge set $\{(g, 0), (sg, 1) \mid g \in T, s \in S\}$. In this paper, we investigate cubic bi-Cayley graphs of finite nonabelian simple groups. We give several sufficient or necessary conditions for a bi-Cayley graph to be semisymmetric, and construct several infinite families of cubic semisymmetric graphs.

Key Words: (T -)semisymmetric graph, symmetric graph, Cayley (di)graph, bi-Cayley graph.

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1 Introduction

All graphs considered here are assumed to be connected, undirected, finite and simple unless stated otherwise. For a graph Γ , we use $V(\Gamma)$, $E(\Gamma)$, and $A := \text{Aut}(\Gamma)$ to denote its vertex set, edge set and full automorphism group respectively. A graph Γ is said to be T -semisymmetric if it is regular and T acts transitively on its edge set but not on its vertex set, where T is a subgroup of A . In particular, an A -semisymmetric graph is called a *semisymmetric* graph. The class of semisymmetric graphs was introduced by Folkman in [5]. He constructed several infinite families of such graphs and posed eight open problems. Afterwards, Bouwer, Titov, Klin, Iofinova, A. A. Ivanov, A. V. Ivanov, Du, Xu and others did much work on semisymmetric graphs (see [1, 2, 12, 11, 8, 9, 4] etc.). They gave new constructions of such graphs by combinatorial or group-theoretical methods. By now, the answers to most of Folkman's problems are known. As we can see, in recent papers on semisymmetric graphs, group-theoretical constructions played a significant role. In 1985, Iofinova and Ivanov classified biprimitive semisymmetric cubic graphs (see [8]) using group-theoretical methods. In 2000, Du and Xu classified semisymmetric graphs of order $2pq$ by using some deep results on finite simple groups (see [4]). Let Γ be a T -semisymmetric graph. It is well-known that Γ is a bipartite graph with two bipartition sets, say U and W , of equal size and T acts transitively on both bipartition sets. Clearly, for every $v \in V(\Gamma)$, the stabilizer T_v acts transitively on the neighborhood $\Gamma_1(v)$ of v in Γ . Let

$$A^+ = \{a \in A \mid U^a = U, W^a = W\}.$$

Then $A = A^+$ or $|A : A^+| = 2$ depending on whether or not there exists an automorphism of Γ which interchanges the two bipartition sets. These two cases imply that Γ is a symmetric graph and a semisymmetric graph, respectively.

When we studied T -semisymmetric graphs of prime degree, we found a class of graphs whose full automorphism groups have a subgroup acting regularly on both bipartition sets. In this paper, we shall consider semisymmetric cubic graphs with this property.

Now let us mention several definitions which will be used in the following sections. For a group T , and a subset S of T such that $1_T \notin S$, the *Cayley digraph* $\text{Cay}(T, S)$ of T with respect to S is the digraph with vertex set T and arc set $\{(x, sx) \mid x \in T, s \in S\}$. If S is inverse-closed, that is, if $S = S^{-1} := \{s^{-1} \mid s \in S\}$, then (x, y) is an arc if and only if (y, x) is an arc. In this case, we identify two arcs (x, y) and (y, x) with an undirected edge $\{x, y\}$, and get an undirected graph. We call this graph a *Cayley graph* of T . For a group T , and a subset S (possibly, containing the identity element 1_T) of T , the *bi-Cayley graph* $\text{BCay}(T, S)$ of T with respect to S is the bipartite

graph with vertex set $T \times \{0, 1\}$ and edge set $\{(g, 0), (sg, 1) \mid g \in T, s \in S\}$. Let $\Gamma = \text{BCay}(T, S)$. Each $g \in T$ induces an automorphism of Γ as follows:

$$R(g) : (x, 0) \mapsto (xg, 0), (x, 1) \mapsto (xg, 1), \forall x \in T.$$

Set $R(T) = \{R(g) \mid g \in T\}$. Finally, a bi-Cayley graph $\Gamma = \text{BCay}(T, S)$ is said to be *normal* if $R(T)$ is normal in $\text{Aut}(\Gamma)$.

