Face antimagic labelings of plane graphs P_a^b

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Abstract

Suppose G is a finite plane graph with vertex set $V(G)$, edge set $E(G)$ and face set $F(G)$. The paper deals with the problem of labeling the vertices, edges and faces of a plane graph G in such a way that the label of a face and labels of vertices and edges surrounding that face add up to a weight of that face. A labeling of a plane graph G is called *d-antimagic* if for every number s, the s-sided face weights form an arithmetic progression of difference d. In this paper we investigate the existence of d-antimagic labelings for special class of plane graphs.

1 Introduction

In this paper, we shall only consider the finite, simple undirected graphs. Given a simple undirected graph G, the vertex set is denoted by $V(G)$ and the edge set is denoted by $E(G)$ (for other graph theoretic notations, see [18] and [19]).

A *labeling* of *type* $(1,1,1)$ assigns labels from the set $\{1,2,...,|V(G)|+|E(G)|+|F(G)|\}$ to the vertices, edges and faces of plane graph G in such a way that each vertex, edge and face receives exactly one label and each number is used exactly once as a label.

If we label only vertices (respectively edges, faces) we call such a labeling a *vertex* (respectively *edge, face*) *labeling* and alternatively the labeling is said to be of *type* $(1,0,0)$ (respectively *type* $(0,1,0)$, *type* $(0,0,1)$).

A labeling of *type* $(1,1,0)$ is a bijection from the set $\{1,2,...,|V(G)|+$

$|E(G)|\}$ to the vertices and edges of plane graph G.

The *weight* of a face under a labeling is the sum of the labels (if present) carried by that face and the edges and vertices surrounding it.

A labeling of a plane graph G is called *d-antimagic* if for every number s, the set of s-sided face weights is $W_s = \{a_s, a_s + d, a_s + 2d, ..., a_s + (f_s - 1)d\}$ for some integers a_s and d, $d \geq 0$, where f_s is the number of s-sided faces. We allow different sets W_s for different s.

Other types of antimagic labelings were investigated by Hartsfield and Ringel in [14] and by Bodendiek and Walther in [9].

If $d = 0$ then Ko-Wei Lih in [15] called such labelings *magic* (*face magic*). Ko-Wei Lih studied face magic labelings for *wheels*, *friendship graphs* and *prisms*. 0-antimagic labelings of type $(1,1,1)$ for *m-antiprisms, fans, bipyramids, Mőbius ladders, grids graphs* and *honeycomb* are described in [1, 2, 3, 4, 5]. 1-antimagic labeling for certain classes of plane graphs are given in [10, 11, 12]. d-antimagic labelings of prisms and generalized Petersen graphs $P(n, 2)$ can be found in [6, 8, 16]. Other results about face-antimagic labelings are shown in [7, 13, 17].

2 Plane graph P_a^b

Let a and b be integers, $a \geq 3$ and $b \geq 2$. Let $y_1, y_2, \ldots, y_a$ be fixed verties, we connect the vertices y_i and y_{i+1} by means of b internally disjoint paths p_i^j of length $i+1$ each, $1 \leq i \leq a-1$, $1 \leq j \leq b$. Let $y_i, x_{i,j,1}, x_{i,j,2}, \ldots, x_{i,j,i}, y_{i+1}$ be the vertices of path p_i^j. The resulting graph embedded in the plane is denoted by P_a^b, where $V(P_a^b) = \{y_i : 1 \leq i \leq a\} \bigcup \bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{x_{i,j,k} : 1 \leq k \leq i\}$ and $E(P_a^b) = \bigcup_{i=1}^{a-1} \{y_i x_{i,j,1} : 1 \leq j \leq b\} \bigcup \bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{x_{i,j,k} x_{i,j,k+1} : 1 \leq k \leq i-1\} \bigcup \bigcup_{i=1}^{a-1} \{x_{i,j,i} y_{i+1} : 1 \leq j \leq b\}$.

