GENERALIZED BOOKS AND C_m-SNAKES ARE PRIME GRAPHS

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ABSTRACT. A graph G on n vertices has a prime labeling if its vertices can be assigned the distinct labels $1, 2, ..., n$ such that for every edge xy in G, the labels of x and y are relatively prime. In this paper, we show that generalized books and C_m snakes all have prime labelings. In the process, we demonstrate a way to build new prime graphs from old ones.

1. INTRODUCTION

We say a graph G on n vertices has a prime labeling if its vertices can be assigned distinct integers $1, 2, ..., n$ such that for every edge xy in G, the labels of x and y are relatively prime. We call G a prime graph if it has a prime labeling. We denote by (a, b) the greatest common divisor of two integers a and b.

As in [4], we define R_n to be the graph on n vertices with vertex set $V(R_n) = \{v_1, v_2, ..., v_n\}$ and edge set $E(R_n) = \{v_iv_j : (i, j) = 1\}$. For a fixed n, this gives the maximum prime graph on n vertices.

Theorem 1. *A graph G on n vertices has a prime labeling if and only if G is a spanning subgraph of R_n.*

The proof of this theorem is obvious. Furthermore, it is clear from the definition that every spanning subgraph of a prime graph is itself a prime graph.

Many classes of graphs are known to be prime (see [1]), and in 1980, Entringer conjectured that all trees are prime graphs. We present several new classes of prime graphs, generalized books and C_m-snakes, by presenting a process for building new prime graphs from existing ones.

2. EDGE AMALGAMATIONS AND GENERALIZED BOOKS

Definition 2. *Suppose we have a finite collection $\{G_i\}$ of graphs, each with a fixed vertex v_{oi}, called a terminal. Then the amalgamation $Amal\{(G_i, v_{oi})\}$ is formed by taking the union of all the G_i and identifying their terminals.*

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In [3], Lee, Wui, and Yeh show that the amalgamation of any finite collection of paths and cycles is a prime graph, regardless of which terminals are chosen. We present an analogous result for the edge amalgamations of finite collections of paths and cycles.

Definition 3. *Let $\{(G_i, x_iy_i)\}$ be a finite collection of graphs, each with a fixed edge which is oriented. Then the edge amalgamation $Edgeamal\{(G_i, x_iy_i)\}$ is formed by taking the union of all the G_i and identifying their fixed edges, all with the same orientation.*

When we consider the edge amalgamations of cycles, we have a generalization of the book graph $S_n \times P_2$. When $\{G_i\}$ is a collection of cycles, we call $Edgeamal\{(G_i, x_iy_i)\}$ a generalized book. The spine xy of the generalized book is the edge we obtain from the identification of the edges x_iy_i, and each cycle G_i containing this edge is called a page. For each page $G_i = xyv_1v_2...v_{k_i}x$ of length $k_i + 2$, we say that $v_1v_2...v_{k_i}$ is the nonspine path of the cycle.

Note that for edge amalgamations of collections of cycles, the choice of edges and orientations is irrelevant. For this reason, we simply use $Edgeamal\{G_i\}$ to denote the edge amalgamation of a collection of cycles.

Lemma 4. *Given a finite collection $\{G_i\}$ of even cycles, $G = Edgeamal\{G_i\}$ is a prime graph.*

Proof. Let xy be the spine of G. Assign the labels 1 to x and 2 to y. Label the vertices of nonspine path of G_1 consecutively with the integers $3, 4, ..., |G_1|$, so that the vertex labeled 3 is adjacent to y and the vertex labeled $|G_1|$ is adjacent to x.

