Sparse Semi-Magic Squares and Vertex-magic Labelings

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Abstract

We introduce a generalisation of the traditional magic square which proves useful in the construction of magic labelings of graphs. An order n *sparse semi-magic square* is an $n \times n$ array containing the entries $1, 2, \cdots, m$ (for some $m < n^2$) once each with the remainder of its entries 0, and its rows and columns have a constant sum k. We discover some of the basic properties of such arrays and provide constructions for squares of all orders $n \geq 3$. We also show how these arrays can be used to produce vertex-magic labelings for certain families of graphs.

1 Introduction

There has been interest in magic squares for many centuries and in many cultures. They and their various generalisations are arguably the most popular topic of study for those who enjoy recreational mathematics. However, from time to time they have also been found useful in some application to another area of mathematics. One such application was found recently in the construction of certain labelings of graphs. In [5] the notion of a *vertex-magic total labeling* (VMTL) for a graph was introduced. For a simple graph G with v vertices and e edges, this is a one-to-one assignment of the labels $1, 2, \cdots, v + e$ to the vertices and edges of G so that the sum of labels of any vertex and its incident edges is a constant. In that paper, an $n \times n$ magic square was used to construct a VMTL for the complete bipartite graph $K_{n,n}$ for all $n > 1$. An up-to-date full account of VMTL's is given in [6].

In fact, the construction used in [5] did not make any use of the diagonals of the square, so that in this paper we will be interested only in squares in which the row-sums and column-sums have the same constant value; these have been called *semi-magic squares.* As a tool in constructing other VMTLs, we would like an array which is like an semi-magic square but allows some of the entries to be 0. We will call such an array a *sparse semi-magic square.* More precisely, for any $n \geq 3$, an $n \times n$ array will be an order n sparse semi-magic square if it contains the entries $1, 2, \cdots, m$ (for some $m < n^2$) once each with the remainder of its entries 0, and its rows and columns have a constant sum k (the *magic constant*). We will abbreviate *sparse semi-magic square* as *SMS*

Since no description of these of arrays seems to be readily available in the literature, we will derive some of their basic properties, and provide constructions for several infinite families of squares, including every $n \geq 3$. In the last section, we will show how a sparse semi-magic square is used to construct a VMTL.

2 Basic Properties

Any magic square is semi-magic, thus semi-magic squares exist for every order $n \geq 3$. We note that any permutation of the rows or columns of an semi-magic square will leave a square which is semi-magic. Apart from rotation, reflection and row or column permutation, the order 3 array on the left below is the unique order 3 semi-square.

6	1	8
7	5	3
2	9	4

5	0	7
6	4	2
1	8	3

Proposition 1 *Sparse semi-magic squares exist for all orders $n \geq 3$.*

Proof. Suppose $M = (M_{i,j})$ is any order n semi-magic square and let k be its magic constant. Let S be the array with $S_{i,j} = M_{i,j} - 1$. Then S has constant row and column sums (equal to $k - n$) and contains exactly one 0 entry. ∎

We will say that the array S is *derived* from the array M. The array shown on the right above is derived from the other and it is easily checked to be the unique sparse semi-magic square of order 3. Conversely, it is clear that any SMS with exactly one zero can be regarded as being derived from an semi-magic square.

We originally pictured SMSs as having one or more diagonals filled with 0s, so that the number of 0s would be a multiple of n, and that there would be the same number of 0s in each row and each column. We soon discovered that neither of these conditions need hold. Let us express the number of non-zero entries in an $n \times n$ SMS as $m = nd - r$ where $0 \leq r < n$. Then we make the following definitions:

Definition *An $n \times n$ sparse semi-magic square whose non-zero entries are the numbers $1, 2, \cdots, nd - r$ has* density d *and* deficiency r. *Such a square will be denoted $S_n(d, r)$.*

The first example below illustrates an $S_5(3, 0)$ and the other is an $S_5(4, 1)$.

1	10	13	0	0
9	11	0	0	4
14	0	0	2	8
0	0	5	7	12
0	3	6	15	0

0	18	15	2	3
1	10	0	19	8
17	0	7	0	14
9	4	0	12	13
11	6	16	5	0

The traditional magic square can be regarded as an array $S_n(n, 0)$.

Definition *An $n \times n$ sparse magic square with the same number of 0s in each row and column is called* regular.

Thus a regular square has deficiency 0 and if k is the number of zeros in each row and column, it has density $d = n - k$. The $S_5(3, 0)$ square on the left above is regular. Finally, we will call an $S_n(d, 1)$ square *almost regular* if it has precisely d positive entries in all but one row and one column, and $d-1$ positive entries in a single row and a single column. The $S_5(4, 1)$ array pictured above is almost regular. Reasoning as in the proposition above, we can always obtain an almost regular square by subtracting 1 from all the non-zero entries of a regular square. We will also call these *derived* squares. The $S_5(4, 1)$ square above illustrates that not every almost regular square is derived from a regular one (For it to be derived, the short row and column would have to intersect in a 0 entry). The next theorem shows that there is a limit on the number of 0s in a SMS.

