Blocking sets in small finite linear spaces*

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Abstract

We classify all finite linear spaces on at most 15 points admitting a blocking set. There are no such spaces on 11 or fewer points, one on 12 points, one on 13 points, two on 14 points, and five on 15 points. The proof makes extensive use of the notion of the weight of a point in a 2-coloured finite linear space, as well as the distinction between minimal and non-minimal 2-coloured finite linear spaces. We then use this classification to draw some conclusions on two open problems on the 2-colouring of configurations of points.

1   Introduction

1.1   Finite linear spaces and blocking sets

By a *finite linear space* (or f.l.s.) S we mean a triple consisting of a finite set of *points* (denoted by capital letters $A, B, \ldots$), a finite set of *lines* (denoted oy lower case letters $\ell, m, \ldots$), and an *incidence relation* $\in$ between the set of points and the set of lines ($P \in \ell$ being read as "*P lies on ℓ*" or "*ℓ passes through P*"), such that the following properties hold:

I1 For any two distinct points P and Q there is a unique line passing through P and Q (denoted by PQ).

I2 Each line passes through at least two points.

I3 There exist three non-collinear points.

By I1 and I2 we may identify a line with the set of points lying on it. By I3 we do not allow a finite linear space to be "one-dimensional". Of course we could have omitted I3 (as is done by certain authors) and then we would only have to change the formulations of certain statements. Other equivalent terms used in the literature are simple matroid of rank 3, and

pairwise balanced incomplete block design. A *proper finite linear space* is an f.l.s. with each line passing through at least 3 points. Proper finite linear spaces are important when considering generalizations of the Sylvester-Gallai Theorem (see Section 1.4). A proper f.l.s. is also called an SG design in [22]. Our terminology follows recent papers such as [19, 7, 8].

A *blocking set* of a finite linear space S is a subset B of S such that each line of S contains a point of B and a point of $S \setminus B$. There is extensive literature on blocking sets of block designs (see [1, 26] and their references), and in particular of affine and projective planes (see [2, Chapter 8], [20, Chapter 13], [5, 10, 27] and their references). Not as much has been done on blocking sets for general linear spaces. Cameron [13] mentions in passing that blocking sets may be studied in general finite linear spaces. Hahn has two papers [17, 18] on how the existence of certain types of blocking sets in finite linear spaces forces the space to be almost a projective or affine plane. Hoffman, Lindner and Phelps [21] and Franek, Griggs, Lindner and Rosa [15] show that a linear space on v points with four points on each line that admits a blocking set exists whenever $v \equiv 1, 4 \pmod{12}$. Batten, Coolsaet and Street [4] consider the existence of blocking sets in finite linear spaces where each line passes through either 2 or 4 points. They show that apart from the above-mentioned cases $v \equiv 1, 4 \pmod{12}$, such spaces also exist for $v \equiv 2 \pmod{12}$, and do not exist for all remaining congruence classes, except perhaps $v \equiv 10 \pmod{12}$, where they conjecture that no such finite linear spaces admit blocking sets. Ling [23] proves that an f.l.s. with five points on each line that admits a blocking set exists whenever $v \equiv 1, 5 \pmod{20}$. See also [14, 9, 16] for related results.

Our main result is a classification of all finite linear spaces of at most 15 points that admit blocking sets.

Theorem 1.

1. *No finite linear space on ≤ 11 points admits a blocking set.*

2. *The punctured projective plane of order 3 is the only f.l.s. on 12 points with a blocking set.*

3. *The projective plane of order 3 is the only f.l.s. on 13 points with a blocking set.*

4. *There are exactly two finite linear spaces on 14 points with blocking sets. Both are subsets of the projective plane of order 4.*

5. *There are exactly five finite linear spaces on 15 points with blocking sets. All except one are subsets of the projective plane of order 4, with the exceptional one not embeddable in any Desarguesian projective space. Furthermore, all except one are extensions of one of the two 14-point spaces with blocking sets.*

It is rather surprising that there are so few finite linear spaces with blocking sets up to 15 points. In all of them the blocking sets are unique up to complements and automorphisms of the space, with the exception of one of the 15-point spaces which admits two essentially different blocking sets. In Section 2 we describe the finite linear spaces mentioned in the above theorem and then give a precise formulation of the theorem (see Theorem 2). The proof of the theorem is an extensive case analysis, as is to be expected. The two main tools used in the proof are the notions of *minimal* two-coloured f.l.s. (see Section 1.3) and of the *weight* of a point in a two-coloured f.l.s. that we introduce below (see Lemma 7 and the definition preceding it).

Note that finite linear spaces up to 12 points have been enumerated [6], with 13 points still out of reach. A brute-force computer proof of our result is therefore not yet feasible.

1.2   General results

We now give an overview of general results on the existence of blocking sets which we use in the proof of our classification.