2 Preliminaries

Let T be a finite group and S be a subset of T . Set $\Gamma = \text{BCay}(T, S)$. By [4], we know that $\text{BCay}(T, S)$ is connected if and only if $\langle SS^{-1} \rangle = T$ if and only if $\langle S^{-1}S \rangle = T$, and that a bipartite graph X is a bi-Cayley graph if and only if there exists a subgroup of $\text{Aut}(X)$ which acts regularly on both bipartition sets. Let $\alpha \in \text{Aut}(\text{Cay}(T, S \setminus \{1_T\}))$. Then α induces an automorphism

$$(x, 0) \mapsto (x^\alpha, 0), (x, 1) \mapsto (x^\alpha, 1), \forall x \in T$$

of $\text{BCay}(T, S)$. So we can identify $\text{Aut}(\text{Cay}(T, S \setminus \{1_T\}))$ with a subgroup of $A = \text{Aut}(\Gamma)$. Furthermore, we have the following lemma.

Lemma 2.1 *Let T be a finite group and S be a subset of T .*

- (1) *If $1_T \notin S$ and the Cayley digraph $\text{Cay}(T, S)$ is arc-transitive, then the bi-Cayley graph $\text{BCay}(T, S)$ is either semisymmetric or symmetric;*
- (2) *If S is an orbit of some subgroup of $\text{Aut}(T)$, then $\text{BCay}(T, S)$ is edge-transitive.*
- (3) *If $\beta \in \text{Aut}(T)$, then $\text{BCay}(T, S) \cong \text{BCay}(T, S^\beta)$;*
- (4) *If $\text{Cay}(T, S \setminus \{1_T\}) \cong \text{Cay}(T, S^{-1} \setminus \{1_T\})$, then $\text{BCay}(T, S)$ is vertex-transitive;*
- (5) *If $S^\sigma = S^{-1}g$ for some $\sigma \in \text{Aut}(T)$ and some $g \in T$, then $\text{BCay}(T, S)$ is vertex-transitive.*

Proof (1) and (2) are obvious.

(3) We define β^* as follows:

$$\beta^* : (x, 0) \mapsto (x^\beta, 0), (x, 1) \mapsto (x^\beta, 1), \forall x \in T.$$

It is easy to check that β^* is an isomorphism from $\text{BCay}(T, S)$ to $\text{BCay}(T, S^\beta)$.

(4) Let ρ be an isomorphism from $\text{Cay}(T, S \setminus \{1_T\})$ to $\text{Cay}(T, S^{-1} \setminus \{1_T\})$ such that $1_T^\rho = 1_T$. We define ρ^* as follows:

$$\rho^* : (x, 0) \mapsto (x^\rho, 1), \quad (x, 1) \mapsto (x^\rho, 0), \quad \forall x \in T.$$

It is easy to prove that ρ^* is an automorphism of $\text{BCay}(T, S)$. It follows that (4) is true.

(5) Let us define σ^* as follows:

$$\sigma^* : (x, 0) \mapsto (gx^\sigma, 1), \quad (x, 1) \mapsto (x^\sigma, 0), \quad \forall x \in T.$$

Then σ^* is an automorphism of $\text{BCay}(T, S)$. It follows that $\text{BCay}(T, S)$ is vertex-transitive. \square

By Lemma 2.1(4), we know that the bi-Cayley graphs of abelian groups are vertex-transitive.

