Every Plane Graph with Girth at least 4 without 8- and 9-circuits is 3-Choosable[1]

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Abstract

The choice number of a graph G, denoted by $\chi_l(G)$, is the minimum number k such that if we give lists of k colors to each vertex of G, there is a vertex coloring of G where each vertex receives a color from its own list no matter what the lists are. In this paper, we show that $\chi_l(G) \leq 3$ for each plane graph of girth at least 4 which contains no 8- and 9-circuits.

Key words and phrases: circuit, girth, choosable, plane graph;
AMS 2000 Subject Classifications: 05C15, 05C78;

1 Introduction

All graphs considered in this paper are finite, simple plane graphs. $G = (V, E, F)$ denotes a plane graph, with V, E and F being the set of vertices, edges and faces of G respectively. We use $b(f)$ to denote the boundary of a face f, and use $N(f)$ to denote the set of faces adjacent to f. A face is incident with all vertices and edges on $b(f)$. The degree of a vertex u, denoted by $d(u)$, is the order of $N(u)$, the set of vertices adjacent to u. The degree of a face f, denoted by $d(f)$, is the number of edges incident with it, where cut edges are counted twice. A k-vertex (k-face) is a vertex (face) of degree k. If $r \leq k$ or $3 \leq k \leq r$, then a k-vertex (k-face) is called an r^+- or r^--vertex (r^+- or r^--face), respectively. A k-circuit is a circuit on k vertices. The vertex set of a circuit C will also be denoted by C. The girth of G is the length of a shortest circuit of G.

Let f be an h-face. f is called a *light* h-face if all incident vertices are 3^--vertices, and is called a *non-light* h-face otherwise. If f is a non-light h-face, then f is called a *minimal* h-face if all vertices on $b(f)$ except one 4-vertex are 3^--vertices, and a *non-minimal* h-face otherwise.

A color list $L = \{L(v) : v \in V\}$ is a family of color sets assigned to each vertex of G. An L-coloring of G is an assignment to each vertex $v \in V$ from $L(v)$ such that adjacent vertices receive distinct colors. A graph G is called k-choosable if G admits an L-coloring for each color-list L with

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k colors in each list. The choice number of G, denoted by $\chi_l(G)$, is the minimum k such that G is k-choosable.

Thomassen proved that every planar graph is 5-choosable[6]. Examples of plane graphs which are not 4-choosable were given by Voigt in [9]. Voigt and Wirth[11] also presented a 3-colorable non-4-choosable plane graph. Lam et al[3, 4] proved that plane graphs without i-circuits, for $i = 3, 4, 5$ or 6, are 4-choosable. Xu[13, 14] proved that each graphs embedded on surfaces of positive characteristic and in which no two triangles share a common vertex is 4-choosable(Wang and Lih[12], proved the same result on plane graphs).

On 3-choosability, Thomassen proved that every plane graph of girth at least 5 is 3-choosable[7], Alon and Tarsi proved that every planar bipartite graph is 3-choosable[1]. In [10], Voigt and Wirth gave some plane graphs of girth 4 which are not 3-choosable. Lam et al[5] proved that every plane graph with girth at least 4 and contains no 5- and 6-circuits, or contains no 7- and 8-circuits is 3-choosable. Xu [15] proved that every toroidal graph of girth at least 4 which contains no 5-, 6- and 7-circuits, or contains no 6-, 7-, and 8-circuits is 3-choosable.

We concern here with a similar problem, the 3-choosability of plane graphs without 3-circuits. We show that $\chi_l(G) \leq 3$ for all plane graphs without 3-, 8- and 9-circuits.

A result of Alon and Tarsi[1] will be used in the proofs. Let $\overrightarrow{G}$ be a digraph, a spanning subdigraph $\overrightarrow{H}$ of $\overrightarrow{G}$ is called an eulerian subdigraph if $d^+(x) = d^-(x)$ for each $x \in V(H)$. An eulerian subdigraph is called even(odd) if it contains even(odd) number of arcs.