It is well known that the projective plane of order 3 has a blocking set (see Example 1). It is easily seen that the punctured projective plane of order 3 still has a blocking set (Example 2). This is a minimal example: we show that an f.l.s. on at most 11 points does not admit a blocking set. More generally, if an f.l.s. admits a blocking set, the blocking set must have size ≥ 6 (Lemma 6). Also, for each $v \geq 12$, there exists an f.l.s. on v points admitting a blocking set (Proposition 1). An f.l.s. with a blocking set of 6 points has at most 13 points (Proposition 2).

It is a folklore result that an f.l.s. with at most three points on a line does not admit a blocking set. Thus a necessary condition for an f.l.s. to admit a blocking set is that the f.l.s. has a line with at least 4 points. A further necessary condition is that there are at least 4 lines passing through *each* point (Lemma 5), and more generally, that the f.l.s. is not the union of 3 lines.

1.3   MR geometries: Definitions

Since the complement of a blocking set is also a blocking set, we find it more convenient to formulate our results in terms of 2-colourings. A 2-*colouring* χ of the finite linear space S is a function assigning the colour red or green to each point of S. In all the figures we use o for red and • for green. A line is *monochromatic* if all points incident with it have the same colour. A *proper* (*weak*) 2-colouring χ is a 2-colouring with the property that no line is monochromatic. An *MR geometry* (S, χ) is a properly 2-coloured finite

linear space. Thus each colour class of an MR geometry (S, χ) forms a blocking set of the underlying finite linear space S. Conversely, any blocking set of S determines a proper 2-colouring. Thus an MR geometry is exactly the same as a finite linear space with a blocking set that is singled out. An *isomorphism* between two MR geometries is an isomorphism between the underlying finite linear spaces such that points of the same colour are mapped to points of the same colour.

An MR geometry (S', χ') is a *one-point extension* of the MR geometry (S, χ) if S is a subset of S' with one point less than S', and the restriction of χ' to S is χ. Note that in the sequel, when we refer to extensions, we will always mean one-point extensions. An MR geometry (S, χ) is *minimal* if it is not a one-point extension of any MR geometry, equivalently, if for all $P \in S$ such that $S \setminus \{P\}$ is non-collinear, the restriction of χ to $S \setminus \{P\}$ is not a proper 2-colouring of $S \setminus \{P\}$. Thus an MR geometry is minimal if we cannot obtain an MR geometry by deleting any single point.

1.4 The Sylvester-Gallai and Motzkin-Rabin theorems

Our motivation for finding small MR geometries arose from questions related to the following theorem (see [24] or [25] for a proof):

Motzkin-Rabin Theorem. *If a finite non-collinear set S of points in the real plane (affine or projective) is 2-coloured, then there exists a line ℓ such that $\ell \cap S$ has size at least two and is monochromatic.*

In the above terminology this theorem asserts that no MR geometry can be embedded into the real projective plane. It is an open question whether an MR geometry can be embedded into the complex projective plane.

Problem 1. *Does there exist an MR geometry that can be embedded into the complex projective plane?*

Since none of the MR geometries on at most 15 points are embeddable in the projective plane over a division ring of characteristic $\neq 2$, we have the following corollary to Theorem 1:

Corollary 1. *If there exists a finite set S of non-collinear points in the projective plane over a division ring of characteristic $\neq 2$ that can be 2-coloured such that no line determined by two points in S is monochromatic, then S has size at least 16.*

A theorem closely related to the Motzkin-Rabin theorem is the Sylvester-Gallai theorem [11]:

Sylvester-Gallai Theorem. *For any finite non-collinear set S of points in the real plane (affine or projective) there exists a line ℓ such that $\ell \cap S$ has size two.*

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In terms of finite linear spaces, this theorem asserts that no proper finite linear space (see 1.1) can be embedded into the real plane. These two theorems have proofs that are very closely related (see [25]), leading to the question whether the one implies the other, given some reduced set of axioms. For example, if one assumes the axioms I1, I2, I3 and one of the theorems, does the other follow from it? First of all it is clear that the Motzkin-Rabin theorem does not imply the Sylvester-Gallai theorem, as the Fano plane PG(2, 2) shows. The converse implication is the following open question:

Problem 2. *Does an MR geometry always contain a proper linear space as a subspace?*

Our classification implies that if the answer to this question is negative, then a counterexample would need at least 16 points. Indeed, all MR geometries on at most 15 points that are not themselves proper, contain a Fano plane.

Corollary 2. *If there exists an MR geometry that does not contain a proper f.l.s., then it must have size at least 16.*

For a survey on these and related problems, see [25].