Lemma 2.2 *Let T be a finite group and S be a subset of T . For each $g \in T$, we have*

$$\text{BCay}(T, S) \cong \text{BCay}(T, gS) \cong \text{BCay}(T, Sg).$$

Proof We define ϕ and ψ as follows:

$$\begin{aligned} \phi : (x, 0) &\mapsto (x, 0), & (x, 1) &\mapsto (gx, 1), \quad \forall x \in T; \\ \psi : (x, 0) &\mapsto (g^{-1}xg, 0), & (x, 1) &\mapsto (g^{-1}xg, 1), \quad \forall x \in T. \end{aligned}$$

It is easy to check that ϕ is an isomorphism from $\text{BCay}(T, S)$ to $\text{BCay}(T, gS)$, and that ψ is an isomorphism from $\text{BCay}(T, S)$ to $\text{BCay}(T, g^{-1}Sg)$. It follows that

$$\text{BCay}(T, S) \cong \text{BCay}(T, gS) \cong \text{BCay}(T, g^{-1}(gS)g) = \text{BCay}(T, Sg).$$

\square

3 Cubic Bi-Cayley Graphs

By Lemma 2.2, we may, sometime for convenience, assume that $S = \{1_T, a, b\}$ for $a, b \in T$ when we consider cubic bi-Cayley graphs.

Theorem 3.1 *Let T be a finite nonabelian group and $S = \{1_T, a, b\}$ be a subset of T such that $T = \langle a, b \rangle$, and let $\Gamma = \text{BCay}(T, S)$. Suppose that $R(T)$ is normal in $\text{Aut}(\Gamma)$. If Γ is edge-transitive, then Γ is symmetric if and only if there exists $\alpha \in \text{Aut}(T)$ such that $S^\alpha = S^{-1}$.*

Proof First, we assume that Γ is symmetric. Then there exists $\rho \in \text{Aut}(\Gamma)$ such that $(1_T, 0)^\rho = (1_T, 1)$, and $(1_T, 1)^\rho = (1_T, 0)$. It follows that ρ interchanges two sets $\{(a, 1), (b, 1)\}$ and $\{(a^{-1}, 0), (b^{-1}, 0)\}$. We consider the action of ρ on the set $\{(1_T, 0), (1_T, 1), (a^{-1}, 0), (b^{-1}, 0), (a, 1), (b, 1)\}$. Then one of the following statements holds:

- (1) $\rho : (1_T, 0) \longleftrightarrow (1_T, 1), (a^{-1}, 0) \longleftrightarrow (a, 1), (b^{-1}, 0) \longleftrightarrow (b, 1)$;
- (2) $\rho : (1_T, 0) \longleftrightarrow (1_T, 1), (a^{-1}, 0) \longleftrightarrow (b, 1), (b^{-1}, 0) \longleftrightarrow (a, 1)$;
- (3) $\rho : (1_T, 0) \longleftrightarrow (1_T, 1), (a^{-1}, 0) \mapsto (a, 1) \mapsto (b^{-1}, 0) \mapsto (b, 1) \mapsto (a^{-1}, 0)$;
- (4) $\rho : (1_T, 0) \longleftrightarrow (1_T, 1), (a^{-1}, 0) \mapsto (b, 1) \mapsto (b^{-1}, 0) \mapsto (a, 1) \mapsto (a^{-1}, 0)$.

Since $R(T)$ is normal in $\text{Aut}(\Gamma)$, for any given $x \in T$, there exists unique $x^* \in T$ such that $R(x^*) = \rho^{-1}R(x)\rho$. Let $\alpha : x \mapsto x^*, \forall x \in T$. It is easy to check that α is an automorphism of T .

If (1) holds, then

$$\begin{aligned} (a^{-1}(a^{-1})^\alpha, 0) &= (a^{-1}, 0)^{R((a^{-1})^*)} = (a^{-1}, 0)^{\rho^{-1}R(a^{-1})\rho} \\ &= (a, 1)^{R(a^{-1})\rho} = (1_T, 1)^\rho = (1_T, 0) \implies (a^{-1})^\alpha = a. \end{aligned}$$

Similarly, we have $a^\alpha = a^{-1}$, $(b^{-1})^\alpha = b$, and $b^\alpha = b^{-1}$. If (2) holds, we have

$$\alpha : a^{-1} \longleftrightarrow b, b^{-1} \longleftrightarrow a.$$

If (3) holds, we have $a^{-1} = a^\alpha$, and $b^{-1} = b^\alpha$. But

$$(b^{-1}(a^{-1})^\alpha, 0) = (b^{-1}, 0)^{\rho^{-1}R(a^{-1})\rho} = (a, 1)^{R(a^{-1})\rho} = (1_T, 0) \implies (a^{-1})^\alpha = b.$$

It follows that $a = b$, and that T is a cyclic group, a contradiction. (4) also leads to a contradiction as above. So we have $S^\alpha = S^{-1}$.