The face set $F(P_a^b)$ contains $b-1$ (2i+2)-sided faces, $1 \leq i \leq a-1$, and one external infinite face. Let $v = |V(P_a^b)| = \frac{ab(a-1)}{2} + a$, $e = |E(P_a^b)| = \frac{b(a-1)(a+2)}{2}$ and $f = |F(P_a^b)| = (a-1)(b-1)+1$.

Kathiresan and Ganesan [12] described d-antimagic labelings of type $(1,1,1)$ for the plane graph P_a^b, $a \geq 2$, $b \geq 2$ and $d \in \{0,1,2,3,4,6\}$.

In this paper, we investigate the existence of d-antimagic labelings of type $(1,1,1)$ for P_a^b for many other values of parameter d.

3 Using known labelings

Let $\lfloor n \rfloor$ be the greatest integer smaller than or equal to n.

Kathiresan and Ganesan [12] defined vertex and edge labelings of the plane graph P_a^b, $a \geq 3$, $b \geq 2$, in the following way:

$$\alpha_1(y_i) = \frac{b}{2}(i-1)(i-2) + i \quad \text{if} \quad 1 \leq i \leq a.$$

If $1 \leq i \leq a-1$ and $1 \leq j \leq b$ then

$$\alpha_1(x_{i,j,k}) = \begin{cases} \frac{bi(i-1)}{2} + i + 1 + \frac{j+1}{2} & \text{if } i \text{ and } j \text{ are odd, } k=1 \\ \frac{bi(i-1)}{2} + \lfloor \frac{b+1}{2} \rfloor + i + 1 + \frac{j}{2} & \text{if } i \text{ is odd, } j \text{ is even, } k=1 \\ \frac{bi(i-1)}{2} + i + 1 + j & \text{if } i \text{ is even and } k=1 \\ \frac{bi(i-1)}{2} + kb + i + 2 - j & \text{if } k \text{ is even, } 2 \leq k \leq i \\ \frac{bi(i-1)}{2} + (k-1)b + i + 1 + j & \text{if } k \text{ is odd, } 3 \leq k \leq i \end{cases}$$

$$\beta_1(y_i x_{i,j,1}) = \tfrac{b}{2}((i+1)i-2) + j \quad \text{if } 1 \leq i \leq a-1 \text{ and } 1 \leq j \leq b.$$

If $1 \leq i \leq a-1$ and $1 \leq j \leq b$ then

$$\beta_1(x_{i,j,k} x_{i,j,k+1}) = \begin{cases} \frac{bi(i+1)}{2} + kb + 1 - j & \text{if } k \text{ is odd, } 1 \leq k < i \\ \frac{bi(i+1)}{2} + (k-1)b + j & \text{if } k \text{ is even, } 2 \leq k < i \end{cases}$$

$$\beta_1(x_{i,j,i} y_{i+1}) = \begin{cases} \frac{bi(i+3)}{2} + 1 - j & \text{if } i \text{ is odd} \\ \frac{b((i+3)i-2)}{2} + \frac{j+1}{2} & \text{if } i \text{ is even and } j \text{ is odd} \\ \frac{bi(i+3)}{2} - \lfloor \frac{b}{2} \rfloor + \frac{j}{2} & \text{if } i \text{ and } j \text{ are even.} \end{cases}$$

It was proved in [12] that the vertex labeling α_1 and the edge labeling $v+\beta_1$ and the face labeling with values in the set $\{v+e+1, v+e+2, \dots, v+e+f\}$ combine to 0-antimagic and 2-antimagic labeling of type (1,1,1).

Let us denote the weights of the $(2i+2)$-sided faces of P_a^b under a vertex labeling α and an edge labeling β as follows:

If $1 \leq i \leq a-1$ and $1 \leq j \leq b-1$ then

$$w(f_{i,j}) = \alpha(y_i) + \beta(y_i x_{i,j,1}) + \alpha(x_{i,j,1}) + \beta(x_{i,j,1} x_{i,j,2}) + \alpha(x_{i,j,2}) + \ldots +$$
$$+\alpha(x_{i,j,i}) + \beta(x_{i,j,i} y_{i+1}) + \alpha(y_{i+1}) + \beta(y_{i+1} x_{i,j+1,i}) + \alpha(x_{i,j+1,i}) +$$
$$+\beta(x_{i,j+1,i} x_{i,j+1,i-1}) + \alpha(x_{i,j+1,i-1}) + \ldots + \alpha(x_{i,j+1,1}) + \beta(x_{i,j+1,1} y_i).$$

We will denote the external infinite face by f_∞.