1.5   Overview of the paper

In the next section we give a description of all MR geometries on 15 and fewer points. In Section 3 we introduce special notation and prove general properties of MR geometries. The remainder of the paper contains the proof of the classification. In the proof we distinguish between minimal and non-minimal MR geometries. The non-minimal geometries on n points may be generated by extending the MR geometries on $n - 1$ points. The results are indicated in the next section with proofs in the Appendix. It then remains to enumerate the minimal MR geometries. Since in Section 3 it is shown that there is no MR geometry on < 12 points, an MR geometry on 12 points must be minimal. In Section 4 we show that there is a unique minimal MR geometry on 12 points (the punctured projective plane of order 3), and that there is no minimal MR geometry on 13 points. In Section 5 we show that there are only two minimal MR geometries on 14 points, and in Section 6 that there is only one minimal MR geometry on 15 points, thus completing the proof of the classification (Theorems 1 and 2).
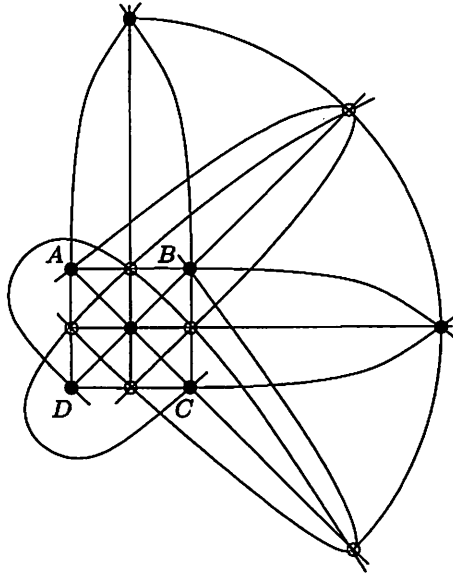
Figure 1: $\mathrm{MR}_{13} = \mathrm{PG}(2,3)$—The projective plane of order 3

2   Examples of small MR geometries

We denote the projective plane of order $q = 2, 3, 4$ by $\mathrm{PG}(2, q)$. See Figure 1 for a standard drawing of $\mathrm{PG}(2,3)$. It will be convenient for us to represent $\mathrm{PG}(2,4)$ as follows: The set of points is $\{i_j : i \in \mathbb{Z}_7, j \in \mathbb{Z}_3\}$, and the lines are

$$\{i_j, (i+1)_j, (i+3)_j, f(i)_{j+1}, f(i)_{j+2}\}, \quad (i \in \mathbb{Z}_7, j \in \mathbb{Z}_3),$$

where f is defined by

i	0	1	2	3	4	5	6
$f(i)$	6	3	0	5	1	2	4

See Figure 2. Clearly each $\{i_j : i \in \mathbb{Z}_7\}$ is a Baer subplane isomorphic to the Fano plane $\mathrm{PG}(2, 2)$.

Proposition 1. *For any $v \geq 12$ there exists an MR geometry on v points.*

Proof. Consider a complete quadrilateral in a projective plane $\mathbb{P}^2$ of order q with sides $\ell_1, \ell_2, \ell_3, \ell_4$, and set $P_{ij} = \ell_i \cap \ell_j$. Colour all points on ℓ_1 and ℓ_2 green, except for P_{14} and P_{23} which we colour red, and colour all the points on ℓ_3 and ℓ_4 red except for P_{13} and P_{24} which we colour green.

$$
\begin{array}{lll}
0_0 1_0 3_0 6_1 6_2 & 0_1 1_1 3_1 6_2 6_0 & 0_2 1_2 3_2 6_0 6_1 \\
1_0 2_0 4_0 3_1 3_2 & 1_1 2_1 4_1 3_2 3_0 & 1_2 2_2 4_2 3_0 3_1 \\
2_0 3_0 5_0 0_1 0_2 & 2_1 3_1 5_1 0_2 0_0 & 2_2 3_2 5_2 0_0 0_1 \\
3_0 4_0 6_0 5_1 5_2 & 3_1 4_1 6_1 5_2 5_0 & 3_2 4_2 6_2 5_0 5_1 \\
4_0 5_0 0_1 1_1 1_2 & 4_1 5_1 0_1 1_2 1_0 & 4_2 5_2 0_2 1_0 1_1 \\
5_0 6_0 1_0 2_1 2_2 & 5_1 6_1 1_1 2_2 2_0 & 5_2 6_2 1_2 2_0 2_1 \\
6_0 0_0 2_0 4_1 4_2 & 6_1 0_1 2_1 4_2 4_0 & 6_2 0_2 2_2 4_0 4_1
\end{array}
$$

Figure 2: PG$(2,4)$—The projective plane of order 4

We then already have that any line of $\mathbb{P}^2$ passes through a green and a red point, except for the line $P_{14}P_{23}$ which does not pass through a green point, and $P_{13}P_{24}$ which does not pass through a red point. This is rectified by choosing $A \in P_{14}P_{23}$ and $B \in P_{13}P_{24}$ with $A \neq B$, and colouring A green and B red. This is possible if $q \geq 3$.

We now have a properly 2-coloured subset of $4q$ points of $\mathbb{P}^2$ with the property that every line of $\mathbb{P}^2$ already passes through a green and a red point. We may add points and colour them arbitrarily and still have an MR geometry. We may thus find an MR geometry for all v satisfying $4q \leq v \leq q^2 + q + 1$. By taking $q = 2^k$ for $k \geq 3$, we cover all $v \geq 32$. By taking $q = 3, 4, 5$, we cover all remaining $v \geq 12$ except 14 and 15. These two cases are covered by Examples 3 to 9 below. $\qquad \Box$

Example 1 (MR_{13} = PG$(2,3)$—The projective plane of order 3).
Any complete quadrilateral (see A, B, C, D in Figure 1) and its three diagonal points ($AB \cap CD, AC \cap BD, AD \cap BC$) form a blocking set. We thus obtain an MR geometry with six points of one colour and seven of the other. We denote it by MR_{13}.