Conversely, by Lemma 2.1(5), Γ is vertex-transitive, and hence symmetric. □

Let Γ be a connected cubic G -semisymmetric graph. By [6], for any $v \in V(\Gamma)$, the order of vertex-stabilizer G_v is $3 \cdot 2^s$ for some $s \leq 7$. In particular, $A_v^+ \leq 3 \cdot 2^s$. Suppose that $A = \text{Aut}(\Gamma)$ has a subgroup T acting transitively on both bipartition sets. Then $|A^+ : T| = 3 \cdot 2^t$ for some $t \leq 7$. We consider the permutation representation of A^+ acting on the right cosets of T by right multiplication. Then A^+/T_{A^+} is isomorphic to a subgroup of the symmetric group $S_{3 \cdot 2^t}$, where T_{A^+} is the core of T in

A^+ (i.e. $T_{A^+} = \bigcap_{x \in A^+} T^x$). If T is a finite nonabelian simple group, then either T is normal in A^+ and hence in A , or $T \leq A_{3 \cdot 2^t}$. It follows that the connected edge-transitive cubic bi-Cayley graphs for most of nonabelian simple finite groups are normal. More precisely, we have the following theorem.

Theorem 3.2 *Let T be a nonabelian simple finite group, and let α be an automorphism of T with $|\alpha| = 3$. For $a \in T$, set $S = \{a, a^\alpha, a^{\alpha^2}\}$. Suppose that $T = \langle S^{-1}S \rangle$. If $\Gamma = \text{BCay}(T, S)$ is not normal then $|T| \leq (3 \cdot 2^7)!$, and T is one of the following groups:*

$$A_n, \quad n = 5, 6, 7, 8, 9, 15, 31, 63, 127;$$

$$M_{11}, M_{12}, M_{22}, J_1, J_2;$$

$$\text{PSL}_2(q), \quad q = 7, 11, 13, \text{PSp}_4(3), \text{PSU}_4(3), \text{PSU}_3(3);$$

$$\text{PSL}_n(q) \quad (n \leq 7), \text{PSU}_n(q) \quad (n \leq 15), \text{PSp}_{2m}(q) \quad (m = 2, 3), \\ \text{P}\Omega_6^+(q), \text{P}\Omega_{2m}^-(q) \quad (m = 2, 3, 4), G_2(q), \text{and } {}^2B_2(q), \text{ where } q \text{ is a} \\ \text{power of } 2.$$

Proof First, we prove the following statement by induction on the order of the group.

Suppose that a finite group G has a nonabelian simple subgroup T , and that $|G : T| = 3^l 2^s$ for $l = 0$ or 1 , and $|G : N_G(T)| = 2^t$. If T is not normal in G , then either $T \cong A_{2^r-1}$ for some $r \leq s$, or T has a non-trivial representation of degree at most s over the field $GF(2)$.

Let N be a minimal normal subgroup of G . Then $T \leq N$ or $T \cap N = 1$. By checking the order of G , we know that N is either a nonabelian simple group or an elementary abelian group. First, we assume that $T \leq N$. Then $|N : N_N(T)|$ is a power of 2. If $|N_N(T) : T|$ is a power of 2, then $|N : T| = 2^r$ for some $r \leq s$. It follows from [7] that $T \cong A_{2^r-1}$. Assume that $|N_N(T) : T|$ is not a power of 2. Then there exists an element $x \in N_N(T) \setminus T$ with $x^3 \in T$. Then $H = T \langle x \rangle$ is a subgroup of N and the index of H in N is a power of 2. It follows that $H \cong A_{2^{r_1}-1}$ for some $r_1 \leq s$. On the other hand, H has a subgroup T of index 3, which is impossible. Now we assume that $T \cap N = 1$. If $N \leq C_G(T)$, we consider the quotient group G/N . Note T is a characteristic subgroup of TN . It follows that TN is not normal in G , and hence TN/N is not normal in G/N . By induction, either $T \cong TN/N \cong A_{2^{r_2}-1}$, or T has a non-trivial representation of degree at most s over the field $GF(2)$. If $N \not\leq C_G(T)$, then $|N|$ must be a power of 2. Consider the conjugate action of T on N , we can get a non-trivial representation of degree at most s over the field $GF(2)$.

