Disjoint Partial Triple Systems of Different Orders

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Abstract

It is noted that Teirlinck's "transposition argument" for disjoint Steiner triple systems applies more generally to certain partial triple systems of different orders. A corollary on the number of blocks common to two Steiner triple systems of different orders is also given.

For elaboration on the definitions and elementary facts cited herein, the reader is referred to the comprehensive reference [1].

A *partial triple system* of *order* v, or PTS(v), is a pair $(V, \mathcal{B})$ with V a v-set of *points* and $\mathcal{B}$ a set of 3-subsets of V, called *blocks*, such that every pair of distinct elements of V occurs in at most one triple. The *leave* of such a PTS(v) is the graph L with vertex set V and for $x, y \in V$, $x \neq y$, we have $\{x, y\}$ an edge of L if and only if $\{x, y\}$ is contained in no block of $\mathcal{B}$. For convenience, the term *reduced leave* will be used to refer to the leave of a PTS(v), but with all isolated vertices removed.

A *Steiner triple system* of *order* v, abbreviated STS(v), is a PTS(v) whose leave has v isolated vertices. The number of blocks in an STS(v) is $v(v-1)/6$. For an STS(v) to exist, it is necessary that this number of blocks be integral and that v be odd; thus a necessary (and sufficient) condition for the existence of an STS(v) is that $v \equiv 1$ or $3 \pmod 6$. Such orders v are said to be *admissible*.

An *incomplete Steiner triple system* of *order* v with *hole size* t, or ISTS(v,t), is an ordered triple $(V,T,\mathcal{B})$ with T a t-subset of V such that $(V,\mathcal{B})$ is a PTS(v) whose reduced leave consists of a complete graph on the points in T. It is known that an ISTS(v,t) exists if and only if v and t are odd, $\binom{v}{2}-\binom{t}{2}$ is divisible by 3, and $v \geq 2t+1$. Often v (and hence t) is assumed to be admissible. In this case, an STS(v) can be formed by taking the union of the block sets of an ISTS(v,t) and an STS(t) (on the appropriate sets of points). When convenient, an STS(v) may be viewed as an ISTS$(v,1)$.

If $\sigma \in \mathcal{S}_V$ is a permutation on the points V, and $\mathcal{B}$ is a set of blocks, define $\sigma\mathcal{B} = \{\{\sigma(a),\sigma(b),\sigma(c)\} : \{a,b,c\} \in \mathcal{B}\}$. The transposition in $\mathcal{S}_V$ which interchanges points $x,y \in V$ (and fixes all other points in V) is denoted (x,y).

In [4], Teirlinck uses transpositions in a clever argument to prove that for any STS(v) $(V,\mathcal{B}_1)$, there exists an STS(v) $(V,\mathcal{B}_2)$ with $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$. Motivated by the recent work [2], a similar method is used for the case of triple systems of different orders.

Theorem 1. *Let $t < u < v$ be odd positive integers with $u \geq 2t+1$ and $\binom{v}{2} \equiv \binom{u}{2} \equiv \binom{t}{2} \pmod 3$. Suppose further that $t < v-u$. Given any ISTS(u,t) $(U,T,\mathcal{B}_1)$, there exists an ISTS(v,t) $(V,T,\mathcal{B}_2)$ with $V \supset U$ such that $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$.*

Proof: Let $(V,T,\mathcal{B}_2)$ be an ISTS(v,t) with $|\mathcal{B}_1 \cap \mathcal{B}_2|$ achieving the minimum over all such ISTS. Suppose $A \in \mathcal{B}_1 \cap \mathcal{B}_2$. There must be some $a \in A \setminus T$. For $x \in V \setminus (A \cup T)$, define

$$\Delta(x) = (\mathcal{B}_1 \cap (a,x)\mathcal{B}_2) \setminus (\mathcal{B}_1 \cap \mathcal{B}_2).$$

Note that for $x,y \in V \setminus (A \cup T)$, $x \neq y$, we have

$$\begin{aligned}\Delta(x) \cap \Delta(y) &= (\mathcal{B}_1 \cap (a,x)\mathcal{B}_2 \cap (a,y)\mathcal{B}_2) \setminus (\mathcal{B}_1 \cap \mathcal{B}_2)\\ &= \mathcal{B}_1 \cap ((a,x)\mathcal{B}_2 \cap (a,y)\mathcal{B}_2 \setminus \mathcal{B}_2)\\ &= \mathcal{B}_1 \cap \emptyset\\ &= \emptyset.\end{aligned}$$