It is well-known that PG$(2,3)$ can be embedded into a Desarguesian projective plane iff the underlying division ring has characteristic 3. It is a proper finite linear space. MR_{13} is not minimal and it cannot be extended to a 14-point MR geometry (Lemma 8). $\qquad \Box$

Example 2 (MR_{12} = PG$(2,3)^*$—The punctured projective plane of order 3).
Remove any diagonal point of the seven-point blocking set above to obtain a 2-coloured punctured projective plane with six points of each colour, which we denote by MR_{12}.

It is also well-known that the punctured PG$(2,3)$ can be embedded into a Desarguesian projective plane iff the underlying division ring has characteristic 3. It is a proper finite linear space. By Lemma 6, MR_{12} is minimal. It is easily seen that there is only one way to extend MR_{12} to an MR geometry on 13 points, viz. by putting back the diagonal point. $\qquad \Box$
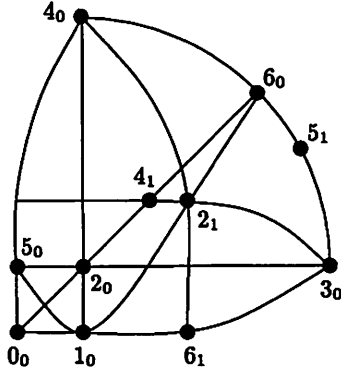
Figure 3: Proof of Lemma 1

All the MR geometries on 14 and 15 points have subsets isomorphic to the following subset of $PG(2,4)$:

Lemma 1. *Let $\mathbb{D}$ be a division ring, and S the subspace of $PG(2,4)$ with points $\{i_0 : i \in \mathbb{Z}_7\} \cup \{2_1, 4_1, 5_1, 6_1\}$. If S can be embedded in $\mathbb{P}^2(\mathbb{D})$, then $\mathbb{D}$ has characteristic 2 and $x^2 + x + 1$ has a root in $\mathbb{D}$ (equivalently, $\mathbb{D}$ contains the field on four elements), and the embedding of S can be extended to an embedding of $PG(2,4)$ in $\mathbb{P}^2(\mathbb{D})$.*

For any two embeddings of S there is an automorphism of $\mathbb{P}^2(\mathbb{D})$ mapping the one embedded $PG(2,4)$ onto the other.

Proof. We use projective coordinates. Without loss of generality we may assume that $0_0 = [0,0,1], 1_0 = [1,0,1], 5_0 = [0,1,1], 3_0 = [1,0,0], 4_0 = [0,1,0]$ (by applying an appropriate projective transformation of $\mathbb{P}^2(\mathbb{D})$). See Figure 3. Then the collinearities $1_0 2_0 4_0$ and $2_0 3_0 5_0$ force $2_0 = [1,1,1]$. The collinearity $3_0 4_0 6_0 5_1$ implies that 6_0 and 5_1 are at infinity. Also, $0_0 2_0 6_0$ forces $6_0 = [1,1,0]$ and $1_0 5_0 6_0$ forces $6_0 = [1,-1,0]$. It follows that $1 = -1$ and the characteristic of $\mathbb{D}$ is 2. (Up to now we have merely created the standard coordinatization of the Fano plane $\{i_0 : i \in \mathbb{Z}_7\}$.)

Since $2_1 \notin 0_0 3_0$, $2_1 \notin 0_0 5_0$, and $2_1 \notin 3_0 4_0$, we have $2_1 = [a,b,1]$ for some $a,b \neq 0$. By $2_1 4_0 6_1$ and $0_0 3_0 6_1$ we then have $6_1 = [a,0,1]$ (hence $a \neq 1$). By $0_0 2_1 5_1$ and $2_0 5_1 6_1$ we have that $0_0 2_1$ and $2_0 6_1$ are parallel in the affine plane $\{[x,y,1] : x,y \in \mathbb{D}\}$. Thus $ba^{-1} = (a+1)^{-1}$, hence $b = (a+1)^{-1}a$. By $0_0 2_0 4_1 6_0$ we have $4_1 = [x,x,1]$ for some $x \neq 0,1$. By $2_1 4_1 3_0$ we then obtain $4_1 = [b,b,1]$. Since $4_1 \notin 1_0 4_0$, we have $b \neq 1$. By $1_0 4_1 5_1$ we have that $1_0 4_1$ is also parallel to $0_0 2_1$ and $2_0 6_1$, thus $b(b+1)^{-1} = ba^{-1}$, and $b = a+1$. Substituting into the previous equation $b = (a+1)^{-1}a$ we obtain $(a+1)^{-1}a = a+1$ from which follows $a^2 + a + 1 = 0$. Now the projective

coordinates of all the points of S are uniquely determined, as well as the remaining points of PG(2, 4). □

Example 3 (MR_{14}^1—complement of a Baer subplane of PG(2, 4)).