Theorem 1 *If $S_n(d, r)$ is a sparse semi-magic square, then $d \geq 3$.*

Proof. At most one row (or column) can contain a single positive entry, since if there were two such rows their row sums would be different. Also if an entry is alone in its row, it must be alone in its column for the row and column sums to be equal.

If $d = 1$, then either some row is empty, or every row contains a single positive entry. In both cases the row sum is not constant.

If $d = 2$, then $r = 0$ or $r = 1$, otherwise either there is a zero row (or column) or there are two rows (columns) with a single positive entry. In both cases the row sum is not constant.

(Case 1: $r = 1$): Then at least one row contains a single entry and that entry appears in a column by itself. So $n-1$ rows contain precisely 2 entries, and each of these entries must be in a column with only one other entry. But since the row-sum and column sum are equal, that entry's row partner and column partner must be equal. This is impossible since the entries are all different.

(Case 2: $r = 0$): At least $n-2$ rows contain exactly 2 entries. Consider the entries in any one of these rows. Neither can be in a column by itself, and at most one can be in a column with 2 other entries. Hence at least one of them is in a column with only one other entry. This would imply that its row-partner and column-partner are equal, which is impossible. ■

The two following theorems show that the spectrum of values of density and deficiency which can actually arise in a sparse semi-magic square will depend on its order.

Theorem 2 *For any sparse semi-magic square $S_n(d,0)$ or $S_n(d,1)$, if n is even then d is even.*

Proof. First let $r = 0$. The set of entries is $\{1, \cdots, nd\}$ and then the magic constant is

$$k = \frac{1}{n}\frac{dn(dn+1)}{2} = \frac{d(dn+1)}{2}.$$

If n is even, then $dn+1$ is odd and so d must be even.

The proof is similar for $r = 1$. ■

Corollary 1 *For any regular or almost regular SMS, if n is even, then d is even.*

Theorem 3 *For any sparse semi-magic square $S_n(d,r)$, if n is a prime power, then $r = 0$ or $r = 1$.*

Proof. The magic constant is

$$k = \frac{(nd-r)(nd-r+1)}{2n}.$$

Hence

$$(nd-r)(nd-r+1) \equiv 0 (\operatorname{mod} 2n) \tag{1}$$

and so, in particular

$$r(r-1) \equiv 0 (\operatorname{mod} n). \tag{2}$$

Since $n = p^\alpha$ and r and $r-1$ are relatively prime, we have $r \equiv 0 (\operatorname{mod} p^\alpha)$ or $r - 1 \equiv 0 (\operatorname{mod} p^\alpha)$. But since $r < n = p^\alpha$, then $r = 0$ or $r = 1$. ■

Corollary 2 *For $n = 4$, the only SMS are the semi-magic squares and the trivial almost regular semi-magic squares derived from them.*

Proof. Since n is a prime power, $r = 0$ or $r = 1$. But n is even, so d is even and by the previous theorem $d = 4$. For $r = 0$ we have an semi-magic square. If $r = 1$, there is just 1 cell containing 0, so the array is a derived square. ∎

It follows from the theorems that 5 is the smallest order for which a non-trivial SMS can exist and 2 examples are given above. It is worth pointing out that squares exist which are neither regular nor almost regular. Shown is an $S_6(4,4)$

0	17	2	9	3	4
0	7	8	6	1	13
5	0	0	0	12	18
0	0	15	20	0	0
16	0	0	0	19	0
14	11	10	0	0	0

which has a magic constant of 35.

3 Squares with Minimum Density

By theorem 1, the sparsest possible squares have density 3, so it is worth examining these in more detail. The first 2 theorems show that in this case the deficiency is considerably restricted.

Theorem 4 *For any sparse semi-magic square $S_n(d,r)$, if $d = 3$ then $r = 0$ or $r = 1$ or*

$$\frac{1}{2}(1+\sqrt{8n+1}) \leq r, \qquad n \text{ odd,}$$
$$\frac{1}{2}(1+\sqrt{4n+1}) \leq r, \qquad n \text{ even}$$

Proof. As in the proof of Theorem 3, we have $(nd-r)(nd-r+1) \equiv n^2d^2 - 2ndr + nd + r^2 - r \equiv 0(\bmod 2n)$. For $d = 3$ and n even, this simplifies to $r(r-1) \equiv 0(\bmod n)$ and thus if $r \neq 0, 1$ we have $r^2 - r \geq n$, whence $r \geq \frac{1}{2}(1+\sqrt{4n+1})$. For odd n, this simplifies to $r(r-1) \equiv 0(\bmod 2n)$. Then $r^2 - r \geq 2n$ and so $r \geq \frac{1}{2}(1+\sqrt{8n+1})$. ∎

