A 4-isosceles 7-point Set with Both Circle and Linear Restrictions *

Xianglin Wei and Ren Ding
College of Mathematics, Hebei Normal University
Shijiazhuang 050016, People's Republic of China
wxlmath@126.com rending@hebtu.edu.cn

Abstract

A finite planar set is *k-isosceles* for $k \geq 3$, if every k-point subset of the set contains a point equidistant from the other two. This paper gives a 4-isosceles set consisting of 7 points with no three on a line and no four on a circle.

A finite planar set is said to be *k-isosceles* for $k \geq 3$, if every $k-$point subset of the set includes a 3-set which forms an isosceles triangle, i.e., contains a point equidistant from the other two. In [1] Fishburn discussed 3-isosceles planar sets and 4-isosceles planar sets. At the end of his paper, he put forward several open questions about 4-isosceles planar sets. Two of them are as follows: Let $\mathcal{F}$ denote a 4-isosceles planar set.

Problem 1: Is there a 6-point set $\mathcal{F}$ with no four points on a circle and no three points on a line ?

Problem 2: Is there a 7-point set $\mathcal{F}$ with no four points on a circle ?

[2] gave affirmative answers to the two questions. In this article we propose a new 7-point set $\mathcal{F}$ with no four points on a circle and meanwhile with no three points on a line. The conclusion is stronger than that in [2].

Theorem 1. *There exists a 7-point set $\mathcal{F}$ with no four points on a circle and no three points on a line.*

Proof. Let $T_0 = \triangle ABC$ be an equilateral triangle with edge length 1 and with center at O. See Figure 1. Construct an equilateral triangle $T_1 = \triangle DEF$ such that

$$|AD| = |BE| = |CF| = |AB| = 1,$$

*This research was supported by National Natural Science Foundation of China (10571042), NSF of Hebei (A2005000144) and the SF of Hebei Normal University (L2004202).

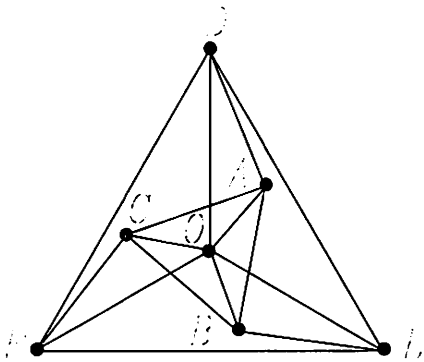

Figure 1: 7-point set $\mathcal{F}$ with no four points on a circle and no three points on a line.

$$\frac{\pi}{2} < \angle DAC = \angle FCB = \angle EBA < \frac{2\pi}{3}.$$

Thus triangles T_0 and T_1 have the same center O. See Figure 1. We prove that $F = \{A, B, C, D, E, F, O\}$ is the 7-point set as required.

By our construction it is easy to check that $\mathcal{F}$ is 4-isosceles with no three points on a line. It remains to prove that no four points of $\mathcal{F}$ are on a circle. $\mathcal{F}$ has thirty five 4-point subsets. It is obvious that the convex hull of each of the following twenty 4-point subsets is a triangle:

$$\{A,B,C,O\}, \{A,B,D,E\}, \{A,B,E,F\}, \{A,B,F,O\}, \{A,C,D,E\},$$

$$\{A,C,D,F\}, \{A,C,E,O\}, \{A,D,E,F\}, \{A,D,E,O\}, \{A,E,F,O\},$$

$$\{B,C,D,F\}, \{B,C,D,O\}, \{B,C,E,F\}, \{B,D,E,F\}, \{B,D,F,O\},$$

$$\{B,E,F,O\}, \{C,D,E,F\}, \{C,D,E,O\}, \{C,D,F,O\}, \{D,E,F,O\}.$$

So each of the twenty subsets is nonconcyclic. Now we prove that each of the remaining fifteen 4-point subsets is nonconcyclic (abbreviated as "nc"):

