Private Domination Trees

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Abstract

For a subset of vertices S in a graph G, if $v \in S$ and $w \in V - S$, then the vertex w is an *external private neighbor of* v (with respect to S) if the only neighbor of w in S is v. A dominating set S is a *private dominating set* if each $v \in S$ has an external private neighbor. Bollóbas and Cockayne (Graph theoretic parameters concerning domination, independence and irredundance. *J. Graph Theory* 3 (1979) 241-250) showed that every graph without isolated vertices has a minimum dominating set which is also a private dominating set. We define a graph G to be a *private domination graph* if every minimum dominating set of G is a private dominating set. We give a constructive characterization of private domination trees.

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1 Introduction

For notation and graph theory terminology we in general follow [2, 4]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. The open neighborhood of vertex $v \in V$ is denoted by $N(v) = \{u \in V \mid uv \in E\}$, while its closed neighborhood is given by $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. A set is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G, and a dominating set of cardinality $\gamma(G)$ is called a *$\gamma(G)$-set*. If $v \in S$ and $w \in V - S$, then the vertex w is an *external private neighbor of* v (with respect to S) if $N(w) \cap S = \{v\}$. The *external private neighborhood* $\text{epn}(v, S)$ is the set of external private neighbors of v with respect to S. A dominating set S is a *private dominating set*, which we denote as PDS, if each $v \in S$ has an external private neighbor, that is, $\text{epn}(v, S) \neq \emptyset$ for all $v \in S$. The following classic result in domination is due to Bollóbas and Cockayne [1].

Theorem 1 (Bollóbas and Cockayne [1]) *Every graph without isolated vertices has a minimum dominating set which is also a PDS.*

For example, consider the path $P_4 = a, b, c, d$. The set $\{a, d\}$ is a private $\gamma(P_4)$-set, while $\{b, d\}$ is a $\gamma(P_4)$-set that is not private because the vertex d has no external private neighbor with respect to $\{b, d\}$. On the other hand, every $\gamma(P_5)$-set is a private dominating set.

If G is a graph for which every $\gamma(G)$-set is a PDS, then we say that G is a *private domination graph*, or just a *PD-graph*. In this paper, we characterize the PD-trees.

2 PD-Trees

First we give some more terminology. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. A *leaf* of G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. A support vertex that is adjacent to at least two leaves we call a *strong support vertex*.

We begin with three straightforward observations.

Observation 2 *A path P_n is an PD-tree if and only if* $n \equiv 0, 2 \pmod 3$.

Observation 3 *If T is a PD-tree with a $\gamma(T)$-set S, then for each vertex $v \in S$, v is an isolate in $T[S]$ or $|epn(v, S)| \geq 2$.*

Proof. Let T be a PD-tree with $\gamma(T)$-set S. Suppose that $v \in S$ has a neighbor in S and $\text{epn}(v, S) = \{u\}$. Then $(S - \{v\}) \cup \{u\}$ is a $\gamma(T)$-set where u has no external private neighbor, contradicting the fact that T is a PD-tree. □

Observation 4 *If T is a tree of order at least 3, then there is a $\gamma(T)$-set that contains all the support vertices of T, and every strong support vertex is in every $\gamma(T)$-set.*

We shall use the following proposition from [3].

Theorem 5 [3] *For any tree T with order $n \geq 3$, $\gamma(T - v) > \gamma(T)$ if and only if v is in every $\gamma(T)$-set.*

3 Characterization

Let $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, and $\mathcal{T}_4$ be the following four operations on a nontrivial tree T.

> **Operation $\mathcal{T}_1$.** Attach a vertex to a vertex of T that is in some $\gamma(T)$-set.
>
> **Operation $\mathcal{T}_2$.** Attach a leaf of a path P_2 to a vertex v of T, where v is not in any $\gamma(T)$-set and every $\gamma(T - v)$-set is a PDS of $T - v$.
>
> **Operation $\mathcal{T}_3$.** Attach a leaf of a path P_3 to a vertex of T.
>
> **Operation $\mathcal{T}_4$.** Attach the center of a path P_3 to a vertex of T that is in every $\gamma(T)$-set.

We note that Mynhardt [6] uses an innovative pruning technique to characterize the vertices that are contained in every, in some, or in no $\gamma(T)$-set of a tree T. Let $\mathcal{T}$ be the family defined by $\mathcal{T} = \{T \mid T$ is obtained from P_2 by a finite sequence of operations $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, and $\mathcal{T}_4\}$. We show first that every tree in the family $\mathcal{T}$ is a PD-tree.