Theorem 5 *For any sparse semi-magic square $S_n(d,r)$, if $d = 3$ then $r = 0$ or $r = 1$ or*

$$r \leq \frac{1}{2}(1 - 6n + \sqrt{48n^2 - 16n + 1}).$$

Proof. Observe that we have $3n - r$ integers to distribute among n^2 cells. The magic constant is $k = \frac{1}{2n}(3n-r)(3n-r+1)$ which is greater than the largest entry, so no row or column can contain only a single entry. Hence every row and column contains at least two entries. If an entry in a 2-row also belonged to a 2-column, its row and column partners would have to be equal (to maintain constant sum). This is impossible since the entries are all different, so the 2-rows and 2-columns contain no common elements and there are at least r 2-rows and r 2-columns. Let $r + \delta_r$ and $r + \delta_c$ be the number of 2-rows and 2-columns respectively. Then there are $2r + \delta_r + \delta_c$ pairs of positive integers each of which sums to k. Now the $2(2r + \delta_r + \delta_c)$ largest entries sum to

$$S = \frac{2(2r+\delta_r+\delta_c)}{2}(2(3n-r) - 2(2r+\delta_r+\delta_c)+1)$$

and we must have $k \leq \frac{1}{2(2r+\delta_r+\delta_c)}S$. Substituting the values for k and S and simplifying gives us

$$r^2 - r + 6nr - 3n^2 + 4n(\delta_r + \delta_c) + n \leq 0.$$

Since $\delta_r, \delta_c \geq 0$, this reduces to $r \leq \frac{1}{2}(1 - 6n + \sqrt{48n^2 - 16n + 1})$. ■

Approximating, we can simplify the above inequality as

$$r \leq (2\sqrt{3} - 3)n$$

or, further, as $r \leq .47n$.

Congruence 1 imposes a strong restriction on the possible values of r. The following theorem singles out certain values of n for which *no* SMS can exist.

Theorem 6 *Let p be any prime. There is no SMS with $n = 2p^\alpha$ and $d = 3$.*

Proof. Let $n = 2p^\alpha$. Using the previous theorem, $r < \frac{1}{2}n = p^\alpha$. By congruence 1 we have after simplifying:

$$r(r-1) \equiv 0 (\operatorname{mod} p^\alpha).$$

Then $p^\alpha \mid r(r-1)$ and since r and $r-1$ are relatively prime, either $p^\alpha \mid r$ or $p^\alpha \mid (r-1)$. Since $r < p^\alpha$ neither is possible unless $r = 0$ or $r = 1$. But since n is even, d must be even by Theorem 2. So $d \geq 4$. ■

In the following table we show the list of all feasible values $r > 1$ which satisfy both the inequalities and the congruence 1, for all $n \leq 100$. Few values of n in this range admit more than a single feasible r. We note that for the triangular numbers $n = \frac{r(r-1)}{2}$, r is a feasible value when $r \equiv 2, 3 (\operatorname{mod} 4)$.

n	r	n	r	n	r	n	r
12	4	44	12	65	26	84	21, 28, 36
15	6	45	10	66	22	85	35
20	5	48	16	69	24	87	30
21	7	51	18	70	15	90	10
24	9	52	13	72	9	91	14
30	6, 10	55	11	75	25	93	31
33	12	56	8	76	20	95	20
35	15	57	19	77	22	96	33
39	13	60	21	78	27	99	45
42	7, 15	63	28	80	16		

From the table, the smallest orders which permit a feasible value of $r > 1$ are 12 and 15, and we provide an example of an $S_{15}(3, 6)$ with the 0's omitted to highlight the sparseness:

39						2	11							
	36					12	4							
		35						10				7		
13			34										5	
		17		29						6				
	16				28						8			
			18					9				25		
				23									26	3
					24				1					27
						38					14			
							37			15				
								33	19					
									32			20		
										31			21	
											30			22

4 Construction of Regular Squares

In this section, we show how to construct an infinite family of sparse semi-magic squares. As one might expect, we deal with those having the most structure, the regular squares. Our construction produces squares containing diagonals composed of 0s. We first define a rectangular array which is the important building block in the construction.

Definition *A* diagonal Kotzig *array is a $d \times n$ rectangular array ($d \leq n$) with the following properties:*

1. *Each row is a permutation of the set $\{1, 2, \cdots, n\}$*
2. *All columns have the same sum.*
3. *All forward diagonals have the same sum.*

Three-row arrays satisfying the first two conditions were used by Kotzig ([3]) to construct edge-magic labelings and there is an account of this in [6] where they are called Kotzig arrays. The first author has constructed a d-row generalisation of these Kotzig arrays and they have been used to construct vertex-magic labelings for complete bipartite graphs ([2]). Our constructions of squares require Kotzig arrays with the additional diagonal condition stated as property 3 above. The theorems below show how to construct such arrays.

It is an easy calculation to show that the constant sum for the diagonals will be the same as the column-sum. The procedure to construct a regular SMS from a diagonal Kotzig array is described in the following theorem.

Theorem 7 *If there exists a $d \times n$ diagonal Kotzig array, then there exists a regular SMS of order n, density d and deficiency* 0.