Let T and S be as the hypotheses in our theorem. Then $|A^+| = |T| \cdot 3 \cdot 2^r$ for some $r \leq 7$. Clearly, $\langle \alpha \rangle \leq N_{A^+}(R(T))$ and hence $|A^+ : N_{A^+}(R(T))|$ is

a power of 2. It follows, from checking the degrees of nontrivial irreducible representations of finite simple groups over fields of characteristic 2 (see, for example, [10]), that our theorem is true. \square

By Lemma 2.2 and Theorem 3.1, 3.2, we have the following corollary.

Corollary 3.3 *Let T be a nonabelian simple finite group, and let α be an automorphism of T such that $|\alpha| = 3$. For $a \in T$, set $S = \{a, a^\alpha, a^{\alpha^2}\}$. Suppose that $T = \langle S^{-1}S \rangle$. If T is not one of simple groups listed in Theorem 3.2, then $\text{BCay}(T, S)$ is semisymmetric if and only if there are no automorphisms of T which map $\{a^{-1}a^\alpha, a^{-1}a^{\alpha^2}\}$ to $\{(a^\alpha)^{-1}a, (a^{\alpha^2})^{-1}a\}$.*

4 Examples

In this section, we shall construct several infinite families of semisymmetric graphs from the bi-Cayley graphs of the alternating group A_n for some special n . Let $S = \{a, a^b, a^{b^2}\}$ such that $\langle a^{-1}a^b, a^{-1}a^{b^2} \rangle = A_n$, where b is an element in A_n of order 3. We set $x = a^{-1}a^b$ and $y = a^{-1}a^{b^2}$. It is well-known that $\text{Aut}(A_n) = S_n$ except for $n = 6$. By the results of last section $\text{BCay}(A_n, S)$ ($n > 9, n \neq 15, 31, 63, 127$) is semisymmetric if and only if $\{x^\sigma, y^\sigma\} \neq \{x^{-1}, y^{-1}\}$ for every $\sigma \in S_n$. In this section we always write a permutation as a product of disjoint cycles.

Example 4.1 $n = 3k+2$, where k is even, and $k > 5$. Let $S = \{a, a^b, a^{b^2}\}$, where

$$\begin{aligned} a &= (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9 \dots 3k\ 3k+1\ 3k+2), \\ b &= (2\ k+3\ k+4)(4\ k+5\ k+6)(5\ k+7\ k+8) \dots \\ &\quad (i\ k+2i-3\ k+2i-2) \dots (k+2\ 3k+1\ 3k+2). \end{aligned}$$

Then $\Gamma = \text{BCay}(A_n, S)$ is a semisymmetric cubic graph.

Proof By calculation, we have

$$\begin{aligned} a^b &= (1\ k+3\ 3\ k+5\ k+7)(k+9\ k+11 \dots k+2i-3 \dots 3k+1\ k+4\ 2\ k+6 \\ &\quad 4\ k+8\ 5\ k+10 \dots i\ k+2i \dots k+1\ 3k+2\ k+2), \\ a^{b^2} &= (1\ k+4\ 3\ k+6\ k+8)(k+10\ k+12 \dots k+i \dots 3k+2\ 2\ k+3 \\ &\quad 4\ k+5 \dots i\ k+2i-3 \dots k+2\ 3k+1). \end{aligned}$$