$$\angle ADC + \angle ABC < \frac{\pi}{3} + \frac{\pi}{3} < \pi \Rightarrow \{A,B,C,D\} \text{ is nc},$$

$$\angle ACB + \angle AEB < \frac{\pi}{3} + \frac{\pi}{3} < \pi \Rightarrow \{A,B,C,E\} \text{ is nc},$$

$$\angle BAC + \angle CFB < \frac{\pi}{3} + \frac{\pi}{3} < \pi \Rightarrow \{A,B,C,F\} \text{ is nc},$$

$$\left.\begin{array}{l} \angle CBF = \frac{1}{2}(\pi - \angle BCF) < \frac{1}{2}(\pi - \frac{\pi}{2}) = \frac{\pi}{4} \\ \angle ADF + \angle ABF < \frac{\pi}{3} + \frac{\pi}{3} + \angle CBF < \pi \end{array}\right\} \Rightarrow \{A,B,D,F\} \text{ is nc},$$

$$\angle ABO + \angle ADO < \frac{\pi}{3} + \frac{\pi}{3} < \pi \Rightarrow \{A,B,D,O\} \text{ is nc,}$$

$$\angle AOB + \angle AEB < \frac{2\pi}{3} + \frac{\pi}{3} = \pi \Rightarrow \{A,B,E,O\} \text{ is nc,}$$

$$\angle AOC + \angle ADC < \frac{2\pi}{3} + \frac{\pi}{3} = \pi \Rightarrow \{A,C,D,O\} \text{ is nc,}$$

$$\left.\begin{array}{l} \angle BAE = \frac{1}{2}(\pi - \angle ABE) < \frac{1}{2}(\pi - \frac{\pi}{2}) = \frac{\pi}{4} \\ \angle EFC + \angle CAE < \frac{\pi}{3} + \frac{\pi}{3} + \angle BAE < \pi \end{array}\right\} \Rightarrow \{A,C,E,F\} \text{ is nc,}$$

$$\angle OFC + \angle CAO < \frac{\pi}{3} + \frac{\pi}{3} < \pi \Rightarrow \{A,C,F,O\} \text{ is nc,}$$

$$\angle OFD + \angle OAC + \angle DAC < \frac{\pi}{6} + \frac{\pi}{6} + \frac{2\pi}{3} = \pi \Rightarrow \{A,D,F,O\} \text{ is nc,}$$

$$\left.\begin{array}{l} \angle DCA = \frac{1}{2}(\pi - \angle DAC) < \frac{1}{2}(\pi - \frac{\pi}{2}) = \frac{\pi}{4} \\ \angle DEB + \angle BCD < \frac{\pi}{3} + \frac{\pi}{3} + \angle DCA < \pi \end{array}\right\} \Rightarrow \{B,C,D,E\} \text{ is nc,}$$

$$\angle OCB + \angle BEO < \frac{\pi}{3} + \frac{\pi}{3} < \pi \Rightarrow \{B,C,E,O\} \text{ is nc,}$$

$$\angle BOC + \angle CFB < \frac{2\pi}{3} + \frac{\pi}{3} = \pi \Rightarrow \{B,C,F,O\} \text{ is nc,}$$

$$\angle ODE + \angle ABO + \angle ABE < \frac{\pi}{6} + \frac{\pi}{6} + \frac{2\pi}{3} = \pi \Rightarrow \{B,D,E,O\} \text{ is nc,}$$

$$\angle OEF + \angle OCB + \angle BCF < \frac{\pi}{6} + \frac{\pi}{6} + \frac{2\pi}{3} = \pi \Rightarrow \{C,E,F,O\} \text{ is nc.}$$

Therefore $\mathcal{F}$ is a 4-isosceles 7-set with no four points on a circle and no three points on a line. The proof is complete. □

References

[1] P.Fishburn, Isosceles planar subsets, *Discrete Comput. Geom.*, **19** (1998), 391-398.

[2] Changqing Xu, Ren Ding, About 4-isosceles planar sets, *Discrete Comput. Geom.*, **27** (2002), 287-290.