Moreover,

$$\begin{aligned}\bigcup_{x \in V\setminus(A\cup T)} \Delta(x) &\subseteq \{B : a \in B \in \mathcal{B}_1 \setminus \{A\}\}\\ &\quad \cup\{\{x,r,s\} \in \mathcal{B}_1 : \{a,r,s\} \in \mathcal{B}_2 \setminus \{A\}\},\end{aligned}$$

and the cardinality of the set on the right is $\leq (u-3)/2+(u-3)/2 = u-3$. By assumption, $v-t-3 > u-3$, so $\Delta(c) = \emptyset$ for some $c \in V \setminus (A \cup T)$. Hence

$$\mathcal{B}_1 \cap (a,c)\mathcal{B}_2 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \setminus \{A\},$$

a contradiction to the minimality of $|\mathcal{B}_1 \cap \mathcal{B}_2|$. □

Remarks: (1) It should be noted that Teirlinck's proof for STS with $u = v$ derives $|\Delta(x)| = 1$ for all x, then requires a more intricate argument. Here, we are content to force $\Delta(c) = \emptyset$ for some c with strict equality in the assumption $t < v - u$.
(2) The same proof applies in a more general setting. Suppose the reduced leave of a PTS(u) has t vertices and its complement is triangle-free (so that $a \in A \setminus T$ can be chosen in the proof). If $t < v - u$ and there is *some* PTS(v) with the same reduced leave, then there is one with disjoint blocks from the PTS(u).
(3) The proof can be viewed algorithmically. If the block set $\mathcal{B}_2$ is chosen arbitrarily, rather than to minimize $|\mathcal{B}_1 \cap \mathcal{B}_2|$, then some $\sigma \in \mathcal{S}_V$ is constructed so that $\mathcal{B}_1 \cap \sigma\mathcal{B}_2 = \emptyset$. Thus the roles of $u < v$ can be reversed in the sense that given any ISTS(v,t) $(V,T,\mathcal{B}_1)$ and a u-set U with $T \subset U \subset V$, there is an ISTS(u,t) $(U,T,\mathcal{B}_2)$ with $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$, again provided t,u,v satisfy the hypotheses.

An application is now considered. For $u \leq v$, let $I(u,v)$ be the set of all possible values of $|\mathcal{B}_1 \cap \mathcal{B}_2|$, where $(U,\mathcal{B}_1)$ and $(V,\mathcal{B}_2)$ are Steiner triple systems of orders u and v, respectively, with $U \subseteq V$. By a counting argument, it is easy to see that there can be no larger value in $I(u,v)$ than $m(u,v) = \frac{1}{6}[v(v-1) - (v-u)(2u+1-v)]$.

Corollary 2. *Suppose $t < u < v$ are admissible with $t \leq \min\{(u-1)/2, v-u-1\}$. Then $I(t,t) \subseteq I(u,v)$.*

Proof: Let $(U,T,\mathcal{B}_1)$ be any ISTS(u,t). By Theorem 1, there exists an ISTS($V,T,\mathcal{B}_2$) with $V \supset U$ and $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$. We may then place on the two copies of T a pair of STS(t) intersecting in any number of blocks in $I(t,t)$. □

For integers $a \leq b$, let $[a,b]$ denote the interval $\{a, a+1, \ldots, b\}$. In [3], it is shown that for $u \neq 9$, $I(u,u) = [0,b] \setminus \{b-1, b-2, b-3, b-5\}$, where $b = u(u-1)/6$. This and Corollary 2 automatically determines roughly the first one-third of the set $I(u,v)$ for $v/u \approx 3/2$. By contrast, the techniques in [2] for determining $I(u,v)$ are successful for $v/u \approx 1$ or 2.

Example 3. Let $t \neq 9$ be admissible, with $b = t(t-1)/6$. Then

$$[0, b] \setminus \{b-1, b-2, b-3, b-5\} \subseteq I(2t+1, 3t+4) \subseteq [0, 3b + (t+1)/2],$$

with the latter containment coming from $m(2t+1, 3t+4) = 3b + (t+1)/2$. It is conjectured in [2] that the upper bound $[0, m(u, v)]$ on $I(u, v)$ is met with equality when $v - u > 2$.

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References

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