Proof. Let $D = (D_{i,j})$ be a $d \times n$ diagonal Kotzig array and let A be the $n \times n$ matrix obtained by appending $n - d$ rows of 0's to the bottom of D. Let B be the $n \times n$ matrix with $b_{i,j} = i - 1$ when $i \leq d$ and $b_{i,j} = 0$ for $i > d$. Finally let $S = A + nB$.

Let $u(S)$ be the matrix obtained from S by moving the entries in column j upward by $j - 1$ positions (mod n), so that the forward diagonals of S become the rows of $u(S)$. More precisely, $u(S)_{i,j} = S_{i+j-1,j}$. We claim that $u(S)$ is the required regular SMS.

Note that the columns of B have constant sum and therefore the columns of $S = A + nB$ will also have a constant sum k. Also the forward diagonals of B have constant sum and so the forward diagonals of $S = A + nB$ will also have constant sum, also equal to k. Since the diagonals of S become the rows of $u(S)$, the rows and columns of $u(S)$ will have constant sum k. It remains to prove that $u(S)$ is regular, i.e. that there are the same number of 0's in each row and column of $u(S)$. Since there were 0's in the bottom $n - d$ rows of both A and B, there will be $n - d$ 0's in each column of $u(S)$. Since each diagonal of S intersected each of the $n - d$ bottom rows (the 0 rows) once, there are $n - d$ 0's on each diagonal of S, and therefore on each row of $u(S)$. ■

So in order to show the existence of a regular $S_n(d, 0)$, we must show how to construct a $d \times n$ diagonal Kotzig array. Several different constructions are required, depending on the values of d and n, and they are provided in the following theorems. The difficulty lies in discovering the array - verifying the row and diagonal sums is usually easy.

Theorem 8 *A $3 \times n$ diagonal Kotzig array exists for all odd $n \geq 3$.*

Proof. A $3 \times n$ diagonal Kotzig array D can be defined as follows

$D_{1,j} = \frac{1}{2}(n+2-j)$	j odd
$D_{1,j} = n+1-\frac{1}{2}j$	j even
$D_{2,1} = n$	
$D_{2,j} = j-1$	$j > 1$
$D_{3,1} = 1$	
$D_{3,j} = \frac{1}{2}(2n+3-j)$	j odd (> 1)
$D_{3,j} = \frac{1}{2}(n+3-j)$	j even

It is easy but tiresome to verify that the columns and forward diagonals all sum to $\frac{1}{2}(3n+3)$. ■

We illustrate this construction for $n = 7$, showing both the diagonal Kotzig array and the resulting SMS. The particular construction described in this theorem produces a square whose main diagonal happens to also add to the magic constant.

$$D = \begin{pmatrix} 4 & 7 & 3 & 6 & 2 & 5 & 1 \\ 7 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 7 & 3 & 6 & 2 & 5 \end{pmatrix}$$

$$S_7(3,0) = \begin{pmatrix} 4 & 8 & 21 & 0 & 0 & 0 & 0 \\ 14 & 18 & 0 & 0 & 0 & 0 & 1 \\ 15 & 0 & 0 & 0 & 0 & 5 & 13 \\ 0 & 0 & 0 & 0 & 2 & 12 & 19 \\ 0 & 0 & 0 & 6 & 11 & 16 & 0 \\ 0 & 0 & 3 & 10 & 20 & 0 & 0 \\ 0 & 7 & 9 & 17 & 0 & 0 & 0 \end{pmatrix}$$

For the following constructions we define the *complement* of a number t to be $n+1-t$. Also the complement of a row will be the row consisting of the complements of its entries (in the same order as the entries).

Theorem 9 *A $d \times n$ diagonal Kotzig array exists for $d \equiv 0 (\mathrm{mod}\, 4)$.*

Proof. Let $d = 4m$. First consider the top $2m$ rows of the array. Each of the first m rows is any permutation of $1, 2, \cdots, n$ (different rows may be different permutations). The next m rows are the complements of the first m rows, shifted horizontally so that the complement of element $D_{i,j}$ appears on the same diagonal as $D_{i,j}$. More precisely, let $D_{i+m,j} = n + 1 - D_{i,j-m}$ for $1 \leq i \leq m$ (where the subscripts are taken mod n). Since the sum of any element and its complement is $n + 1$, each diagonal now sums to $m(n + 1)$.

For the bottom $2m$ rows of D, we just take the complements of the first $2m$ rows (in the same order). Precisely, $D_{i+2m,j} = n + 1 - D_{i,j}$. Since $D_{i+2m,j} + D_{i,j} = n + 1$, each column sums to $2m(n + 1)$.

Since each diagonal in the top $2m$ rows adds to the constant $m(n + 1)$, each diagonal in the bottom $2m$ rows will be the complement of that diagonal and therefore also sum to a constant, namely $2m(n + 1) - m(n + 1) = m(n + 1)$. Thus each diagonal of the whole array sums to $2m(n + 1)$. Therefore the array is diagonal Kotzig. ■

Since each of the first m rows was an arbitrary permutation, the construction actually provides a large family of inequivalent squares.