For the investigation of antimagicness of the plane graph P_a^b, we shall only consider the face weights of $(2i+2)$-sided faces $f_{i,j}$, $1 \le i \le a-1$, $1 \le j \le b-1$. We will not specify the weight of the infinite face f_∞ because there is only one such face.

In this section, we shall present d-antimagic labelings of a plane graph P_a^b for various values of d by using vertex labelings and edge labelings defined by Kathiresan and Ganesan [12].

Lemma 1 *For $a \ge 3$ and $b \ge 2$, the plane graph P_a^b has an a-antimagic labeling and $(a-2)$-antimagic labeling of type $(1,1,1)$.*

Proof

It was proved in [12] that labelings α_1 and $\beta_1 + v$ combine to a labeling of type $(1,1,0)$ and the weights of the $(2i+2)$-sided faces are

$$w(f_{i,j}) = \begin{cases} A - b + \lfloor \frac{b+1}{2} \rfloor + j & \text{if } i \text{ is odd, } 1 \le j \le b-1 \\ A - \lfloor \frac{b}{2} \rfloor + j & \text{if } i \text{ is even and } 1 \le j \le b-1 \end{cases}$$

where

$$A = b(2i^3 + 4i^2 - i) + 6i + 2i^2 + 2 + (i+1)a(ba - b + 2).$$

Now we define face labelings γ_1 and γ_1' as bijections from the set $\{v+e+1, v+e+2, \ldots, v+e+f\}$ onto the faces of P_a^b as follows

$$\gamma_1(f_{i,j}) = v + e + (a-1)(j-1) + i \quad \text{for } 1 \le i \le a-1 \text{ and } 1 \le j \le b-1$$

$$\gamma_1'(f_{i,j}) = v + e + f - (a-1)(j-1) - i$$
$$\text{for } 1 \le i \le a-1 \text{ and } 1 \le j \le b-1$$

$$\gamma_1(f_\infty) = \gamma_1'(f_\infty) = v + e + f.$$

It can be seen that the labelings $\alpha_1, \beta_1 + v, \gamma_1$ and $\alpha_1, \beta_1 + v, \gamma_1'$ give an a-antimagic and $(a-2)$-antimagic labeling of type $(1,1,1)$ respectively. □

We consider vertex labeling α_2 and edge labeling β_2 of P_a^b which were also defined by Kathiresan and Ganesan in [12] and used for proving that P_a^b has a 1-antimagic and 3-antimagic labeling of type $(1,1,1)$.

$$\alpha_2(y_i) = \alpha_1(y_i) \text{ for } 1 \le i \le a$$

If $1 \le i \le a-1$, $1 \le j \le b$ and $1 \le k \le i$ then

$$\alpha_2(x_{i,j,k}) = \begin{cases} \frac{bi(i-1)}{2} + kb + i + 2 - j & \text{for } k \text{ even} \\ \frac{bi(i-1)}{2} + (k-1)b + i + 1 + j & \text{for } k \text{ odd} \end{cases}$$

$$\beta_2(y_i x_{i,j,1}) = \beta_1(y_i x_{i,j,1})$$

$$\beta_2(x_{i,j,k} x_{i,j,k+1}) = \beta_1(x_{i,j,k} x_{i,j,k+1})$$

$$\beta_2(x_{i,j,i} y_{i+1}) = \begin{cases} \frac{bi(i+3)}{2} + 1 - j & \text{if } i \text{ is odd} \\ \frac{b}{2}((i+3)i - 2) + j & \text{if } i \text{ is even.} \end{cases}$$

Lemma 2 *For $a \ge 3$ and $b \ge 2$, the plane graph P_a^b has an $(a+1)$-antimagic labeling and $(a-3)$-antimagic labeling of type $(1,1,1)$.*