For each subsequent cycle in the graph, label the vertices of the nonspine path $v_1v_2...v_{k_i}$ (where v_1 is adjacent to y and v_{k_i} is adjacent to x) as follows:

If m vertices of G have already been labeled, label the nonspine path of G_i consecutively with the integers from $(m + 1)$ to $(m + k_i)$. Now, v_1 is adjacent to only y and v_2, and v_{k_i} is adjacent only to x and v_{k_i-1}. It is clear that $m+1$ is an odd integer, so $(m+1, 2) = 1$, and obviously, $(1, m+k_i) = 1$. Furthermore, the labels of consecutive vertices in our nonspine path have consecutive integers as labels, and these are clearly relatively prime. □

Figure 1(i) shows a prime labeling of $Edgeamal\{C_4, C_4, C_6\}$.

Lemma 5 (Bertrand's Postulate, [2] p.373). *For every positive integer $n > 3$ there is a prime p such that $n < p < 2n - 2$.*

Note that this is a slightly stronger version of Bertrand's Postulate than is usually stated. The original, which states that there is a prime between any integer and its double, is sufficient for the following proof, but not for later purposes.

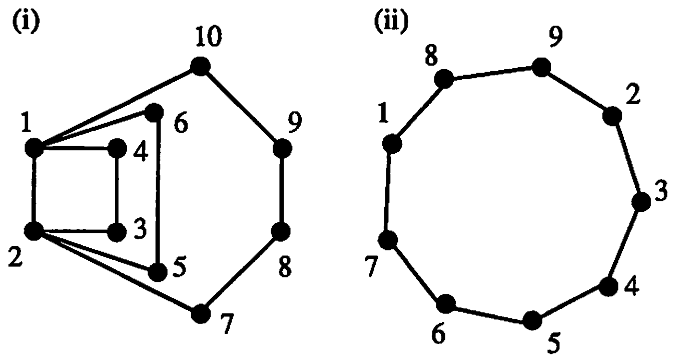

FIGURE 1. The prescribed prime labelings for $Edgeamal(\{C_4, C_4, C_6\})$ and for C_9.

Lemma 6. *Let G be an odd cycle of length $2n-1$, with $n \geq 2$. Then G has a prime labeling in which 1 and p are assigned to adjacent vertices, where p is a prime with $n < p < 2n$.*

Proof. By Bertrand's Postulate, there exists a prime number p such that $n < p < 2n$. This is clearly an odd prime, and we can label our cycle as follows. Fix one vertex and label it 1. Label the remaining vertices in this order around the cycle: $p+1, p+2, ..., 2n-2, 2n-1, 2, 3, ..., p$. The only pairs of adjacent labels which are not consecutive integers are $(1, p+1)$ and $(2n-1, 2)$, both of which are obviously relatively prime pairs.

Thus, odd cycles have prime labelings in which 1 and p are assigned to adjacent vertices. □

Figure 1(ii) shows a prime labeling of C_9.

Lemma 7. *Suppose G is a prime graph on at least three vertices, and let $\{p_1, p_2, ..., p_k\}$ be the set of all primes satisfying $\frac{1}{2}|G| < p_i \leq |G|$. Then permuting the labels $\{1, p_1, p_2, ..., p_k\}$ gives a new prime labeling for G.*

Proof. We simply note that the integers $\{p_1, p_2, ..., p_k\}$ all satisfy $(p_i, j) = 1$, for all $j \in \{1, 2, ..., n\}, j \neq p_i$, and that $(1, j) = 1$ for all j. Since each of these labels is relatively prime to all other labels in our graph, we can permute them arbitrarily to obtain new prime labelings. □

Remark 8. *In particular, if we have a generalized book G with the labels 1 and p on the spine, where $\frac{1}{2}|G| < p \leq |G|$, then G has a prime labeling with 1 and q on the spine, where q is the largest prime less than or equal to $|G|$.*

Lemma 9. *Suppose G is a generalized book with a prime labeling, where the labels 1 and p are assigned to its spine xy, and p is a prime such that $\frac{1}{2}|G| < p \leq |G|$. Then $G' = Edgeamal\{(G,xy),(C_k,ab)\}$ is a prime graph, for any integer k and any edge $ab \in C_k$.*

Proof. Essentially, we wish to add another page to our generalized book G. In G', start by giving the subgraph G a prime labeling which has the labels 1 and p on the edge xy. Let q be the largest prime less than or equal to $|G|$, and switch the labels p and q. By Lemma 7, this new labeling of G is still prime. Assume, without loss of generality, that x is assigned the label 1 and y is assigned the label q. Let $|G| = n$, and complete the labeling as follows.