Lemma 6 *If $T \in \mathcal{T}$, then T is a PD-tree.*

Proof. We proceed by induction on the number $s(T)$ of operations required to construct the tree $T \in \mathcal{T}$. If $s(T) = 0$, then $T = P_2$ and T is a PD-tree. This establishes the base case. Assume, then, that $k \geq 1$ is an integer and that each tree $T' \in \mathcal{T}$ with $s(T') < k$ is a PD-tree. Let $T \in \mathcal{T}$ be a tree with $s(T) = k$. Then, T can be obtained from a tree $T' \in \mathcal{T}$ with $s(T') < k$ by one of the operations $\mathcal{T}_1$, $\mathcal{T}_2$ $\mathcal{T}_3$, and $\mathcal{T}_4$. Applying the inductive hypothesis to the tree T', T' is a PD-tree. If T is a star, then T has order at least three, and so T is a PD-tree. Hence we may assume that $\text{diam}(T) \geq 3$. We now consider four possibilities depending on whether T is obtained from T' by operation $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, or $\mathcal{T}_4$.

Case 1. T is obtained from T′ by operation $\mathcal{T}_1$. Suppose T is obtained from T' by adding a vertex u and the edge uv where $v \in V(T')$ and v is in some $\gamma(T')$-set. Any $\gamma(T')$-set containing v dominates T, and so $\gamma(T) = \gamma(T')$. Let S be a $\gamma(T)$-set. Suppose $u \in S$. Since $\gamma(T') = \gamma(T) = |S|$, the set $S - \{u\}$ is not a dominating set of T', that is, $S \cap N[v] = \{u\}$. But then $(S - \{u\}) \cup \{v\}$ is a $\gamma(T')$-set that is not a PDS of T' (since v has no external private neighbor with respect to this set), contradicting the inductive hypothesis that T' is a PD-tree. Hence, $u \notin S$, and so S is a $\gamma(T')$-set. By the inductive hypothesis, S is a PDS of T', and therefore of T. Hence, T is a PD-tree.

Case 2. T is obtained from T′ by operation $\mathcal{T}_2$. Suppose T is obtained from T' by adding the path u, w and the edge uv, where $v \in V(T')$, v is not in any $\gamma(T')$-set, and every $\gamma(T' - v)$-set is a PDS of $T' - v$. If $\gamma(T' - v) < \gamma(T')$, then v is in some $\gamma(T')$-set, a contradiction. Hence, $\gamma(T' - v) = \gamma(T')$. It follows that $\gamma(T) = \gamma(T') + 1$. Let S be a $\gamma(T)$-set, and let $S' = S \cap V(T')$. Then, $|S'| = |S| - 1 = \gamma(T) - 1 = \gamma(T')$.

Suppose S' does not dominate $V(T')$. Then, S' is a $\gamma(T' - v)$-set (and $S \cap N[v] = \{u\}$), and so, by assumption, S' is a PDS of $T' - v$. Hence, S is a PDS of T. On the other hand, suppose S' dominates $V(T')$. Then, S' is a $\gamma(T')$-set and therefore $v \notin S'$. If $w \in S$, then, since S' is a PDS of T', the set S is a PDS of T. Hence we may assume that $u \in S$. If $\text{epn}(v', S') = \{v\}$ for some vertex $v' \in S'$, then $(S' - \{v'\}) \cup \{v\}$ is a $\gamma(T')$-set containing v, a contradiction. Hence, v is not the unique external private neighbor of any vertex in S'. Thus, S is a PDS of T.

Case 3. T is obtained from T′ by operation $\mathcal{T}_3$. Suppose T is obtained from T' by adding the path u, x, w and the edge uv where $v \in V(T')$.

We show first that $\gamma(T) = \gamma(T')+1$. Any $\gamma(T')$-set can be extended to a dominating set of T by adding to it the vertex x, and so $\gamma(T) \leq \gamma(T') + 1$. On the other hand, let S be a $\gamma(T)$-set. We may assume $S \cap \{u, x, w\} = \{x\}$ (if $w \in S$, then replace w in S by x, while if $u \in S$, then replace u in S by v). Then, $S - \{x\}$ is a dominating set of T', and so $\gamma(T') \leq \gamma(T) - 1$. Consequently, $\gamma(T) = \gamma(T') + 1$.