Proof

Label the vertices and edges of P_a^b by α_2 and $\beta_2 + v$, respectively, resulting in a labeling of type $(1,1,0)$. The weights of the $(2i+2)$-sided faces are

$$w(f_{i,j}) = 2v(i+1) + bi(2i^2 + 4i - 1) - b + 2i^2 + 6i + 2 + 2j$$

$$\text{for } 1 \le i \le a-1 \text{ and } 1 \le j \le b-1.$$

If we complete face labeling γ_1 and γ_1' then the labelings α_2, $\beta_2 + v$, γ_1 and α_2, $\beta_2 + v$, γ_1' combine to $(a+1)$-antimagic and $(a-3)$-antimagic labeling of type $(1,1,1)$, respectively. □

The following labelings α_3 and β_3 were defined in [12].

$$\alpha_3(y_i) = \alpha_1(y_i)$$

for $1 \le i \le a$.

If $1 \le i \le a-1$ and $1 \le j \le b$ then

$\alpha_3(x_{i,j,k}) =$

$$\begin{cases} \frac{bi(i-1)}{2} + i + 1 + \frac{j+1}{2} & \text{if } i \text{ and } j \text{ are odd and } k = 1 \\ \frac{bi(i-1)}{2} + \lfloor \frac{b+1}{2} \rfloor + i + 1 + \frac{j}{2} & \text{if } i \text{ is odd, } j \text{ is even and } k = 1 \\ \frac{bi(i-1)}{2} + i + 1 + j & \text{if } i \text{ is even and } k = 1 \\ \frac{bi(i-1)}{2} + kb + i + 2 - j & \text{if } k \text{ is even and } 4 \le k \le i \\ \frac{b}{2}(i^2 - i + 4) + i + 2 - j & \text{if } i \text{ is odd and } k = 2 \\ \frac{b}{2}(i^2 - i + 2) + i + 1 + j & \text{if } i \text{ is even } k = 2 \\ \frac{bi(i-1)}{2} + (k-1)b + i + 1 + j & \text{if } k \text{ is odd and } 3 \le k \le i. \end{cases}$$

$$\beta_3(y_i x_{i,j,1}) = \beta_1(y_i x_{i,j,1})$$

$$\beta_3(x_{i,j,k} x_{i,j,k+1}) = \beta_1(x_{i,j,k} x_{i,j,k+1})$$

$$\beta_3(x_{i,j,i} y_{i+1}) = \begin{cases} \frac{b}{2}(i^2 + 3i - 2) + j & \text{if } i \text{ is odd} \\ \frac{b}{2}(i^2 + 3i - 2) + \frac{j+1}{2} & \text{if } i \text{ is even and j is odd} \\ \frac{bi(i+3)}{2} - \lfloor \frac{b}{2} \rfloor + \frac{j}{2} & \text{if } i \text{ and } j \text{ are even.} \end{cases}$$

Lemma 3 *For $a \ge 3$ and $b \ge 2$, the plane graph P_a^b has a $(a+4)$-antimagic labeling and $|(a-6)|$-antimagic labeling of type* $(1,1,1)$.

Proof

Kathiresan and Ganesan in [12] showed that combining the vertex labeling α_3 and the edge labeling $\beta_3 + v$ gives a 5-antimagic labeling of type $(1,1,0)$.

Let the weights of the $(2i+2)$-sided faces under a 5-antimagic labeling of type $(1,1,0)$ be

$w(f_{i,j}) = A(i) + 5j$ for $1 \le i \le a-1$ and $1 \le j \le b-1$.

Complete the face labeling with γ_1 and γ_1'. We can see that the resulting labeling determined by labelings α_3, $\beta_3 + v$, γ_1 and α_3, $\beta_3 + v$, γ_1' is $(a+4)$-antimagic labeling and $|(a-6)|$-antimagic labeling of type $(1,1,1)$, respectively.

4 New labelings and results

In this section we construct new vertex and edge and face labelings of plane graph P_a^b and prove d-antimagicness of P_a^b for some values of the parameter d.

Lemma 4 *If $a \geq 3$ and $b \geq 2$ then the plane graph P_a^b has a 6-antimagic labeling of type* $(1,1,0)$.