Theorem A [1] *Let $\overrightarrow{D} = (V, A)$ be a digraph, f be an integer function defined on V such that $f(v) = d^+(v) + 1$ for each $v \in V$. If the number of even eulerian subdigraphs (the null digraph consisting only vertices is also counted as an even eulerian subdigraph) differs from the number of odd eulerian subdigrphs, then $\overrightarrow{D}$ is f-choosable. Moreover, the underlying graph of $\overrightarrow{D}$ is f-choosable.*

2 Preliminary Lemmas and Corollaries

A *minimally non-3-choosable* graph is a graph which is not 3-choosable, but every of its proper induced subgraph is 3-choosable. It is clearly that every vertex of a minimally non-3-choosable graph has degree at least 3.

Lemma 1 *[2] Every circuit of even length is 2-choosable.*

Lemma 2 *Let G be a minimally non-3-choosable graph. Then any $2n$-circuit of G contains at least one 4^+-vertex.*

Proof: Suppose to the contrary that $2 \leq d(v) \leq 3$ for all $v \in C$. Let L be a color-list of G with $|L(v)| = 3$ for all $v \in V(G)$. By assumption, there exists ϕ_0, an L_0-coloring of $G_0 = G \setminus C$, where L_0 is the restriction of L to $V(G_0)$.

Let $L' = \{L'(v_i) : 1 \leq i \leq 2n\}$ where $L'(v_i) = L(v_i) \setminus \{\phi_0(u) : u \in N(v_i) \setminus C\}$. It is clear that $|L'(v_i)| \geq 2$. Since even circuits are 2-choosable, there exists an L'-coloring ϕ' on C. An L-coloring of G immediately follows by combining ϕ_0 and ϕ'. This contradiction ends the proof. ∎

Lemma 3 *Let G be a minimally non-3-choosable graph, C_1 and C_2 two even circuits with exactly one vertex v_0 in common. If $d(v_0) = 4$, then at least one of C_1 and C_2 is a non-minimal circuit.*

Proof: Let $V' = C_1 \cup C_2$. If both C_1 and C_2 are minimal, then all vertices in $V' \setminus \{v_0\}$ are 3-vertices. Let L be a color-list of G with $|L(v)| = 3$ for each $v \in V$, and L_0 the restriction of L to $V \setminus V'$. Then, for any $L_0-coloring$ ϕ_0 of $G \setminus V'$, $|L'(v_0)| = 3$ and $|L'(v)| = 2$ for all $v \in V' \setminus \{v_0\}$, where $L'(v) = L(v) \setminus \{\phi_0(u) : u \in N(v) \setminus V'\}$.

Let G' be the subgraph induced by V'. We give G' an orientation $\overrightarrow{G'}$ by making both C_1 and C_2 into oriented circuits. By Theorem A, it is easy to check that G' admits an L'-coloring ϕ'. ϕ_0 together with ϕ' yields an L-coloring of G. This contradiction completes the proof. ∎

Lemma 4 *Let $G=(V,E)$ be a circuit $v_1v_2v_3 \cdots v_nv_1$ with exactly one chord v_1v_k $(3 \leq k \leq n-1)$, L a color-list with $|L(v_1)|=|L(v_k)|= 3$ and $|L(v_i)|= 2$ where $i \neq 1, k$. Then G is L-colorable.*

Proof: We first choose a color $c(v_1) \in L(v_1) \setminus L(v_n)$ for v_1, then choose for $v_2, v_3, \cdots, v_n$ successively from $L(v_i) \setminus L(v_{i-1})$ whenever $i \neq k$, and from $L(v_k) \setminus \{L(v_{k-1}) \cup L(v_1)\}$ whenever $i = k$. ∎

Corollary 1 *Let G be a minimally non-3-choosable plane graph. If f_1 and f_2 are two light faces, then f_1 and f_2 cannot be adjacent.*

3 Main Result

Theorem 1 *Let G be a plane graph of girth at least 4. Then G is 3-choosable if G contains no 8- and 9-circuits.*

Proof: Suppose that G is a counterexample of minimum order. Then, $\delta(G) \geq 3$. For convenience, a 4-face adjacent to exactly i 4-faces is called a 4_i-face, where i=0, 1 or 2. Because G contains neither 8-circuits nor 9-circuits, we have:

(O_1) G contains neither 4_3-faces nor 4_4-faces. Every 4_2-face must be in a configuration as shown in Fig1(a).