MR_{14}^1 is the subspace of PG(2, 4) induced by two of the Baer subplanes, say

$$\{i_0 : i \in \mathbb{Z}_7\} \cup \{i_1 : i \in \mathbb{Z}_7\}$$

to be definite, where we colour each i_0 green and each i_1 red.

This space is the example of an f.l.s. on 14 points with either 2 or 4 points on a line and admitting a blocking set given in [4]. By Lemma 1, MR_{14}^1 can be embedded into a Desarguesian projective plane iff the underlying division ring has characteristic 2 and $x^2 + x + 1$ has a root, and the embedding is projectively unique. MR_{14}^1 is minimal (proven below in Lemma 8). There are exactly two ways of extending MR_{14}^1 to a 15-point MR geometry: one is called MR_{15}^1 (Example 5), and the other MR_{15}^2 (Example 6). See the Appendix for a proof that these are the only one-point extensions. □

Example 4 (MR_{14}^2).

MR_{14}^2 is the subspace of PG(2, 4) induced by

$$\{i_0 : i \in \mathbb{Z}_7\} \cup \{2_1, 4_1, 5_1, 6_1, 0_2, 1_2, 3_2\},$$

where we colour each i_0 green and the remaining points red. MR_{14}^2 has the same embeddability properties as MR_{14}^1 (Lemma 1), and is minimal (Lemma 8). There are four ways to extend MR_{14}^2: to $MR_{15}^1, MR_{15}^{3r}, MR_{15}^{3g}$, and MR_{15}^4 (Examples 5, 7 and 8). See the Appendix for a proof that there are no other one-point extensions. □

The following four examples on 15 points are extensions of MR geometries on 14 points.

Example 5 (MR_{15}^1).

There are two ways to obtain MR_{15}^1 as an extension of an MR geometry on 14 points. One is the extension $MR_{14}^1 \cup \{6_2\}$. The point 6_2 may be given either colour to give two isomorphic MR geometries. To be definite we colour 6_2 green. The other is the extension $MR_{14}^2 \cup \{6_2\}$, where in this case 6_2 is forced to be red.

The following is an isomorphism between the two representations:

0_0	1_0	2_0	3_0	4_0	5_0	6_0	0_1	1_1	2_1	3_1	4_1	5_1	6_1	6_2
0_2	3_2	4_1	1_2	2_1	5_1	6_2	1_0	0_0	5_0	3_0	4_0	2_0	6_0	6_1

By Lemma 1, MR_{15}^1 has the same embeddability properties as the MR geometries on 14 points. □

Example 6 (MR_{15}^2—closed complement of a Baer subplane of PG(2, 4)).

MR_{15}^2 is the extension of MR_{14}^1 where we add a point ∞ to each of the lines $i_0 i_1$, $i \in \mathbb{Z}_7$ of MR_{14}^1, and colour it green, say.

It is a proper finite linear space, being case (xii) in Brouwer's classification of proper finite linear spaces on 15 points [12]. By Lemma 1, MR_{15}^2 cannot be embedded into a Desarguesian projective plane, since in the (unique) embedding found in the proof of Lemma 1, the lines corresponding to $2_0 2_1, 4_0 4_1, 6_0 6_1$ are not concurrent in $\mathbb{P}^2(\mathbb{D})$. $\quad\square$

Example 7 (MR_{15}^{3r} and MR_{15}^{3g}).

Both MR_{15}^{3r} and MR_{15}^{3g} have the same underlying linear space: $MR_{14}^2 \cup \{4_2\}$. In MR_{15}^{3r} we colour 4_2 red, and in MR_{15}^{3g} we colour it green.

These two MR geometries are not isomorphic (no automorphism of the underlying finite linear space preserves the colour classes). Indeed, MR_{15}^{3r} has a colour class isomorphic to the Fano plane, while no colour class of MR_{15}^{3g} is a Fano plane. By Lemma 1, MR_{15}^{3r} and MR_{15}^{3g} have the same embeddability properties as the MR geometries on 14 points. $\quad\square$

Example 8 (MR_{15}^4).

MR_{15}^4 is the extension $MR_{14}^2 \cup \{3_1\}$, with 3_1 coloured red.

By Lemma 1, MR_{15}^4 has the same embeddability properties as the MR geometries on 14 points. $\quad\square$

Example 9 (MR_{15}^5).

We obtain the underlying f.l.s. of MR_{15}^5 by choosing two lines ℓ and m of PG(2, 4), and removing four points from ℓ and two from m, such that the intersection of ℓ and m is not among the removed points. To be definite, we delete $0_1, 1_1, 3_1, 6_2$ from $\ell = 6_0 6_2$ and $0_2, 3_2$ from $m = 6_0 6_1$, and let the green points be $\{0_0, 1_0, 3_0, 4_0, 5_1, 1_2, 5_2\}$, and the red points $\{2_0, 5_0, 6_0, 2_1, 4_1, 6_1, 2_2, 4_2\}$.