Proof

Define the vertex labeling $\alpha_4 : V(P_a^b) \to \{1,2,\dots,v\}$ and the edge labeling $\beta_4 : E(P_a^b) \to \{1,2,\dots,e\}$ in the following way.

$$\alpha_4(y_i) = \alpha_1(y_i) \text{ for } 1 \leq i \leq a.$$

If i is even, $2 \leq i < a$ and $k = 1$ then

$$\alpha_4(x_{i,j,k}) = \frac{bi(i-1)}{2} + i + 1 + j \text{ for } 1 \leq j \leq b.$$

If i and k are even or i and k are odd, $1 \leq i < a$, $1 \leq k \leq i$ then

$$\alpha_4(x_{i,j,k}) = \frac{bi(i-1)}{2} + i + 1 + (k-1)b + j \text{ for } 1 \leq j \leq b.$$

If i is even and k is odd or i is odd and k is even, $1 \leq i < a$ and $2 \leq k \leq i$, then

$$\alpha_4(x_{i,j,k}) = \frac{bi(i-1)}{2} + i + 2 + kb - j \text{ for } 1 \leq j \leq b.$$

If $1 \leq i \leq a-1$ and $1 \leq j \leq b$ then

$$\beta_4(y_i x_{i,j,1}) = \beta_1(y_i x_{i,j,1})$$

$$\beta_4(x_{i,j,k} x_{i,j,k+1}) = \beta_1(x_{i,j,k} x_{i,j,k+1})$$

$$\beta_4(x_{i,j,i} y_{i+1}) = \frac{bi(i+3)}{2} - b + j.$$

Label the vertices and the edges of P_a^b by α_4 and $\beta_4 + v$, respectively. These labelings realize a labeling of type $(1, 1, 0)$ and the weights of the $(2i + 2)$-sided faces are

$$w(f_{i,j}) = 2v(i + 1) + bi(2i^2 + 4i - 1) + 2i^2 + 6i - 3b + 2 + 6j$$

where $1 \le i \le a - 1$ and $1 \le j \le b - 1$.

We can see that the resulting labeling is 6-antimagic. □

Theorem 1 *For $a \ge 3$, $b \ge 2$ and $d \in \{5, 7, |a - 7|, a + 5\}$, the plane graph P_a^b has a d-antimagic labeling of type $(1, 1, 1)$.*

Proof

Define the face labelings γ_2 and γ_3 from the set $\{v+e+1, v+e+2, \dots, v+e+f\}$ onto the faces of P_a^b as follows

$$\gamma_2(f_{i,j}) = v + e + f - (b - 1)(i - 1) - j$$

$$\gamma_3(f_{i,j}) = v + e + (b - 1)(i - 1) + j$$

$$\gamma_2(f_\infty) = \gamma_3(f_\infty) = v + e + f.$$

If we combine 6-antimagic labeling of type $(1, 1, 0)$ from Lemma 4, and face labeling γ_2 or γ_3, we obtain a 5-antimagic labeling or 7-antimagic labeling of type $(1, 1, 1)$, respectively.

On the other hand, if we complete the 6-antimagic labeling of type $(1, 1, 0)$ by face labelings γ_1 or γ_1', then the resulting labeling is $(a + 5)$-antimagic or $|(a - 7)|$-antimagic labeling of type $(1, 1, 1)$, respectively. □

Lemma 5 *If $a \ge 3$ and $b \ge 2$ then the graph P_a^b has a 3-antimagic labeling of type $(1, 1, 0)$.*