(O_2) G contains no adjacent 5-faces;

(O_3) A 6-face is not adjacent to neither 4-faces nor 5-faces;

(O_4) No 7-face can be adjacent to 4-faces;

(O_5) A 5-face can be adjacent to at most one 4_0-face;

(O_6) A 5-face is not adjacent to any 4_2-face;

(O_7) If a 5-face is adjacent to a 4_1-face, then it must be the situation as shown in Fig1(d);

(O_8) A 4_1-face is adjacent to at most one 5-face;

(O_9) Suppose a 10^+-face f is adjacent to three 4-faces on consecutive edges tu, uv and vw on $b(f)$, then at least one of u and v is a 4^+-vertex.

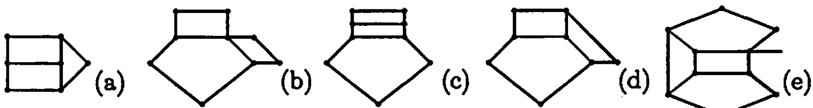

Figure 1: Some configurations of the observations

The proofs of O_1, O_2, O_3, O_4, O_5, O_9 are trivial. If a 5-face adjacent to a 4_2-face, then it must contain a sub-configuration as shown in Figure 1(c), but it yields a 9-circuit, this contradiction give us (O_6). (O_7) and (O_8) can be proved similarly.

Let $\omega(v) = \frac{3d_G(v)}{10} - 1$ if $v \in V(G)$ and $\omega(f) = \frac{d_G(f)}{5} - 1$ if $f \in F(G)$. Applying Euler's formula for plane graphs, $|V|+|F|-|E|= 2$, we have $\sum_{v\in V(G)}(\frac{3\cdot d_G(v)}{10} - 1) + \sum_{v\in F(G)}(\frac{d_G(f)}{5} - 1) = -2$. We will construct a new weight $\omega^*(x)$ by transferring weights from one element to another with the property $\sum_{x\in V\cup F}\omega^*(x) = -2$, and show that $\omega^*(x) \geq 0$ for all $x \in V \cup F$. Then, we get a contradiction and complete the proof.

Weights will be transferred according to the following rules:

(R_1) A face transfers $\frac{1}{30}$ to every incident 3-vertex;

(R_2) A 4-vertex transfers $\frac{1}{10}$ to every incident 4-face, and $\frac{1}{20}$ to every incident 5-face;

(R_3) A 5^+-vertex transfers $\frac{1}{6}$ to every incident 4- or 5-face;

(R_4) A 5-face transfers $\frac{1}{20}$ to every adjacent 4_0-face, and $\frac{1}{15}$ to every adjacent 4_1-face;

(R_5) A 7-face transfers $\frac{1}{20}$ to every adjacent 5-face;

(R_6) Let f be a 10^+-face, $tuvw$ a segment on $b(f)$, and f' a face adjacent to f at uv.

(R_{61}) If f' is a 4-face, f transfers to f':
$\frac{1}{30}$ if f' is a *non-minimal* face;
$\frac{1}{20}$ if f' is a 4_0-face;
$\frac{1}{15}$ if f' is a 4_1-face, or a 4_2-face incident with a 5^+-vertex;
$\frac{1}{12}$ if f' is a 4_2-face and both u and v are 3-vertices;
$\frac{1}{8}$ if f' is a 4_2-face, and neither tu nor vw is incident with a 4-face;
$\frac{1}{10}$ otherwise;

(R_{62}) If f' is a *light* 5-face , f transfers to f',
$\frac{1}{30}$ whenever f' is not adjacent to any 4-face;
$\frac{7}{120}$ whenever f' is adjacent to a 4_0-face;
$\frac{1}{8}$ if f' is adjacent to a 4_1-face;

(R_{63}) If f' is a *non-light* 5-face , f transfers to f',
$\frac{2}{45}$ if f' is a *non-minimal* 5-face;
$\frac{1}{60}$ if f' is not adjacent to any 4-face;
$\frac{1}{30}$ if f' is adjacent to a 4_0-face;
$\frac{7}{60}$ if f' is adjacent to 4_1-face and $d(u) = d(v) = 3$;
$\frac{1}{30}$ if f' is adjacent to a 4_1-face and incident with one 5^+-vertex;
$\frac{1}{20}$ otherwise;

If two or more of the above sub-rules apply, the earliest one takes priority.