Set $x = a^{-1}a^b$, and $y = a^{-1}a^{b^2}$. Then

$$\begin{aligned}
 x &= (1 \ k+10 \ k+11 \ 6 \ k+2 \ 3k+2 \ k+4 \ 3 \ k+6 \ k+7 \ 4 \ k+5 \ 2 \ k+3 \ k+9 \ 5 \ k+8) \\
 &\quad (7 \ k+12 \ k+13) \cdots (i \ k+2i-2 \ k+2i-1) \cdots (k+1 \ 3k \ 3k+1), \\
 y &= (1 \ k+7 \ k+8 \ 6 \ 2 \ k+4 \ 4 \ k+6 \ 5 \ k+5 \ 3 \ k+3 \ 3k+1 \ 3k+2 \ k+10 \ 7 \ k+9) \\
 &\quad (8 \ k+11 \ k+12) \cdots (i \ k+2i-5 \ k+2i-4) \cdots (k+2 \ 3k-1 \ 3k), \\
 (x^9)^{-1}y &= (2)(1 \ 5 \ 4 \ 7 \ k+9 \ k+10 \ k+8 \ k+3 \ 3k-1 \ 3k \ k+2 \ k+4)(k+7) \\
 &\quad (3 \ k+5 \ k+12 \ 8 \ k+11 \ k+6 \ 6) \\
 &\quad (3k+1 \ 3k+2)(9 \ k+13 \ k+14)(10 \ k+15 \ k+16) \cdots (k+2 \ 3k-1 \ 3k), \\
 y^{10} &= (1 \ 3 \ 6 \ 3k+2 \ 4 \ k+9 \ k+5 \ k+8 \ 3k+1 \ k+4 \ 7 \ 5 \ k+7 \ k+3 \ 2 \ k+10 \ k+6) \\
 &\quad (8 \ k+11 \ k+12) \cdots (i \ k+2i-5 \ k+2i-4) \cdots (k+2 \ 3k-1 \ 3k), \\
 xy^{10} &= (2)(3k+2 \ 7 \ 8 \ 9 \ \cdots \ k \ k+1 \ k+2 \ 4 \ k+8 \ 3 \ 1 \ k+6 \ k+3 \ k+5 \\
 &\quad k+10 \ k+12 \ \cdots \ k+2i-4 \ k+2i-2 \ k+2i \ \cdots \ 3k-2 \ 3k \\
 &\quad k+4 \ 6 \ 3k-1 \ 3k-5 \ 3k-9 \ \cdots \ 3k-4i-1 \ 3k-4i-5 \ \cdots).
 \end{aligned}$$

It is easy to see that $\langle x, y \rangle$ is 2-transitive on $\{1, 2, \dots, 3k+2\}$. As x^3 is a 17-cycle, $\langle x, y \rangle = A_{3k+2}$ (see, for example, Theorem 3.3E of [3]). So Γ is connected.

Obviously S is an orbit of the inner automorphism induced by b , and so Γ is edge-transitive.

Assume that Γ is not semisymmetric. Then there is some $\sigma \in S_{3k+2}$ such that $\{x^\sigma, y^\sigma\} = \{x^{-1}, y^{-1}\}$. Note that σ also maps 17-cycles to 17-cycles (under the conjugate action). Consider the symbols appeared in only one 17-cycle. They are $k+11, k+2, 3k+1$, and 7. Then σ either fixes $\{k+11, k+2\}$ and $\{3k+1, 7\}$ setwise or interchanges these two sets. As $k+11$ and $k+2$ has distance 2 in the first 17-cycle, and $3k+1$ and 7 has distance 3 in the second 17-cycle, σ fixes $\{k+11, k+2\}$ and $\{3k+1, 7\}$ setwise. It follows that $x^\sigma = x^{-1}$ and $y^\sigma = y^{-1}$. By checking these two 17-cycles we can easily obtain a contradiction. \square

Example 4.2 $n = 3k+2$, where k is odd, and $k > 3$. Let $S = \{a, a^b, a^{b^2}\}$, where

$$\begin{aligned}
 a &= (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ \cdots \ 3k \ 3k+1 \ 3k+2), \\
 b &= (1 \ k+2 \ k+3)(3 \ k+4 \ k+5) \cdots (i \ k+2i+2 \ k+2i-1) \cdots (k+1 \ 3k \ 3k+1).
 \end{aligned}$$

Then $\Gamma = \text{BCay}(A_n, S)$ is a semisymmetric cubic graph.