For even n not multiples of 4, we have to modify the above construction somewhat. The modification will consist of changing a few rows in the centre of the array, and differs as n is odd or even.

Theorem 10 *A $d \times n$ diagonal Kotzig array exists for $d \equiv 2 \pmod 4$ and n odd.*

Proof. Let $d = 4m + 2$. Consider the top $2m + 1$ rows of the array. We construct the first $2m - 2$ rows as in the proof of Theorem 9. Let row $2m - 1$ be any permutation of $1, 2, \cdots, n$. Define row $2m$ by

$D_{2m,j+1} = \frac{n+1}{2} + D_{2m-1,j}$	$j < \frac{n+1}{2}$
$D_{2m,j+1} = D_{2m-1,j} - \frac{n-1}{2}$	$j > \frac{n-1}{2}$

Then the diagonal sums for these 2 rows will be the consecutive integers $\frac{n+1}{2} + 1$ to $\frac{n+1}{2} + n$. Now we let row $2m + 1$ complement these sums in the following way:

$$D_{2m+1,j+2} = \frac{3}{2}(n + 1) - D_{2m,j+1} - D_{2m-1,j}$$

so that the diagonal sums for the 3 rows are now constant. Thus the diagonal sums for the top $2m + 1$ rows are the constant $\frac{3}{2}(n + 1) + (m - 1)(n + 1)$.

Take as the bottom $2m+1$ rows the complements of the top $2m+1$ rows, exactly as in the previous construction. Then both columns and diagonals sum to $(2m + 1)(n + 1)$ and so the array is diagonal Kotzig. ■

Theorem 11 *A $d \times n$ diagonal Kotzig array exists for $d \equiv 2 (\bmod 4)$ and n even.*

Proof. Let $d = 4m + 2$ and consider the top $2m + 1$ rows of the array. We construct the first $2m-2$ rows as in the proof of Theorem 9. This leaves the rows $2m - 1$ to $2m + 1$ to be filled. In row $2m - 1$ place the integers $1, \cdots, n$ in ascending order. Row $2m$ is the permutation defined by

$D_{2m,j+1} = \frac{1}{2}(n+1-j)$	j odd
$D_{2m,j+1} = \frac{1}{2}(2n+2-j)$	j even

and row $2m + 1$ is the permutation defined by

$D_{2m,j+2} = \frac{1}{2}(2n+1-j)$	j odd
$D_{2m,j+2} = \frac{1}{2}(n+2-j)$	j even

The diagonal sums for these three rows are alternately $\frac{3}{2}n + 1$ and $\frac{3}{2}n + 2$.

Take the complements of these three rows as rows $2m + 2$, $2m + 3$ and $2m + 4$, and the complements of the first $2m - 2$ rows as the last $2m - 2$ rows. Then the column sums in the array will all be equal and the diagonal sums for the bottom half of the array will also alternately sum to $\frac{3}{2}n + 1$ and $\frac{3}{2}n + 2$. To get constant diagonal sums in the array, we need to make the $\frac{3}{2}n + 1$ diagonals from the bottom half align with the $\frac{3}{2}n + 2$ diagonals of the top half. We do this by simply inserting a row of zeroes between rows $2m+1$ and $2m+2$. We then must insert a row of zeroes in the corresponding row of B. ∎

Having constructed the arrays for all even d it is now an easy matter to construct the arrays for odd d:

Theorem 12 *A $d \times n$ diagonal Kotzig array exists for all odd $d \geq 7$ and n odd.*

Proof. By the constructions of Theorems 10 and 11, there exists a $2t \times n$ diagonal Kotzig array D for any $t \geq 2$. From Theorem 8 there exists a $3 \times n$ diagonal Kotzig array D' for any odd n. Create a $2t + 3 \times n$ array E by appending the 3 rows of D' below the rows of D. Since each column of D has a constant sum k and each column of D' has a constant sum k' the columns of the new array E will all sum to $k + k'$. Similarly each diagonal will sum to the same constant $k + k'$. Finally, each row of E is a permutation of $1, \cdots, n$ so E is a diagonal Kotzig array. Since $t \geq 2$, then $d \geq 7$. ∎

The only odd d not covered by Theorems 8 and 12 is $d = 5$. We provide a special construction to cover this case.

Theorem 13 *A $5 \times n$ diagonal Kotzig array exists for all odd $n \geq 5$.*

Proof. We define the array as follows

$$\begin{aligned}
D_{1,j} &= \frac{n+1}{2}(j-1)(\operatorname{mod} n)+1 \\
D_{2,j} &= n+1-j \\
D_{3,j} &= j \\
D_{4,j} &= n+1-j \\
D_{5,j} &= j+\frac{n+1}{2}-D_{1,j}.
\end{aligned}$$

It is clear that each row is a permutation of $1, \cdots, n$ and that the columns sum to $\frac{5}{2}(n+1)$.