Claim 1. $w^*(v) \geq 0$ for every vertex v.
Proof. Let v be a k-vertex. If $k = 3$, then v is incident with three 4^+-faces and therefore $\omega^*(v) = \omega(v) + \frac{3}{30} = 0$.

Suppose $k = 4$. By O_1, v is incident with at most two 4-faces. If v is not incident with any 4-faces, then $\omega^*(v) \geq \omega(v) - \frac{1}{20} \cdot 2 > 0$. If v is incident with exactly one 4-face, then by O_2, the total number of 5-faces incident with v is at most 2, $\omega^*(v) \geq \omega(v) - \frac{1}{10} - \frac{1}{20} \cdot 2 = 0$. If v is incident with two 4-faces, then by O_1, O_5 and O_7, v is not incident with any 5-faces, $\omega^*(v) = \omega(v) - \frac{1}{10} \cdot 2 = 0$.

If $k \geq 5$, according to the analysis as above, at least two of the faces incident with v are not 5^--faces. Therefore $\omega^*(v) \geq \omega(v) - \frac{k-2}{6} = \frac{2k-10}{15} \geq 0$. ∎

Let f be an h-face of G. ($h = 4, 5, 6, 7, 10^+$).
Claim 2. $w^*(f) \geq 0$ if $h = 4$.
Proof. By O_3, O_4, f is adjacent only to 4-, 5-, or 10^+-faces. By R_4 and R_{61}, if f is a 4_0-face or a 4_1-face, the weight transferred to it from a 5-face is equal to that transferred from a 10^+-face. So, we may assume that f is adjacent to 4-faces or 10^+-faces.

If $b(f)$ has least two 4^+-vertices, then by R_1, R_2 and R_{61}, f transfers at most $2 \cdot \frac{1}{30}$ to its incident vertices, receives at least $2 \cdot \frac{1}{10}$ from its incident vertices, and receives at least $2 \cdot \frac{1}{30}$ from its adjacent faces. Therefore, $\omega^*(f) \geq \omega(f) - \frac{2}{30} + \frac{2}{10} + \frac{2}{30} = 0$.

If $b(f)$ contains a 5^+-vertex and three 3-vertices, then by R_{61}, the total weight transferred from adjacent 10^+-faces is $\frac{4}{20}$, $\frac{3}{15}$ or $\frac{2}{15}$, depending on whether f is a 4_0-, 4_1- or 4_2-face respectively. Therefore, $\omega^*(f) \geq \omega(f) - \frac{3}{30} + \frac{1}{6} + \frac{2}{15} = 0$.

Now, assume that f is a minimal 4-face. Then by R_1, R_2 and R_{61}, $\omega^*(f) \geq \omega(f) - \frac{3}{30} + \frac{1}{10} + \frac{4}{20} = 0$ whenever f is a 4_0-face, and $\omega^*(f) \geq \omega(f) - \frac{3}{30} + \frac{1}{10} + \frac{3}{15} = 0$ whenever f is a 4_1-face. If f is a 4_2-face, then f must be adjacent to two 10^+-faces at two consecutive edges. Suppose $b(f) = tuvwt$ and f is adjacent to two 10^+-faces f_1 and f_2 at edges uv and vw respectively. Then t is a 3-vertex. If $d(w) = 4$, then $d(u) = d(v) = 3$ and f_2 cannot be adjacent to any 4-faces at the two edges adjacent to vw, and by R_{61}, the weight transferred from f_1 and f_2 across uv and vw are $\frac{1}{12}$ and $\frac{1}{8}$ respectively, $\omega^*(f) \geq \omega(f) - \frac{3}{30} + \frac{1}{10} + \frac{1}{12} + \frac{1}{8} > 0$. A similar conclusion can be reached if $d(u) = 4$. If $d(v) = 4$, then the weights transferred from f_1 and f_2 across uv and vw are both $\frac{1}{10}$, and so $\omega^*(f) \geq \omega(f) - \frac{3}{30} + \frac{1}{10} + \frac{2}{10} = 0$.∎

Claim 3. $w^*(f) \geq 0$ if $h = 5$.

Proof. By the choice of G, f is adjacent to only 4- or 7- or 10^+-face. We divide the proof into three cases depending on the type of f.