Proof First, we have

$$\begin{aligned}
 a^b &= (1 \ k+5 \ 3 \ k+7 \ 4 \ k+9 \ \cdots \ i \ k+2i+1 \ \cdots \ k \ 3k+1 \\
 &\quad k+1 \ 3k+2 \ k+2 \ 2 \ k+4 \ k+6 \ \cdots \ k+2i-2 \ \cdots \ 3k-2 \ 3k \ k+3), \\
 a^{b^2} &= (1 \ k+2 \ 3 \ k+4 \ \cdots \ i \ k+2i-2 \ \cdots \ k \ 3k-2 \ k+1 \ 3k \ 3k+2 \\
 &\quad k+3 \ 2 \ k+5 \ \cdots \ k+2i-1 \ \cdots \ 3k+1).
 \end{aligned}$$

Set $x = a^{-1}a^b$, $y = a^{-1}a^{b^2}$, and $z = x^{-1}y = (a^b)^{-1}a^{b^2}$. Then

$$\begin{aligned}
 x &= (1 \ k+2 \ 3k+2 \ k+1 \ 3k+1 \ k+3 \ 2 \ k+5 \ k+6 \ 3 \ k+4) \\
 &\quad (4 \ k+7 \ k+8) \cdots (i \ k+2i-1 \ k+2i) \cdots (k \ 3k-1 \ 3k), \\
 y &= (1 \ k+3 \ 3 \ k+5 \ 4 \ k+4 \ 2 \ k+2 \ 3k \ 3k+1 \ 3k+2) \\
 &\quad (5 \ k+6 \ k+7) \cdots (i \ k+2i-4 \ k+2i-3) \cdots (k+1 \ 3k-2 \ 3k-1), \\
 z &= (1 \ 2 \ 3 \ k+7 \ k+4 \ k+5 \ k+2 \ k+3 \ 3k+2 \ 3k \ k+1) \\
 &\quad (4 \ k+9 \ k+6) \cdots (i \ k+2i+1 \ k+2i-2) \cdots (k \ 3k+1 \ 3k-2), \\
 zx^5 &= (1)(2 \ k+1 \ k+3 \ k+5)(3 \ 4 \ \cdots \ k)(k+2 \ k+4)(k+6 \ k+8 \ \cdots \\
 &\quad k+2i \ k+2i+2 \ \cdots \ 3k+2 \ 3k-1 \ 3k-5 \ 3k-9 \ \cdots) \cdots.
 \end{aligned}$$

Together with the formula of x^{11} we know that (x, y) is 2-transitive on the set $\{1, 2, \dots, 3k+2\}$. As x^3 is a 11-cycle, $(x, y) = A_{3k+1}$. So Γ is connected. Obviously S is an orbit of the inner automorphism induced by b , and so Γ is edge-transitive.

Assume that Γ is not semisymmetric. Then there is some $\sigma \in S_{3k+2}$ such that $\{x^\sigma, y^\sigma\} = \{x^{-1}, y^{-1}\}$. Note that σ also maps 11-cycles to 11-cycles. Since the symbols appeared in only one 11-cycle are $k+1, k+6, 4$, and $3k$, the permutation σ either fixes $\{k+1, k+6\}$ and $\{4, 3k\}$ setwise or interchanges these two sets. As $k+1$ and $k+6$ has distance 5 in one 11-cycle, and 4 and $3k$ has distance 4 in the other 11-cycle, σ fixes $\{k+1, k+6\}$ and $\{4, 3k\}$ setwise. It follows that $x^\sigma = x^{-1}$ and $y^\sigma = y^{-1}$. Checking these two 11-cycles we can easily obtain a contradiction. \square