We need to verify that the diagonals also have the same sum. There are 6 cases: odd $j < n-3$, even $j < n-3$, $j = n-3$, $j = n-2$, $j = n-1$, $j = n$. We prove the first; let $j = 2t-1$. We have

$$\begin{aligned}
D_{1,j} &= \frac{n+1}{2}(2t-2)(\operatorname{mod} n)+1 = t \\
D_{2,j+1} &= n+1-(j+1) = n+1-2t \\
D_{3,j+2} &= j+2 = 2t+1 \\
D_{4,j+3} &= n+1-(j+3) = n-1-2t \\
D_{5,j+4} &= (j+4)+\frac{n+1}{2}-D_{1,j+4} \\
&= (2t+3)+\frac{n+1}{2}-[\frac{n+1}{2}(2t+2)(\operatorname{mod} n)+1] \\
&= t+1+\frac{n+1}{2}.
\end{aligned}$$

Then the diagonal sum is

$$\begin{aligned}
&D_{1,j}+D_{2,j+1}+D_{3,j+2}+D_{4,j+3}+D_{5,j+4} \\
&= \frac{5}{2}(n+1)
\end{aligned}$$

as required. The other cases are similar and the proofs are omitted. ■

We conclude this section with the following theorem which summarises the previous results:

Theorem 14 *An order n and density d SMS with deficiency 0 exists whenever (1) $n \geq 4$ is even and $d \geq 4$ is even, and (2) $n \geq 3$ is odd and $d \geq 3$.*

Corollary 3 *An order n and density d SMS with deficiency 1 exists whenever (1) $n \geq 4$ is even and $d \geq 4$ is even, and (2) $n \geq 3$ is odd and $d \geq 3$.*

5 Vertex-magic Labelings of Graphs

In this section , we present an example showing how a SMS can be used to construct a VMTL. We begin by describing the kind of labeling problem that motivated our study in the first place. Let G be a disconnected graph with e edges and having 2 disconnected components whose vertex sets are $\{v_1, \cdots, v_n\}$ and $\{u_1, \cdots, u_n\}$. Suppose G has a vertex-magic labeling λ with magic constant k. Form a new connected graph H by adding all the edges $u_i v_j$ for $1 \leq i, j \leq n$. In other words we insert the edges of a complete bipartite graph between the two components of G. Extend λ to a labeling for H by assigning numbers to these new edges using an $n \times n$ magic square M (which has magic constant k'), as follows:

$$\lambda'(u_i v_j) = M_{i,j} + 2n + e.$$

The labels now form the consecutive set $1, \cdots, 2n+e+n^2$. Since $\lambda(u_i) = k$ and the new edges meeting at vertex u_i sum to $k' + n(2n+e)$, then $\lambda'(u_i) = k + k' + n(2n + e)$ for all u_i. The same calculation holds for the v_i, so λ' is a vertex magic labeling for H.

An example of this construction is shown in Figure 1 where we insert the edges of a $K_{3,3}$, labeled using a 3×3 magic square, between the components of a labeled $2P_3$.

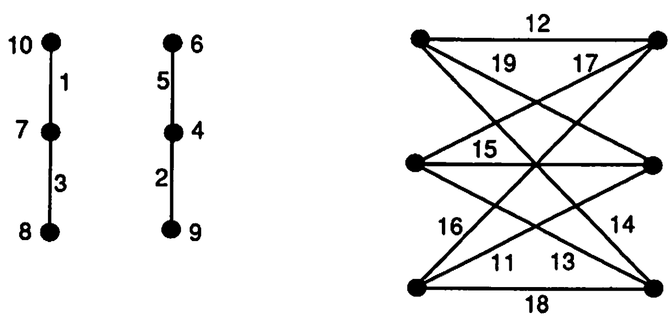

Figure 1: A magic square used to join $2PC_3$.

The above construction produces one new connected graph from a disconnected graph with two components. If we could arrange to insert the edges of other labeled incomplete bipartite graphs between the components, then we could construct a variety of new labeled graphs from the single starter. The SMS allows us to do this. The construction is identical to that above, except that the entry $S_{i,j} = 0$ indicates that the edge $u_i v_j$ is omitted from the bipartite graph. In Figure 2 we show a labeling of $2P_5$ for which $v + e = 18$, and the edges to be inserted between the components, labeled by adding 18 to the entries of the $S_5(3,0)$ shown in section 2.

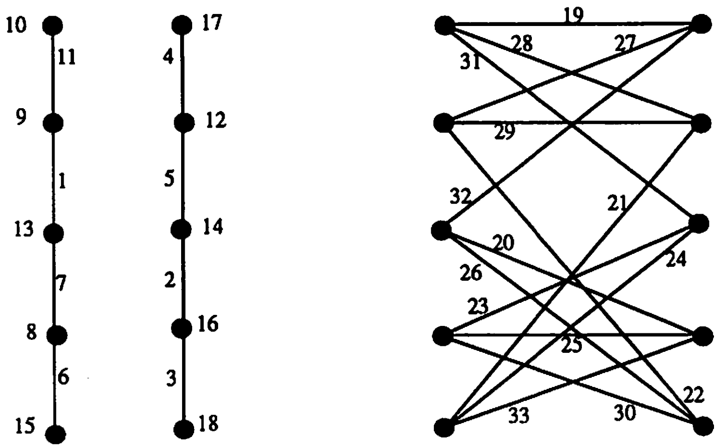

Figure 2: An $S_5(3,0)$ used to join $2P_5$.