Case 1 f is a *light* 5-face.

If $N(f)$ contains no 4-faces, by R_5 and R_{62}, the worst situation for $\omega^*(f)$ occurs whenever $N(f)$ contains no 7-faces, then by R_1 and R_{62}, $\omega^*(f) \geq \omega(f) - \frac{5}{30} + \frac{1}{30} \cdot 5 = 0$

If $N(f)$ contains a 4_0-face(as shown in Fig 2(a)), by R_5 and R_{62}, the worst situation for $\omega^*(f)$ occurs whenever the number of 10^+-faces in $N(f)$ is as small as possible. By O_4, $N(f)$ contains at most two 7-faces, so $\omega^*(f) \geq \omega(f) - \frac{5}{30} - \frac{1}{20} + \frac{1}{20} \cdot 2 + \frac{7}{120} \cdot 2 = 0$ by R_1, R_4, R_5 and R_{62}.

If $N(f)$ contains a 4_1-face, again by R_5 and R_{62}, the worst situation for $\omega^*(f)$ occurs whenever f is contained in a configuration as shown in Fig2 (b) by O_4, and by R_1, R_4, R_5 and R_{62}, $\omega^*(f) \geq \omega(f) - \frac{5}{30} - \frac{1}{15} \cdot 2 + \frac{1}{20} + \frac{1}{8} \cdot 2 = 0$.

Case 2 f is a *minimal* 5-face incident with a 4-vertex and four 3-vertices.

If $N(f)$ contains no 4-faces, By R_5 and R_{63}, the worst situation occurs whenever f is only adjacent to 10^+-faces, then by R_1, R_2 and R_{63}, $\omega^*(f) \geq \omega(f) - \frac{4}{30} + \frac{1}{20} + \frac{1}{60} \cdot 5 = 0$

If $N(f)$ contains a 4_0-face, the worst situation occurs whenever f is contained in a configuration as shown in Fig2(c), then by R_1, R_2, R_4 and R_{63}, $\omega^*(f) \geq \omega(f) - \frac{4}{30} - \frac{1}{20} + \frac{1}{20} + \frac{1}{30} \cdot 4 = 0$.

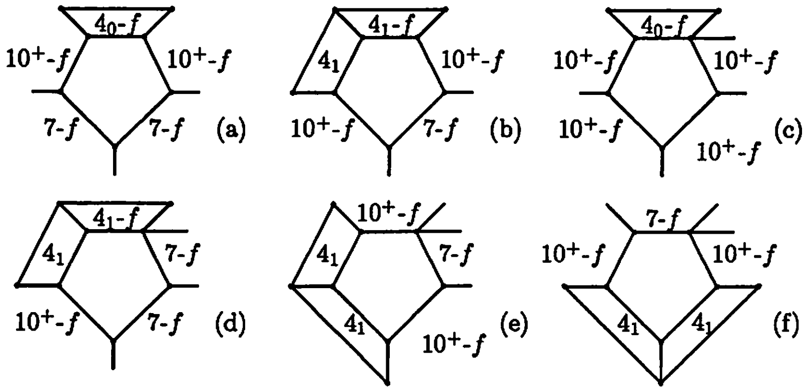

Figure 2: Some configurations while h=5

If $N(f)$ contains a 4_1-face, all three possible configurations that may contain f are shown in Fig2 (d), (e) and (f). In every configuration, one can find an edge, say xy, on $b(f)$ and a 10^+-face f' such that $d(x) = d(y) = 3$ and one of x and y is shared by f, f' and a 4_1-face. By R_{63}, such a 10^+-face f' transfers $\frac{7}{60}$ to f. By R_1, R_4, R_2, R_5 and R_{63}, $\omega^*(f) \geq \omega(f) - \frac{4}{30} - \frac{1}{15}\cdot 2 + \frac{1}{20} + \frac{1}{20}\cdot 2 + \frac{7}{60} = 0$.

Case 3 f is a *non-minimal* 5-face.

If $b(f)$ contains at least two 4^+-vertices, since f get $\frac{1}{20}$ and $\frac{2}{45}$ from every adjacent 7-face and 10^+-face respectively, and $N(f)$ contains at most two 4_1-faces, by R_1, R_4, R_2, and R_{63}, $\omega^*(f) \geq \omega(f) - \frac{3}{30} - \frac{1}{15}\cdot 2 + \frac{1}{20}\cdot 2 + \frac{2}{45}\cdot 3 = 0$.