Proof

Let $\alpha_5 : V(P_a^b) \to \{1, 2, \dots, v\}$ be the vertex labeling and $\beta_5 : E(P_a^b) \to \{1, 2, \dots, e\}$ be the edge labeling of P_a^b where

$$\alpha_5(y_i) = \alpha_1(y_i)$$

and

$$\alpha_5(x_{i,j,k}) = \alpha_3(x_{i,j,k}).$$

If $1 \le i \le a-1$, $1 \le j \le b$ and $1 \le k < i$ then

$$\beta_5(y_i x_{i,j,1}) = \frac{bi(i+1)}{2} + 1 - j$$

$$\beta_5(x_{i,j,k}x_{i,j,k+1}) = \begin{cases} \frac{bi(i+1)}{2} + (k-1)b + j & \text{if } k \text{ is odd} \\ \frac{bi(i+1)}{2} + kb - j + 1 & \text{if } k \text{ is even} \end{cases}$$

$$\beta_5(x_{i,j,i}y_{i+1}) = \begin{cases} \frac{bi(i+3)}{2} - j + 1 & \text{if } i \text{ is odd} \\ \frac{bi(i+3)}{2} - \frac{j}{2} + 1 & \text{if } i \text{ and } j \text{ are even} \\ \frac{bi(i+3)}{2} - \lfloor\frac{b}{2}\rfloor - \frac{j-1}{2} & \text{if } i \text{ is even and } j \text{ is odd.} \end{cases}$$

It can be seen that the labelings α_5 and β_5 are bijections and that α_5 and $\beta_5 + v$ give a labeling of type (1,1,0). The weights of the $(2i+2)$-sided faces form an arithmetical progression with difference $d = 3$. □

Theorem 2 *For $a \ge 3$, $b \ge 2$, the plane graph P_a^b has $|(a-4)|$-antimagic labeling and $(a+2)$-antimagic labeling of type $(1,1,1)$.*

Proof

Define the face labelings γ_4 and γ_5 from the set $\{v+e+1, v+e+2, ldots, v+e+f\}$ onto the faces of P_a^b as follows

$$\gamma_4(f_{i,j}) = \begin{cases} v+e+f-(a-1)(j-1)-i \\ \qquad \text{if } i \text{ is odd, } 1 \le i \le a-1,\ 1 \le j \le b-1 \\ v+e+f-(a-1)(b-j-1)-i \\ \qquad \text{if } i \text{ is even, } 2 \le i \le a-1,\ 1 \le j \le b- \end{cases}$$

$$\gamma_5(f_{i,j}) = \begin{cases} v+e+(a-1)(j-1)+i \\ \qquad \text{if } i \text{ is odd, } 1 \le i \le a-1,\ 1 \le j \le b-1 \\ v+e++(a-1)(b-j-1)+i \\ \qquad \text{if } i \text{ is even, } 2 \le i \le a-1,\ 1 \le j \le b-1 \end{cases}$$

$$\gamma_4(f_\infty) = \gamma_5(f_infty) = v + e + f.$$

Label the vertices and edges and faces by labeling α_5,$\beta_5 + v$ and γ_4 or α_5, $\beta_5 + v$ and γ_5, respectively. From previous lemma follows that the labelings γ_5 and $\beta_5 + v$ give the 3-antimagic labeling of type $(1, 1, 0)$, and thus labelings α_5, $\beta_5 + v$ and γ_4 combine to $(a + 2)$-antimagic labeling and α_5, $\beta_5 + v$ and γ_5 combine to $(|a - 4|)$-antimagic labeling of type $(1, 1, 1)$.□

Lemma 6 *If $a \geq 3$ and $b \geq 2$ then the graph P_a^b has a $(2a - 2)$-antimagic labeling of type $(1, 1, 0)$.*

Proof

Define the bijections α_6 and β_6 as follows:

$\alpha_6 : V(P_a^b) \to \{1, 2, \ldots, v\}$, $\beta_6 : E(P_a^b) \to \{1, 2, \ldots, e\}$,

$$\alpha_6(y_i) = i \text{ for } 1 \leq i \leq a.$$

If $1 \leq i \leq a - 1$, $1 \leq j \leq b$ and $1 \leq k < i$ then

$$\alpha_6(x_{i,j,k}) = \begin{cases} j(a-1) + 1 + i & \text{if } k = 1 \\ a(b+1) + b/2(i^2 - 3i + 2k - 4) + j & \text{if } k \text{ is even} \\ a(b+1) + b/2(i^2 - 3i + 2k - 2) - j + 1 & \text{if } k \text{ is odd, } k \geq 3 \end{cases}$$