6 Sparse Magic Squares

Because of the application to graph labeling, we have been interested in squares with constant row-sums and column-sums; the diagonal sums did not matter. However it is possible to construct sparse squares which are truly magic; that is, have both main diagonals sum to the magic constant as well. In this section, we give an example of one construction which produces such sparse magic squares for certain odd orders.

Let us call a regular sparse semi-magic square to be *centre-complementary* if it has the property that $a_{i,j}+a_{n+1-i,n+1-j} = nd+1$ whenever $a_{i,j} \neq 0$ and the entries are not in the same row. Then column i is the complement of the horizontal reflection of column $n+1-i$. It is clear that if we interchange columns i and j, we can preserve the centre-complementary property by simultaneously interchanging columns $n+1-i$ and $n+1-j$. Our goal is to permute the columns in such a way that each of the main diagonals has d non-zero entries while maintaining the centre-complementarity of the matrix. Then the diagonal sums would equal the magic constant. The next theorem is necessary for constructing centre-complementary squares.

Theorem 15 *For odd n and $3 \leq d \leq n$ there is a centre-complementary $d \times n$ diagonal Kotzig array.*

Proof. The $3 \times n$ and $5 \times n$ constructed previously have the required property. Define a $4 \times n$ array as follows:

$D_{1,j} = j$	$j \leq \frac{n-1}{2}$
$D_{1,j} = j + 1$	$\frac{n+1}{2} \leq j \leq n - 1$
$D_{1,j} = \frac{n+1}{2}$	$j = n$
$D_{2,j} = \frac{n+1}{2}$	$j = 1$
$D_{2,j} = n + 2 - j$	$2 \leq j \leq \frac{n+1}{2}$
$D_{2,j} = n + 1 - j$	$j > \frac{n+1}{2}$
$D_{3,j} = n + 1 - D_{2,n+1-j}$	
$D_{4,j} = n + 1 - D_{1,n+1-j}$	

It is easy to check that this is a Kotzig array. Not only are the forward diagonal sums all equal to the column sum s, but the forward diagonals of the first two rows alone each sum to $\frac{s}{2}$ as they do for the last two rows. It is clear from the definition that the array is centre-complementary.

Given any $n \times t$ diagonal Kotzig array which is centre-complementary, we can sandwich it between the two top rows and the two bottom rows of the $4 \times n$ array to produce a centre-complementary array of size $(4+t) \times n$. Thus from the 3 arrays above, we can produce an array of any odd density d. An array with even density $d = 2\delta$ can be constructed from any $\delta \times n$ array A by appending to A the $\delta \times n$ array A' which is its complement rotated 180 degrees. ∎

The method of construction is as in the proof of Theorem 7, except that we must ensure that there is a suitable permutation of the columns. This is easily done when n is not divisible of 3, so we restrict ourselves to this case. We will make use of the following result:

Lemma 1 *For $n \equiv \pm 1 \pmod 6$, there exists an $n \times n$ diagonal latin square whose entries in centrally symmetric positions are complementary.*

Proof. Let L be the square defined by $L_{i,j} = (2j - i) \pmod n + 1$ for $0 \leq i, j \leq n - 1$. This is easily checked to be latin whenever n is odd. For entries on the main diagonal, we have $L_{i,j} = i$ and these entries are clearly distinct. To show entries on the back diagonal are distinct, suppose $L_{i,n-1-i} = L_{j,n-1-j}$. Then $2n - 2 - 3i \equiv 2n - 2 - 3j \pmod n$, which yields $i = j$ so long as n is relatively prime to 3. Thus L is a diagonal latin square. Now consider the sum of two entries in centrally symmetric positions. We have $L_{i,j} + L_{n-1-i,n-1-j} = (2j - i) \pmod n + (n - 1 - (2j - i)) \pmod n + 2$

which simplifies to

$$\begin{array}{rrcl} (2j-i-n)+(2n-1-(2j+i))+2, & 2j-i & \geq & n \\ (2j-i)+(n-1-(2j-i))+2, & 2j-i & < & n \end{array}$$

both of which simplify to $n+1$. ■

We will make use of the fact that in L the entries in the columns occur consecutively in decreasing order (mod n).

Theorem 16 *A regular sparse magic square exists for every order* $n = 6m \pm 1$ *and density* d *when* $3 \leq d \leq n$.

Proof. Let D be a centre-complementary $d \times n$ diagonal Kotzig array. By Theorem 7, we can construct from D a regular SMS S of order n and density d. The matrix S will have the centre-complementary property and have $n-d$ zeroes appearing in each column. In particular, the zeroes will occur as a consecutive block in each column.