If $b(f)$ contains a 5^+-vertex, f get $\frac{1}{6}$ from the 5^+-vertex. Even if $N(f)$ contains two 4_1-faces and three 10^+-faces, $\omega^*(f) \geq \omega(f) - \frac{4}{30} - \frac{1}{15}\cdot 2 + \frac{1}{6} + \frac{2}{45}\cdot 3 > 0$. ∎

Claim 4. $w^*(f) \geq 0$ if $6 \leq h \leq 7$.

Proof. If $d(f) = 6$, by Lemma 2 and O_3, $b(f)$ contains at least one 4^+-vertex, and $N(f)$ contains only 6^+-faces, then $\omega^*(f) \geq \omega(f) - \frac{5}{30} = \frac{1}{30} > 0$.

If $d(f) = 7$, by O_4, $N(f)$ contains only 5^+-faces. Let r be the number of 5-faces in $N(f)$. Then, by O_2, there are at most $14 - 2r$ 3-vertices on $b(f)$ whenever $r \geq 4$. Therefore, $\omega^*(f) \geq \omega(f) - \frac{7}{30} - \frac{3}{20} > 0$ if $r \leq 3$, and $\omega^*(f) \geq \omega(f) - \frac{14-2r}{30} - \frac{r}{20} = \frac{r-4}{60} \geq 0$ if $r \geq 4$. ∎

Claim 5. $w^*(f) \geq 0$ if $h \geq 10$.

Proof. We assign a *quota* of $\frac{1}{30}$ and $\frac{1}{15}$ to each vertex and edge on $b(f)$, respectively. By the discharging rules, f transfers only to either 3-vertices on $b(f)$, or 4-face or 5-face in $N(f)$. By adjusting the quotas, we will show that the total quotas, $\frac{h}{30} + \frac{h}{15}$, are enough to cover all transfers to incident vertices and adjacent faces, and then $\omega^*(f) \geq \omega(f) - \frac{h}{30} - \frac{h}{15} \geq 0$.

For each 4^+-vertex v on $b(f)$, the quota assigned to v can be donated to the edges incident with v on $b(f)$. For an edge uv on $b(f)$ and a face $\bar{f}$ adjacent to f at uv, if the quota assigned to uv is bigger than the weight transferred from f to $\bar{f}$, then the unused quota can be also donated to the edges adjacent to uv on $b(f)$.

Let tu, uv and vw be three consecutive edges on $b(f)$, and let f_1, f' and f_2 be the faces adjacent to f at tu, uv and vw, respectively. Without loss of generality, we assume that $d(f') = 4$ or 5. Let s be the weight transferred from f to f'. If $s \leq \frac{1}{15}$, we are done. So, we assume that $s > \frac{1}{15}$. Then by R_{61}, R_{62} and R_{63}, f' must be a 4_2-face, or a *light* 5-face adjacent to a 4_1-face, or a *non-light* 5-face adjacent to a 4_1-face with $d(u) = d(v) = 3$.

Case 1: f' is a 4_2-face;

By our assumption $s > \frac{1}{15}$, we have that f' is a 4_2-face.

If $d(u) = d(v) = 3$, then $s = \frac{1}{12}$, f' is adjacent to both f_1 and f_2, and by $(O_3), (O_4)$ and (O_6), one of f_1 and f_2, say f_2, is a 10^+-face, and the weight transferred from f to f_2 across vw is 0. Therefore, $\frac{1}{30}$, half of the unused quota of vw, may be donated to uv, adjusting its quota to $\frac{1}{15} + \frac{1}{30} = \frac{1}{10} > \frac{1}{12} = s$.

Now we may assume by symmetry that $d(u) = 3$ and $d(v) = 4$. By the discharging rules, $s = \frac{1}{8}$ if neither tu nor vw is incident with a 4-face, and $s = \frac{1}{10}$ otherwise by R_{61}.