$$\beta_6(y_i x_{i,j,1}) = \beta_1(y_i x_{i,j,1})$$

$$\beta_6(x_{i,j,k} x_{i,j,k+1}) = \beta_1(x_{i,j,k} x_{i,j,k+1})$$

$$\beta_6(x_{i,j,i} y_{i+1}) = \frac{bi(i+3)}{2} - j + 1.$$

Label the vertices and edges of P_a^b by α_6 and $\beta_6 + v$. By direct computation we obtain that the set of weights of the $(2i + 2)$-sided faces form an arithmetic progression with difference $2a - 2$. Thus the resulting labeling is $(2a - 2)$-antimagic labeling of type $(1, 1, 0)$. □

Theorem 3 *For $a \geq 3$, $b \geq 2$ and $d \in \{2a-3, 2a-1, a-1, 3a-3\}$, the plane graph P_a^b has a d-antimagic labeling of type* $(1,1,1)$.

Proof

In light of Lemma 6, it follows that

- Labelings α_6, $\beta_6 + v$ and γ_2 combine to $(2a-3)$-antimagic labeling of type $(1,1,1)$.
- Labelings α_6, $\beta_6 + v$ and γ_3 combine to $(2a-1)$-antimagic labeling of type $(1,1,1)$.
- Labelings α_6, $\beta_6 + v$ and γ_1 combine to $(3a-3)$-antimagic labeling of type $(1,1,1)$.
- Labelings α_6, $\beta_6 + v$ and γ_1' combine to $(a-1)$-antimagic labeling of type $(1,1,1)$.

□

Lemma 7 *If $a \geq 3$, $b \geq 2$ then plane graph P_a^b has a $(2a+2)$-antimagic labeling of type* $(1,1,0)$.

Proof

We label the vertices of P_a^b by $\alpha_7(y_i) = \alpha_4(y_i)$ and $\alpha_7(x_{i,j,k}) = \alpha_4(x_{i,j,k})$.

If $1 \leq i \leq a-1$, $1 \leq j \leq b$ and $1 \leq k < i$ then we label the edges by $\beta_7 + v$ where

$$\beta_7(y_i x_{i,j,1}) = (j-1)(a-1) + i$$

$$\beta_7(x_{i,j,k} x_{i,j,k+1}) = \begin{cases} \frac{bi(i-1)}{2} + (a-2+k)b + j & \text{if } k \text{ is odd} \\ \frac{bi(i-1)}{2} + (a-1+k)b - j + 1 & \text{if } k \text{ is even} \end{cases}$$

$$\beta_7(x_{i,j,i} y_{i+1}) = \begin{cases} \frac{bi(i+1)}{2} + j + b(a-2) & \text{if } i \text{ is odd} \\ \frac{bi(i+1)}{2} + 1 - j + b(a-1) & \text{if } i \text{ is even.} \end{cases}$$

It is not difficult to check that the values of α_7 are $1, 2, \dots, v$ and the values of β_7 are $1, 2, \dots, e$. By direct computation we obtain that under the labelings α_7 and $\beta_7 + v$, the weights of the $(2i+2)$-sided faces form an arithmetical progression with difference $2a+2$. □

Theorem 4 *For $a \geq 3$, $b \geq 2$ and $d \in \{a+3, 2a+1, 2a+3, 3a+1\}$, the plane graph P_a^b has a d-antimagic labeling of type $(1,1,1)$.*

Proof

In light of Lemma 7, it follows that

- Labelings α_7, $\beta_7 + v$ and γ_2 combine to $(2a+1)$-antimagic labeling of type $(1,1,1)$.
- Labelings α_7, $\beta_7 + v$ and γ_3 combine to $(2a+3)$-antimagic labeling of type $(1,1,1)$.
- Labelings α_7, $\beta_7 + v$ and γ_1 combine to $(3a+1)$-antimagic labeling of type $(1,1,1)$.
- Labelings α_7, $\beta_7 + v$ and γ_1' combine to $(a+3)$-antimagic labeling of type $(1,1,1)$.