We construct a sequence $\{q_i\}$ of primes. Let $q_1 = q$. For $i \geq 1$, first check whether $2q_i > |G'|$. If it is, stop the sequence at q_i. If not, find a prime q_{i+1} satisfying $q_i < q_{i+1} < 2q_i - 2$. Bertrand's Postulate, as stated in Lemma 5, assures the existence of such a prime. Note that for $i \geq 2$, we have $q_i > n$.

Label the vertices of the nonspine path sequentially with the integers $n+1, n+2, ...n+k-2$, starting at the vertex adjacent to y and ending at the vertex adjacent to x. Since $q_i > n$, for $i \geq 2$, the labels $\{q_2, ..., q_d\}$ are all on the nonspine path of our new cycle C_k.

Suppose our sequence of primes is $q_1, q_2, ..., q_d$. Then, for $i = 1, 2, ..., d-1$, take the vertex originally labeled q_{i+1} and change its label to q_i. In addition, change the label on vertex y from q_1 to q_d. We claim this gives a prime labeling of G'.

Clearly, it is sufficient to verify that all the vertices labeled q_i still have relatively prime neighbors, since no other vertices were affected in the construction. Since $\frac{1}{2}|G'| < q_d$ and q_d is prime, q_d is relatively prime to every other label of $|G'|$. Thus, all that remains to be checked are the pairs $(q_{i+1} - 1, q_i)$ and $(q_i, q_{i+1} + 1)$ in our nonspine path. Since q_i is prime, it is sufficient to show that neither $q_{i+1} - 1$ nor $qi + 1 + 1$ is a multiple of q_i. Since we used Bertrand's Postulate in constructing our sequence of primes, we know that $q_i + 1 < q_{i+1} < 2q_i - 2$ for all i. This implies $q_i < q_{i+1} - 1 < 2q_i - 3$ and $q_i + 2 < q_{i+1} + 1 < 2q_i - 1$, so neither $q_{i+1} - 1$ nor $q_{i+1} + 1$ is a multiple of q_i.

We note that for $q_1 = 3$, this argument fails, but the process of building prime generalized books is still valid. There is only one odd cycle for which 3 is the largest prime, namely C_3. If we have an edge amalgamation of cycles where there is an odd cycle of length greater than 3, we may simply begin our process with the longer cycle. If every odd cycle is of length 3, we consider the following two subcases. If every cycle is a C_3, the edge amalgamation is clearly prime; we label the spine with 1 and the largest prime, and all other vertices can be labeled arbitrarily. Otherwise, there is an even cycle in our collection. If we first take the edge amalgamation of any even cycle and a single triangle, $G = Edgeamal\{C_3, C_{2m}\}$, the result is clearly a prime graph, and furthermore, the largest prime label is now greater than 3. More importantly, our graph G has a labeling with 1 and our largest prime p on the spine. We label the cycle C_{2m} with the integers $1, 2p+2, 2p+3, ..., |G|, 2, 3, ..., p$ and we give the nonspine vertex of our triangle the label $p+1$. We then continue with the algorithm described above.

Thus, adding a page to a prime generalized book graph gives a prime graph, and the proof is complete. □

We now present the following result, which is analogous to the results on amalgamations of collections of cycles and paths.

Theorem 10. *All generalized books are prime graphs.*

Proof. Suppose G is a generalized book. We consider the following cases.