Let S be a $\gamma(T)$-set, and let $S' = S \cap V(T')$. Then, $|S'| \leq |S| - 1 = \gamma(T')$. Suppose S' is not a dominating set of T'. Then, S' is a dominating set of $T' - v$ and $S \cap N[v] = \{u\}$. Also, $|S \cap \{x, w\}| = 1$. Hence, $|S'| = |S| - 2 = \gamma(T') - 1$. Therefore, $S' \cup \{v\}$ is a $\gamma(T')$-set with $\text{epn}(v, S' \cup \{v\}) = \emptyset$, contradicting the fact that T' is a PD-tree. Hence, S' is a dominating set of T', and so $\gamma(T') \leq |S'|$. Consequently, $|S'| = |S| - 1 = \gamma(T')$. Thus, S' is a $\gamma(T')$-set and S contains either x or w (to dominate w). Since S' is a PDS of T', the set S is therefore a PDS of T.

Case 4. T is obtained from T′ by operation $\mathcal{T}_4$. Suppose T is obtained from T' by adding the path u, x, w and the edge xv, where $v \in V(T')$ is in every $\gamma(T')$-set. Note that since T' is a nontrivial tree and v is in every $\gamma(T')$-set, the order of T' is at least three.

We show first that $\gamma(T) = \gamma(T')+1$. Any $\gamma(T')$-set can be extended to a dominating set of T by adding to it the vertex x, and so $\gamma(T) \leq \gamma(T') + 1$. On the other hand, let S be a $\gamma(T)$-set, and let $S' = S \cap V(T')$. Then, $x \in S$ by Observation 4, and so $|S'| = |S| - 1 = \gamma(T) - 1$. If S' is a dominating set of T', then $\gamma(T') \leq |S'| = \gamma(T) - 1$. If S is not a dominating set of T', then S' is a dominating set of $T' - v$, and so, by Theorem 5, $\gamma(T') < \gamma(T' - v) \leq |S'| = \gamma(T) - 1$. In both cases, $\gamma(T') \leq \gamma(T) - 1$. Consequently, $\gamma(T) = \gamma(T') + 1$.

Next we show that T is a PD-tree. Again we let S be a $\gamma(T)$-set, and let $S' = S \cap V(T')$. By Observation 4, $x \in S$, and so $|S'| = |S| - 1 = \gamma(T')$. If $v \notin S$, then S' is a dominating set of $T' - v$, and so, by Theorem 5, $\gamma(T') < \gamma(T' - v) \leq |S'|$, a contradiction. Hence, $v \in S$ and S' is a dominating set of T', that is, S' is a $\gamma(T')$-set. Since S' is a PDS of T', the set S is therefore a PDS of T. □

We show next that every PD-tree is in the family $\mathcal{T}$.

Lemma 7 *If T is a PD-tree, then $T \in \mathcal{T}$.*

Proof. We proceed by induction on the order $n \geq 2$ of T. If $n = 2$, then $T = P_2$ and $T \in \mathcal{T}$. This establishes the base case. Assume $n \geq 3$ and that every PD-tree of order less than n is in the family $\mathcal{T}$. Let T be a PD-tree of order n. If T is a star, then it can be obtained from P_2 using repeated applications of operation $\mathcal{T}_1$, so we may assume that $\text{diam}(T) \geq 3$. Consider a longest path $v_0, v_1, v_2, \ldots, v_{\text{diam}(T)}$ in T, and root T at the vertex $v_{\text{diam}(T)}$. Then, v_0 is a leaf and v_1 is a support vertex. We consider three possibilities depending on the degree of v_1.

Case 1. $\deg(v_1) \geq 4$. By our choice of v_0, v_1 has three or more leaf children. Let $T' = T - v_0$. Then v_1 is a strong support in T' and hence is in every $\gamma(T')$-set. Thus every $\gamma(T')$-set is a $\gamma(T)$-set, and so T' is a PD-tree. Applying our inductive hypothesis, $T' \in \mathcal{T}$. Hence, T can be obtained from T' using operation $\mathcal{T}_1$ implying that $T \in \mathcal{T}$.

Case 2. $\deg(v_1) = 3$. By our choice of v_0, v_1 has exactly two leaf children, and every child of v_2 is either a leaf or a support vertex. Since v_1 is a strong support vertex, it is in every $\gamma(T)$-set.

First suppose that v_2 is in some $\gamma(T)$-set S. Since $v_1 \in S$, Observation 3 implies that $|\text{epn}(v_2, S)| \geq 2$ implying that v_2 has at least one leaf child. Suppose that v_2 has exactly one leaf neighbor u. Then, $|\text{epn}(v_2, S)| = 2$ and $v_3 \in \text{epn}(v_2, S)$. Since T is a PD-tree and v_1 is in every $\gamma(T)$-set, the leaf u is in no $\gamma(T)$-set (since otherwise u would have no external private neighbor in such a set). Hence, v_2 is in every $\gamma(T)$-set. Next suppose that v_2 has two or more leaf children. By Observation 4, v_2 is in every $\gamma(T)$-set. Hence if v_2 is in some $\gamma(T)$-set, then v_2 is in every $\gamma(T)$-set.