□

Lemma 8 *If $a \geq 3$, $b \geq 2$ then plane graph P_a^b has a $(4a-2)$-antimagic labeling of type $(1,1,0)$.*

Proof

We label the vertices of P_a^b by $\alpha_8(y_i) = \alpha_6(y_i)$ and $\alpha_8(x_{i,j,k}) = \alpha_6(x_{i,j,k})$ and label the edges by $\beta_8(y_i x_{i,j,1}) = \beta_7(y_i x_{i,j,1})$, $\beta_8(x_{i,j,k} x_{i,j,k+1}) = \beta_7(x_{i,j,k} x_{i,j,k+1})$ and $\beta_8(x_{i,j,i} y_{i+1}) = \beta_7(x_{i,j,i} y_{i+1})$.

It is easy to check that we get the desired results. □

Theorem 5 *For $a \geq 3$, $b \geq 2$ and $d \in \{4a-1, 4a-3, 5a-3, 3a-1\}$, the plane graph P_a^b has a d-antimagic labeling of type $(1,1,1)$.*

Proof

In light of Lemma 8, it follows that

- Labelings α_8, $\beta_8 + v$ and γ_2 combine to $(4a-3)$-antimagic labeling of type $(1,1,1)$.
- Labelings α_8, $\beta_8 + v$ and γ_3 combine to $(4a-1)$-antimagic labeling of type $(1,1,1)$.
- Labelings α_8, $\beta_8 + v$ and γ_1 combine to $(5a-3)$-antimagic labeling of type $(1,1,1)$.
- Labelings α_8, $\beta_8 + v$ and γ_1' combine to $(3a-1)$-antimagic labeling of type $(1,1,1)$.

□

Lemma 9 *If $a \geq 3$ and $b \geq 2$ then plane graph P_a^b has a $(6a-6)$-antimagic labeling of type $(1,1,0)$.*

Proof

We label the vertices of P_a^b by $\alpha_9(y_i) = \alpha_6(y_i)$ and $\alpha_9(x_{i,j,k}) = \alpha_6(x_{i,j,k})$.

Now, we construct the edge labeling $\beta_9 : E(P_a^b) \to \{1, 2, \dots, e\}$ in the following way

$$\beta_9(y_i x_{i,j,1}) = \beta_7(y_i x_{i,j,1}).$$

If $1 \leq i \leq a-1$, $1 \leq j \leq b$ and $1 \leq k < i$ then

$$\beta_9(x_{i,j,i} y_{i+1}) = (a-1)(b+j-1) + i$$

$\beta_9(x_{i,j,k} x_{i,j,k+1}) =$

$$\begin{cases} \frac{bi(i-3)}{2} + (2a-1+k)b - j + 1 & \text{if } i \text{ is odd, } i \geq 3, \text{ and } k \text{ is even} \\ & \text{or if } i \text{ is even and } k \text{ is odd} \\ \frac{bi(i-3)}{2} + (2a-2+k)b + j & \text{if both } i \text{ and } k \text{ are even} \\ & \text{or both } i \text{ and } k \text{ are odd, } i \geq 3. \end{cases}$$

It is not difficult to check that the vertex labeling α_9 and the edge labeling $\beta_9 + v$ give us a $(6a-6)$-antimagic labeling of type $(1,1,0)$. □

Theorem 6 *For $a \geq 3$, $b \geq 2$ and $d \in \{6a-5, 6a-7, 7a-7, 5a-5\}$, the plane graph P_a^b has a d-antimagic labeling of type $(1,1,1)$.*

Proof

In light of Lemma 9, it follows that

- Labelings α_9, $\beta_9 + v$ and γ_2 combine to $(6a - 7)$-antimagic labeling of type $(1, 1, 1)$.
- Labelings α_9, $\beta_9 + v$ and γ_3 combine to $(6a - 5)$-antimagic labeling of type $(1, 1, 1)$.
- Labelings α_9, $\beta_9 + v$ and γ_1 combine to $(7a - 7)$-antimagic labeling of type $(1, 1, 1)$.
- Labelings α_9, $\beta_9 + v$ and γ_1' combine to $(5a - 5)$-antimagic labeling of type $(1, 1, 1)$.

□

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