Case 1: Suppose all cycles of G are even. Then Lemma 4 completes the proof.

Case 2: Suppose G has at least one odd cycle. We wish to induct on the number of pages of G. Lemma 6 establishes the base case, and Lemma 9 shows that we can complete the inductive step. By construction, G will have a prime labeling with 1 and p on the spine, where p is the largest prime such that $p \leq |G|$. □

Corollary 11. *Let $\{G_i\}$ be a finite collection of paths and cycles. Then $G = Edgeamal\{(G_i, x_iy_i)\}$ is a prime graph for all choices of x_iy_i.*

Above, we simply considered generalized books with P_2 as a spine. It is reasonable to look at books with spines of other lengths as well.

Corollary 12. *Let G be a generalized book with P_m as a spine, where $m > 2$. Then G is a prime graph.*

Both corollaries follow directly from the fact that all spanning subgraphs of a prime graph are prime. Furthermore, we can use the construction in Lemma 9 to show that the following generalization holds.

Theorem 13. *Suppose G is a prime graph and that p is the largest prime less than or equal to $|G|$. Let x be the vertex assigned the label 1 and let y be the vertex assigned the label p. If the edge xy is in G, then let $G' = G$. If not, let G' be the graph G with the edge xy added. Then $G^* = Edgeamal\{(G', xy), (C_k, ab)\}$ is a prime graph, for any integer $k \in \mathbb{N}$ and any edge $ab \in C_k$.*

Proof. Set $|G'| = n$, and let p be the largest prime such that $p \leq n$. If $n > 4$ (that is, when $p > 3$) we simply use the labeling process from Lemma 9, constructing a sequence of primes between p and n, inclusive, and then rotating them.

As in Lemma 9, we must verify that the small cases also work, so we check the graphs for which $p = 3$, the graphs on 3 and 4 vertices. If G has 3 vertices, it is a spanning subgraph of C_3, and if G has 4 vertices, it is a spanning subgraph of $Edgeamal\{(C_3), (C_3)\}$. In either case, we may apply Lemma 9 directly.

□

3. Snake Graphs

Definition 14. *A snake graph is formed by taking n copies of a cycle C_m and identifying exactly one edge of each copy to a distinct edge of the path P_{n+1}, which we will call the backbone of the snake. We will use $T_n^{(m)}$ to denote this snake graph.*

In [5], the authors show that triangular and quadrilateral snakes are prime. Here we present a generalization and tie it to the results of Seoud and Youssef [4].

Lemma 15 (Seoud and Youssef, [4]). *Let $A_i = 1 + (i-1)b$, where i and b are both positive integers. Then $(A_i, A_{i+1}) = 1$.*

Theorem 16. *All snake graphs $T_n^{(m)}$ are prime.*

Proof. We label the first cycle C_k sequentially with the integers $1, 2, ..., k$ so that the labels 1 and k are on the backbone of the snake. Label the remaining vertices of the adjacent cycle sequentially with the integers $k+1, k+2, ..., 2k-1$ so that the label $2k-1$ is on the backbone. Continue in the same manner, labeling the i^{th} cycle C_k with the integers $1+(i-1)(k-1), 2+(i-1)(k-1), ..., 1+i(k-1)$. It is easy to see that we have a path labeled sequentially, with additional edges between the vertices labeled $1+(i-1)(k-1)$ and $1+i(k-1)$. By Lemma 15, these pairs of labels are relatively prime. □

In [4], the authors show that the star (m,n)-gon $S_n^{(m)}$ is prime for all $m, n \geq 3$. It is easy to see that $S_n^{(m)}$ can be formed from $T_n^{(m)}$ by identifying the first and last vertices of the path P_{n+1} used to construct $T_n^{(m)}$. These vertices are labeled 1 and $|T_n^{(m)}|$ in $T_n^{(m)}$ and in $S_n^{(m)}$ the identified vertices take the label 1, so the graph remains prime.

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