Let L be any $n \times n$ diagonal latin square whose centrally symmetric entries are complementary. We will replace certain entries of L by 0s and use the positions of these 0s to define the permutation of columns in S. To obtain a SMS of density d, we need $n-d$ consecutive 0s in each column. If d is even, let $n-d = 2u+1$ and set the entries $\frac{1}{2}(n+1)-u, \cdots, \frac{1}{2}(n+1)+u$ to 0. If d is odd so that $n-d = 2u$, we set the entries $1, \cdots, u$ and $n+1-u, \cdots, n$ to 0. In both cases, we have a consecutive block of $n-d$ 0s since the entries in each column of L are consecutive $(\bmod\, n)$. Now we move column i of S to the unique position where its block of consecutive 0s corresponds with the block of 0s of a column of L. Column $n+1-i$ of S must be moved correspondingly since there are 0s in centrally symmetric positions of L, but as described above, this will preserve the centre-complementary character of S. ■

We illustrate the construction with an example of a magic $S(5,4,0)$. The matrix A consists of a centrally complementary 4×5 diagonal Kotzig array with a row of 0s inserted between the top two rows and the bottom two rows. Then $A + 5B =$

$$\begin{pmatrix} 1 & 2 & 4 & 5 & 3 \\ 3 & 5 & 4 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 5 & 4 & 2 & 1 & 3 \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix} + 5 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 4 & 5 & 3 \\ 8 & 10 & 9 & 7 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 15 & 14 & 12 & 11 & 13 \\ 18 & 16 & 17 & 19 & 20 \end{pmatrix} \longrightarrow \begin{pmatrix} 15 & 16 & 4 & 7 & 0 \\ 18 & 2 & 9 & 0 & 13 \\ 1 & 10 & 0 & 11 & 20 \\ 8 & 0 & 12 & 19 & 3 \\ 0 & 14 & 17 & 5 & 6 \end{pmatrix}$$

The latin square from Lemma 1 determines the permutation of columns of the array as follows:

$$\begin{pmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} x & 0 & x & x & x \\ x & x & x & x & 0 \\ x & x & 0 & x & x \\ 0 & x & x & x & x \\ x & x & x & 0 & x \end{pmatrix}$$

which yields the sparse magic square

$$\begin{pmatrix} 16 & 0 & 4 & 15 & 7 \\ 2 & 13 & 9 & 18 & 0 \\ 10 & 20 & 0 & 1 & 11 \\ 0 & 3 & 12 & 8 & 19 \\ 14 & 6 & 17 & 0 & 5 \end{pmatrix}.$$

The construction described in the proof of Theorem 16 actually allows us to produce two sparse magic squares simultaneously, one of density d by setting $n - d$ of the numbers $1, \cdots, n$ equal to 0 in the latin square, and the other of density $n - d$ by setting the d complementary numbers of the latin square equal to 0. The non-zero entries in one square will occur in precisely the positions where the 0 entries occur in the other square. We give an example of 7×7 squares, S_1 and S_2:

$$\begin{pmatrix} 0 & 18 & 7 & 0 & 0 & 8 & 0 \\ 0 & 0 & 9 & 0 & 0 & 21 & 3 \\ 0 & 0 & 17 & 6 & 0 & 0 & 10 \\ 2 & 0 & 0 & 11 & 0 & 0 & 20 \\ 12 & 0 & 0 & 16 & 5 & 0 & 0 \\ 19 & 1 & 0 & 0 & 13 & 0 & 0 \\ 0 & 14 & 0 & 0 & 15 & 4 & 0 \end{pmatrix} \quad \begin{pmatrix} 23 & 0 & 0 & 5 & 20 & 0 & 10 \\ 3 & 21 & 0 & 12 & 22 & 0 & 0 \\ 13 & 25 & 0 & 0 & 2 & 18 & 0 \\ 0 & 1 & 15 & 0 & 14 & 28 & 0 \\ 0 & 11 & 27 & 0 & 0 & 4 & 16 \\ 0 & 0 & 7 & 17 & 0 & 8 & 26 \\ 19 & 0 & 9 & 24 & 0 & 0 & 6 \end{pmatrix}$$

Since the row and column sums (and diagonals) of S_1 are constant, as they are for S_2, we have the interesting consequence that the two sparse squares can be superimposed to form a traditional (full density) magic square (so long as we add 21 to every entry of the second square to ensure that the numbers range from 1 to 49). In this case we get the square

$$\begin{pmatrix} 44 & 18 & 7 & 26 & 41 & 8 & 31 \\ 24 & 42 & 9 & 33 & 43 & 21 & 3 \\ 34 & 46 & 17 & 6 & 23 & 39 & 10 \\ 2 & 22 & 36 & 11 & 35 & 49 & 20 \\ 12 & 32 & 48 & 16 & 5 & 25 & 37 \\ 19 & 1 & 28 & 38 & 13 & 29 & 47 \\ 40 & 14 & 30 & 45 & 15 & 4 & 27 \end{pmatrix}.$$

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