Let T' be the tree formed by removing v_1 and its children from T. Since every $\gamma(T')$-set can be extended to a $\gamma(T)$-set by adding to it the vertex v_1, it follows that v_2 is in every $\gamma(T')$-set (otherwise, there exists a $\gamma(T)$-set that does not contain v_2). Thus, T' is a PD-tree, and by our inductive hypothesis, $T' \in \mathcal{T}$. Since T can be obtained from T' using operation $\mathcal{T}_4$, $T \in \mathcal{T}$.

Next, assume that v_2 is not in any $\gamma(T)$-set. Observation 4 implies that v_2 is not a support vertex, and so every child of v_2 is a support vertex. Let $T' = T - v_0$. Since v_1 is a support vertex in T', there exists a $\gamma(T')$-set containing v_1, and so $\gamma(T) = \gamma(T')$. Every $\gamma(T')$-set containing v_1 is a $\gamma(T)$-set, and hence is a PDS of T'. Suppose S' is a $\gamma(T')$-set that does not contain v_1. Then $x \in S'$, where x is the leaf neighbor of v_1 in T'. If $v_1 \notin \text{epn}(x, S')$ in T', then $v_2 \in S'$. But then $(S' - \{x\}) \cup \{v_1\}$ is a $\gamma(T)$-set containing v_2, a contradiction. Therefore every $\gamma(T')$-set contains v_1 and is therefore a PDS in T', and so T' is a PD-tree. By the inductive hypothesis, $T' \in \mathcal{T}$. Thus, T is a PD-tree because it can be obtained from T' using operation $\mathcal{T}_1$.

Case 3. $\deg(v_1) = 2$. Then, v_0 is the only leaf neighbor of v_1. If v_2 is in some $\gamma(T)$-set S, then $(S - \{v_1\}) \cup \{v_0\}$ is a $\gamma(T)$-set that is not a PDS of T, a contradiction since T is a PD-tree. Hence, v_2 is not in any $\gamma(T)$-set.

First assume that $\deg(v_2) \geq 3$, and let $T' = T - \{v_0, v_1\}$. By our choice of v_0, every child of v_2 is a leaf or a support vertex. Observation 4 implies that v_2 is not a support vertex, and so it must be the case that every child of v_2 is a support vertex. Thus the restriction of any $\gamma(T)$-set that contains all the support vertices of T (such a set exists by Observation 4) to the tree T' is a dominating set of T', and so $\gamma(T') \leq \gamma(T) - 1$. On the other hand, any $\gamma(T')$-set can be extended to a dominating set of T by adding to it the vertex v_1, and so $\gamma(T) \leq \gamma(T') + 1$. Consequently, $\gamma(T) = \gamma(T') + 1$.

If v_2 is in some $\gamma(T')$-set, then adding v_1 to this set produces a $\gamma(T)$-set containing v_2, a contradiction. Thus, no $\gamma(T')$-set includes v_2. If S' is a $\gamma(T')$-set that is not a PDS of T', then $S' \cup \{v_1\}$ is a $\gamma(T)$-set that is not a PDS of T, a contradiction. Hence every $\gamma(T')$-set is a PDS of T', that is, T' is a PD-tree. Applying our inductive hypothesis, $T' \in \mathcal{T}$. Moreover, if there exists a $\gamma(T' - v_2)$-set S' that is not a PDS of $T' - v_2$, then $S' \cup \{v_1\}$ is a $\gamma(T)$-set that is not a PDS of T, a contradiction. Hence every $\gamma(T' - v_2)$-set is a PDS in $T' - v_2$. Therefore, T can be formed from T' using operation $\mathcal{T}_2$, and hence, T is in $\mathcal{T}$.

We may assume that $\deg(v_2) = 2$. Let $T' = T - \{v_0, v_1, v_2\}$. Let S' be a $\gamma(T')$-set. Now every $\gamma(T')$-set S' is a PDS of T', for otherwise $S' \cup \{v_1\}$ is a $\gamma(T)$-set that is not a PDS of T, a contradiction. It follows that T' is a PD-tree, and by our inductive hypothesis, $T' \in \mathcal{T}$. Now T can be obtained from T' using operation $\mathcal{T}_3$. □

As an immediate consequence of Lemmas 6 and 7, we have our main result.

Theorem 8 *A tree $T \in \mathcal{T}$ if and only if T is a PD-tree.*

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