Example 4.3 $n = 3k + 1$, where k is odd, and $k \geq 7$, $k \neq 10, 42$. Let $S = \{a, a^b, a^{b^2}\}$, where

$$\begin{aligned}
 a &= (1 \ 2 \ 3)(4 \ 5 \ 6 \ 7 \ 8 \ \cdots \ 3k \ 3k+1), \\
 b &= (2 \ k+2 \ k+3) \cdots (i \ k+2i-2 \ k+2i-1) \cdots (k+1 \ 3k \ 3k+1).
 \end{aligned}$$

Then $\Gamma = \text{BCay}(A_n, S)$ is a semisymmetric cubic graph.

Proof By calculation, we have

$$\begin{aligned}
 a^b &= (1 \ k+2 \ k+4)(k+6 \ k+8 \ \cdots \ k+2i-2 \ k+2i \ \cdots \ 3k \\
 &\quad k+3 \ 2 \ \cdots \ k+2i-1 \ i \ \cdots \ 3k+1 \ k+1), \\
 a^{b^2} &= (1 \ k+3 \ k+5)(k+7 \ k+9 \ \cdots \ k+2i-1 \ \cdots \ 3k+1 \ 2 \ k+2 \\
 &\quad 3 \ k+4 \ \cdots \ i \ k+2i-2 \ \cdots \ k+1 \ 3k).
 \end{aligned}$$

Set $x = a^{-1}a^b$, and $y = a^{-1}a^{b^2}$. Then

$$\begin{aligned}
 x &= (1 \ k+7 \ k+8 \ 4 \ k+1 \ 3k+1 \ k+3 \ k+4 \ 2 \ k+2 \ k+6 \ 3 \ k+5)(5 \ k+9 \ k+10) \\
 &\quad \dots (i \ k+2i-1 \ k+2i) \dots (k \ 3k-1 \ 3k), \\
 y &= (1 \ k+4 \ k+5 \ 4 \ 2 \ k+3 \ 3 \ k+2 \ 3k \ 3k+1 \ k+7 \ 5 \ k+6)(6 \ k+8 \ k+9) \\
 &\quad \dots (i \ k+2i-4 \ k+2i-3) \dots (k+1 \ 3k-2 \ 3k-1), \\
 yx^6 &= (1)(3k+1 \ k+4)(2 \ k+5 \ k+2 \ 3k \ 3 \ k+8 \ k+9 \ 6) \\
 &\quad (k+1 \ 3k-2 \ 3k-1 \ k+6 \ k+3)(4 \ k+7 \ 5) \\
 &\quad (7 \ k+10 \ k+11) \dots (i \ k+2i-4 \ k+2i-3) \dots (k \ 3k-4 \ 3k-3).
 \end{aligned}$$

Together with the formula of x^3 we know that that $\langle x, y \rangle$ is 2-transitive on the set $\{1, 2, \dots, 3k+1\}$. As x^3 is a 13-cycle, $\langle x, y \rangle = A_{3k+1}$. So Γ is connected. Obviously S is an orbit of the inner automorphism induced by b , and hence Γ is edge-transitive.

Assume that Γ is not semisymmetric. Then there is some $\sigma \in S_{3k+1}$ such that $\{x^\sigma, y^\sigma\} = \{x^{-1}, y^{-1}\}$. Note that σ also maps 13-cycles to 13-cycles. Since the symbols appeared in only one 13-cycle are $k+8, k+1, 3k$, and 5, the permutation σ either fixes $\{k+8, k+1\}$ and $\{3k, 5\}$ setwise or interchanges these two sets. Since $k+8$ and $k+1$ has distance 2 in one 13-cycle, and $3k+1$ and 7 has distance 3 in the other 13-cycle, the permutation σ fixes $\{k+8, k+1\}$ and $\{3k, 5\}$ setwise. It follows that $x^\sigma = x^{-1}$ and $y^\sigma = y^{-1}$. By checking these two 13-cycles, we can easily obtain a contradiction. \square

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