If $s = \frac{1}{8}$, then neither f_1 nor f_2 is a 4-face. Since $d(u) = 3$, $d(f_1) \geq 10$ by (O_6), and $\frac{1}{2} \cdot \frac{1}{15} = \frac{1}{30}$, half of the unused quota of tu, may be donated to uv. Since f' is a 4_2-face and $d(f_1) \geq 10$, $d(f_2) \geq 10$ and half of the quota of vw may be donated to uv also. Therefore, the quota of uv can be adjusted to $\frac{1}{15} + 2 \cdot \frac{1}{30} > \frac{1}{8} = s$.

If $s = \frac{1}{10}$, then f_1 must be a 4-face. If $d(f_2) = 4$, f_2 cannot be a *minimal* face by Lemma 3 and the weight transferred from f to f_2 is at most $\frac{1}{30}$ by R_{61}, then both $\frac{1}{60}$ that is half of the unused quota of v and $\frac{1}{60} = \frac{1}{2} \cdot (\frac{1}{15} - \frac{1}{30})$ that is half of the unused quota of vw may be donated to uv, and hence the quota of uv can be adjusted to $\frac{1}{15} + \frac{1}{60} + \frac{1}{60} = \frac{1}{10} = s$. If $d(f_2) \neq 4$, then f, f_1, f' and f_2 must be the configuration as shown in Figure 3. By R_{63}, the weight transferred across from f to f_2 is at most $\frac{1}{20}$, here, f_2 is a minimal 5-face adjacent to a 4_1-face. Therefore, both $\frac{1}{30}$ that is unused quota of v and $\frac{1}{2} \cdot (\frac{1}{15} - \frac{1}{20}) = \frac{1}{120}$ that is half of the unused quota of vw may be donated to uv, and the quota of uv can be adjusted to $\frac{1}{15} + \frac{1}{30} + \frac{1}{120} = \frac{13}{120} > \frac{1}{10} = s$.

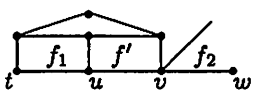

Figure 3: The situation of $d(u) = 3, d(v) = 4, d(f_1) = 4$ and $d(f_2) \neq 4$

Before proving the following case, we first give a useful observation.

Observation 10 *Let f_0, f_1, f' and f_2 be the faces adjacent to f at four consecutive edges st, tu, uv and vw on $b(f)$. If there is a configuration as* *shown in Figure 4 with a light 5-face f', then uv can get at least $\frac{1}{30}$ from the edges and vertices on the path from s to u on $b(f)$.*

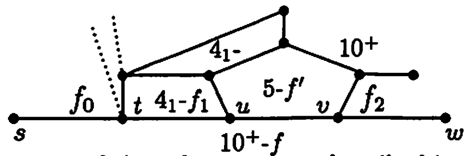

Figure 4: Subconfiguration as described in Observation 10

Proof: By Lemma 2, f_1 must be incident with a 4^+-vertex. We will prove this Observation according to the degree of t.

If $d(t) = 3$, $d(f_0) \geq 10$ by O_1 and O_8, and by R_{61}, f transfers $\frac{1}{15}$ to f_1, then $\frac{1}{2} \cdot \frac{1}{15} = \frac{1}{30}$ that is half of the unused quota of st can be donated to uv.

If $d(t) = 4$ and f_1 is a *non-minimal* 4-face, then by R_{61}, the weight transferred from f to f_1 is at most $\frac{1}{30}$, and hence both $\frac{1}{60}$ that is half of the unused quota of t and $\frac{1}{2} \cdot (\frac{1}{15} - \frac{1}{30}) = \frac{1}{60}$ that is half of the unused quota of tu can be donated to uv.

If $d(t) = 4$ and f_1 is a *minimal* 4-face, then f_0 must be either a *non-minimal* 4-face by Lemma 3 or a 5^+-face. If f_0 is a 5^+-face, then $\frac{1}{30}$ that is the unused quota of t can be donated to uv. If f_0 is a *non-minimal* 4-face(see Figure 5), then $\frac{1}{60} = \frac{1}{2} \cdot (\frac{1}{15} - \frac{1}{30})$ that is half of the unused quota of st by R_{61}, and $\frac{1}{2} \cdot \frac{1}{30}$ that is half of the unused quota of t can be donated to uv.