This is the unique minimal MR geometry on 15 points (Proposition 6 in Section 6). MR_{15}^5 is a proper finite linear space, being case (vii) in Brouwer's classification [12]. By Lemma 1, MR_{15}^5 has the same embeddability properties as the MR geometries on 14 points. $\quad\square$

The relationships between the different MR geometries is shown in Figure 4.

We may now state our main result in an exact form.

Theorem 2. *Up to isomorphism, the only MR geometries on $v \leq 15$ points are*

$$MR_{12}, MR_{13}, MR_{14}^1, MR_{14}^2, MR_{15}^1, MR_{15}^2, MR_{15}^{3r}, MR_{15}^{3g}, MR_{15}^4, MR_{15}^5.$$
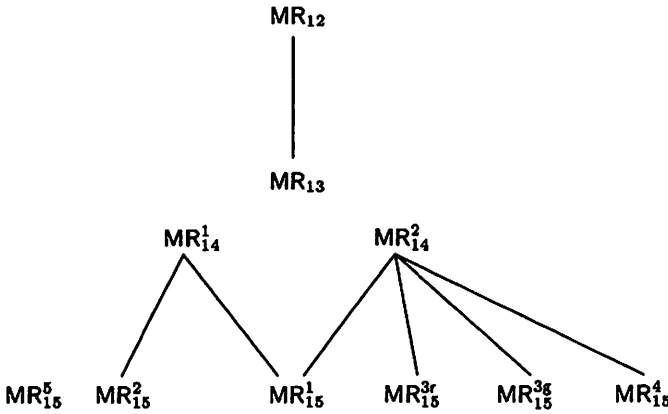
Figure 4: The MR geometries on ≤ 15 points

3 Preparations

A *k-line* in an f.l.s. is a line of size k. The *degree* of a point P, denoted by $\deg(P)$, is the number of lines passing through P. A *d-point* is a point of degree d.

We often draw the (P,Q)-*grid* of a finite linear space. This is the grid formed by the lines through P except PQ drawn vertically, and the lines through Q except PQ drawn horizontally. The line PQ is then intuitively considered to be at infinity. In particular, this shows that there are at most $(\deg(P) - 1)(\deg(Q) - 1)$ points not on PQ.

We denote the number of green points in a subset A of an MR geometry by $g(A)$, and the number of red points by $r(A)$. An $[\begin{smallmatrix} a \\ b \end{smallmatrix}]$-*line* in an MR geometry is a line passing through exactly a green and b red points. We always assume that $a, b \geq 1$. If we do not want to specify the number of red or green points we put a $*$ in its position. For example, a $[\begin{smallmatrix} * \\ 2 \end{smallmatrix}]$-line is a line passing through at least one green point and exactly two red points. Suppose that the lines passing through P are $[\begin{smallmatrix} g_i \\ r_i \end{smallmatrix}]$-lines ($i = 1, 2, \ldots, k$) where $k = \deg(P)$. Then the *neighbourhood type* (*ntype* for short) of P is $[\begin{smallmatrix} g_1 & g_2 & \ldots & g_k \\ r_1 & r_2 & \ldots & r_k \end{smallmatrix}]$. Note that we consider such a matrix to denote the same ntype if the columns are permuted. We use the same convention regarding $*$'s as before. For example, $[\begin{smallmatrix} * & * & 1 \\ * & 1 & 2 \end{smallmatrix}]$ denotes the ntype of a point incident with a $[\begin{smallmatrix} * \\ * \end{smallmatrix}]$-line, a $[\begin{smallmatrix} * \\ 1 \end{smallmatrix}]$-line, and a $[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}]$-line.

Lemma 2. *Let G be a green point in an MR geometry (S, χ) with ntype*

$$\begin{bmatrix} x & g_1 & & g_t & 1 & & 1 \\ y & * & \cdots & * & * & \cdots & * \end{bmatrix}.$$

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Then $ty \leq (t-1)(g(S) - x)$. *(In particular, $t \geq 2$.)*

Proof. Denote the $[\begin{smallmatrix} g_i \\ * \end{smallmatrix}]$-lines by ℓ_i ($i = 1, \ldots, t$) and the $[\begin{smallmatrix} x \\ y \end{smallmatrix}]$-line by ℓ. Choose a red point $R_i \in \ell_i$ for each $i = 1, \ldots, t$. Let $S_1, \ldots, S_y$ be the red points on ℓ. Let $g'_i = g_i - 1$. Then $g(S) = x + \sum_{i=1}^{t} g'_i$. For a fixed $j = 1, \ldots, t$, the set $R_j S_1 \cup R_j S_2 \cup \cdots \cup R_j S_y$ contains y distinct green points that are also in the set $\bigcup_{\substack{i=1 \\ i \neq j}}^{t} \ell_i$. Therefore, $y \leq \sum_{i \neq j} g'_i$, and

$$ty \leq \sum_{j=1}^{t} \sum_{i \neq j} g'_i = (t-1) \sum_{i=1}^{t} g'_i = (t-1)(g(S) - x). \qquad \square$$

The following lemma is very useful in shortening the case analysis when determining the minimal MR geometries of a certain size.

Lemma 3. *In a minimal MR geometry (S, χ), each green point G passes through a $[\begin{smallmatrix} 1 \\ b \end{smallmatrix}]$-line for some $b \geq 2$, hence $\deg(G) \leq r(S) - 1$. Consequently, each line passes through at most $r(S) - 1$ points. (An analogous statement holds for red points.)*

Proof. By minimality, if we remove G, then either $S \setminus \{G\}$ is collinear, or the 2-colouring is not proper anymore. In the first case, S is a near-pencil (i.e. $S \setminus \{x\}$ is collinear for some x), which is not 2-colourable, a contradiction. In the second case there is a monochromatic line $\ell \setminus \{G\}$ in $(S, \chi) \setminus \{G\}$. Since ℓ is not monochromatic in (S, χ), we must have $G \in \ell$. Hence all points (at least 2) of $\ell \setminus \{G\}$ are red. Since there is a red point on any line through G, we then obtain $\deg(G) \leq r(S) - 1$.

Secondly, given any $[\begin{smallmatrix} a \\ b \end{smallmatrix}]$-line ℓ, there clearly must be a green point $G \notin \ell$. Then $\deg(G) \geq a + b$. $\qquad \square$

The above proof also gives

Lemma 4. *An MR geometry contains a minimal MR geometry.* $\qquad \square$

Lemma 5. *The degree of any point in an MR geometry is at least 4.*

Proof. Suppose that a red point P has degree at most 3. Let ℓ_1 and ℓ_2 be two lines passing through P. Let G_i be a green point on ℓ_i ($i = 1, 2$). Let Q be a red point on $G_1 G_2$, and let $\ell_3 = PQ$. Then for each red $R \in \ell_2 \setminus \{P\}$ there is a green point $G = G_R \in QR \cap \ell_1$ (Figure 5). The function $R \mapsto G_R$ is injective. Thus $r(\ell_2) - 1 \leq g(\ell_1 \setminus \{G_1\}) = g(\ell_1) - 1$. By using a green point on ℓ_3 we obtain in a similar way an injection $G \mapsto R_G$ from the green points of ℓ_1 to the red points on $\ell_2 \setminus \{P\}$. Thus $g(\ell_1) \leq r(\ell_2) - 1$, a contradiction. $\qquad \square$
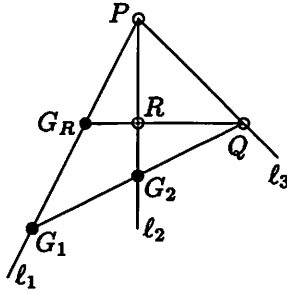
Figure 5: Proof of Lemma 5

Although we will not use it, the above lemma can easily be strengthened to show that an MR geometry is not contained in the union of any three lines.

The following lemma shows why an MR geometry has to have at least 12 points.

Lemma 6. *In an MR geometry* (S, χ),

    1. for any line ℓ there are at least 3 points of each colour not on ℓ, and

    2. there are at least 6 points of each colour.

Proof. Let G be any green point on ℓ. Let ℓ_1, ℓ_2, ℓ_3 be three other lines through G (by Lemma 5). Then ℓ_1, ℓ_2, ℓ_3 each contains a red point $\notin \ell$. In a similar way we can find three green points $\notin \ell$.

Next we find six green points. If we choose two green points and let ℓ be the line through them, by the first part of the proof we obtain $g(S) \geq 5$. If some three green points are collinear, then we would similarly obtain $g(S) \geq 6$, and then we are finished. We may now assume that $g(S) = 5$ and that no three green points are collinear, and find a contradiction.

By Lemma 4 we may assume without loss of generality that (S, χ) is minimal. Thus for every red point R there is an $\begin{bmatrix} a \\ 1 \end{bmatrix}$-line passing through R with $a \geq 2$, and also $\deg(R) \leq g(S) - 1 = 4$ (Lemma 3). By Lemma 5 we then have $\deg(R) = 4$ and $a = 2$. Thus R has ntype $\begin{bmatrix} 1 & 1 & 1 & 2 \\ * & * & * & 1 \end{bmatrix}$. It follows that there are exactly as many red points as pairs of green points: $r(S) = \binom{g(S)}{2} = 10$.

By the pigeon-hole principle there are at least three red points $\neq R$ on one of the $\begin{bmatrix} 1 \\ * \end{bmatrix}$-lines through R. Let G be the green point on this line. Then GR is a $\begin{bmatrix} 1 \\ y \end{bmatrix}$-line with $y \geq 4$. Apply Lemma 2 to G to obtain $4t \leq ty \leq (t-1)(g(S) - 1) = 4(t-1)$, a contradiction.