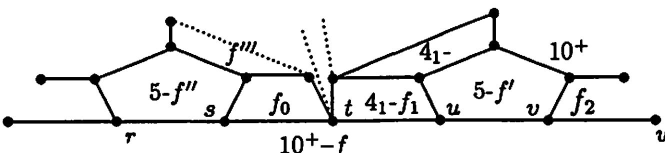

Figure 5: Subconfiguration as described in Observation 10

If $d(t) = 5$, and either $d(f_0) \geq 5$ or $d(f_0) = 4$ but $d(f'') \neq 5$ or $d(f''') \neq 4$(see Figure 5), then $\frac{1}{30}$, the unused quota of t can be donated to uv. In the case that $d(t) = 5$, $d(f_0) = 4$, $d(f'') = 5$ and $d(f''') = 4$, we have, by O_6(a 5-face is not adjacent to any 4_2-face) and O_8(a 4_1-face is adjacent to at most one 5-face), all faces incident with t are 6^+-faces except f_0 and f_1. Therefore, $|F_4(t)| = 2$ and $|F_5(t)| = 0$, $\omega^*(t) = \omega(t) - \frac{2}{6} = \frac{1}{6}$ by (R_3), and $\frac{1}{12} = \frac{1}{2} \cdot \frac{1}{6}$ that is the unused weight of t can be donated to uv.

Now we consider the case when $d(t) \geq 6$. We call a symmetric configuration induced by f, f_0 and f_1 as shown in Figure 5 as a *butterfly*. By an easy calculation, one may find that if t is incident with l *butterflies*, then the number of 6^+-faces incident with t is at least $l + 2$, and hence

$\omega^*(t) \geq \omega(t) - \frac{d(t)-l-2}{6} = \frac{4d+5l-20}{30} > \frac{l}{6}$. By a similar argument to that of $d(t) = 5$, the unused weight and quota of t are enough to cover the demands as claimed in Observation 10. ∎

Now, we return back to the proof of our main theorem.

Case 2: f' is a 5-face adjacent to 4_1-faces. Since G contains neither 8-circuits nor 9-circuits, at most one of f_1 and f_2 is a 4-face.

By R_{62} and R_{63}, f transfers $\frac{1}{8}$ to f' if f' is a *light* 5-face, and transfers $\frac{7}{60}$ to f' if f' is a *non-light* 5-face with $d(u) = d(v) = 3$.

If neither f_1 nor f_2 is a 4-face, then $d(f_1) \geq 10$ and $d(f_2) \geq 10$(as shown in Figure 6(a)), both $\frac{1}{2} \cdot \frac{1}{15} = \frac{1}{30}$ that is half of the unused quota of tu and $\frac{1}{30}$ that is half of the unused quota of vw may be donated to uv, and then the quota of uv can be adjusted to $\frac{1}{15} + \frac{1}{30} + \frac{1}{30} > \frac{1}{8} = s > \frac{7}{60}$.

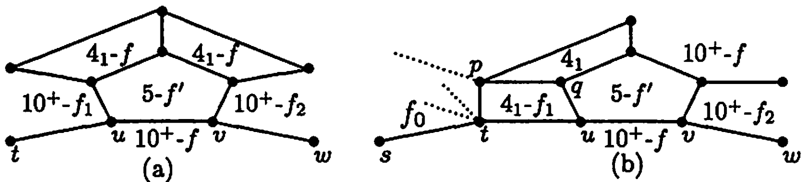

Figure 6: Two configurations of Case 2

If one of f_1 and f_2, say f_1 by symmetry, is a 4-face, then $d(f_2) \geq 10$(as shown in Figure 6(b)). By Observation 10, uv can be donated $\frac{1}{30}$ from the vertices and edges on the left of u. By adding $\frac{1}{2} \cdot \frac{1}{15}$ that is half of the unused quota of edge vw, we get that the quota of uv can also be adjusted to $\frac{1}{15} + \frac{1}{30} + \frac{1}{30} > \frac{1}{8} = s$.

In all of the above cases, the weight transferred from f to f' across uv is less or equal to adjusted quota. This ends the proof of Claim 5.

By Claims 1 to 5, we get that $\omega^*(x) \geq 0$ for each $x \in V \cup F$, i.e.,

$$0 \leq \sum_{x \in V \cup F} \omega^*(x) = \sum_{x \in V \cup F} \omega(x) = -2.$$

This contraction completes the proof. ∎

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