Therefore, $g(S) > 5$. Similarly, $r(S) \geq 6$. $\qquad\square$

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The following generalization of the counting argument for finding the number of red points in the previous proof will be crucial in the determination of the minimal MR geometries. We first define the weight of a point. Let P be a point with ntype $[\begin{smallmatrix} g_1 & \dots & g_k \\ r_1 & & r_k \end{smallmatrix}]$. Then the *weight* of P is defined as

$$
w(P) = \begin{cases} \sum_{i=1}^{k} \binom{r_i}{2}/g_i & \text{if } P \text{ is green,} \\ \sum_{i=1}^{k} \binom{g_i}{2}/r_i & \text{if } P \text{ is red.} \end{cases}
$$

For each point P in a minimal MR geometry we have $w(P) \geq 1$ by Lemma 3.

Lemma 7. *In any MR geometry* (S, χ),

$$
\sum_{G \text{ green}} w(G) = \binom{r(S)}{2}.
$$

Proof. Interchange the order of summation and use axiom I1:

$$
\sum_{\text{green } G} w(G) = \sum_{\text{green } G} \sum_{\ell \ni G} \frac{\binom{r(\ell)}{2}}{g(\ell)}
$$

$$
= \sum_{\ell} \sum_{\text{green } G \in \ell} \frac{\binom{r(\ell)}{2}}{g(\ell)}
$$

$$
= \sum_{\ell} \binom{r(\ell)}{2} = \binom{r(S)}{2}. \qquad \square
$$

The following is our first application of Lemma 7.

Proposition 2. *An MR geometry with exactly six green points has at most* 13 *points.*

Proof. Denote the total number of points by v, and the number of red 4-points by f. Since there are only six green points, for any red point R we have $w(R) \leq 3$ if $\deg(R) = 4$, and $w(R) \leq 1$ if $\deg(R) \geq 5$. By Lemma 7 we thus obtain

$$
15 = \binom{g}{2} = \sum_{R \text{ red}} w(R) \leq 3f + v - 6 - f, \tag{1}
$$

hence

$$
v + 2f \geq 21. \tag{2}
$$

Let G_1, G_2, G_3 be non-collinear green points (see Lemma 6), R_1 a red point on $G_2 G_3$, and R_2 a red point on $G_1 G_3$. Then $\deg(R_1), \deg(R_2) \leq 5$.

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Let R_1R_2 be an $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$-line. Let G be a green point on R_1R_2 and R a red point on GG_1. Then $\deg(R) \leq 5$ as well, and considering the (R_1, R_2)-grid,

$$5 \geq \deg(R) \geq x + y \geq v - 16, \tag{3}$$

and

$$v \leq 21. \tag{4}$$

Suppose $f = 0$. Then by (2) and (4), $v = 21$, by (3) R_1R_2 is a 5-line, and by (1) the weight of each red point is 1. Thus any red point lies on a $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$-line, and we may repeat the above argument for any two red points R_1 and R_2, to obtain that any line passing through at least two red points must be a 5-line. It also follows that $\deg(R) = 5$ for any red R, by choosing R_1 and R_2 such that $R \notin R_1R_2$. Thus any red point must have ntype $\left[\begin{smallmatrix} 1 & 1 & 1 & 1 & 2 \\ r_1 & r_2 & r_3 & r_4 & 1 \end{smallmatrix}\right]$ with each $r_i \in \{1, 4\}$. Then $v \leq 19$, a contradiction.

Thus $f \geq 1$. Let R_1 be a red 4-point, G_1 and G_2 two red points not collinear with R_1 (see Lemma 6), R_2 a red point on G_1G_2, and R_1R_2 an $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$-line. Then $\deg(R_2) \leq 5$ and as before, by considering the (R_1, R_2)-grid we obtain

$$5 \geq x + y \geq v - 12.$$

Then $v \leq 17$, and by (2), $f \geq 2$. Now let R_3 be a second red 4-point, and R_1R_3 an $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$-line. By considering the (R_1, R_3)-grid, $5 \geq a + b \geq v - 9$, and $v \leq 14$.

Suppose $v = 14$. Then we must have $a + b = 5$, and by (2), $f \geq 4$. Since each red point not on the 5-line R_1R_3 must have degree at least 5, we have $f \leq b \leq 4$. Thus $f = 4$, R_1R_3 is a $\left[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}\right]$-line, and all red points on R_1R_3 are 4-points. By Lemma 2 applied to the green point G on R_1R_3 we obtain for $t = \deg(G) - 1$ that $4t \leq (t - 1)(6 - 1)$, hence $t \geq 5$. This gives at least $4 + 5$ red points, a contradiction.

Thus $v \leq 13$. $\qquad\square$

It is also possible to prove that an f.l.s. with a blocking set of 7 points has at most 21 points. The projective planes of order 3 (Example 1) and 4 show that these two results are sharp. In general, using the techniques of Proposition 2, we only get that an f.l.s. with a blocking set of g points has at most $g^2 - cg$ points. As the Desarguesian projective planes of square order show, there exist finite linear spaces with a blocking set of size $g = q + \sqrt{q} + 1$ and with $q^2 + q + 1 = g^2 - g\sqrt{4g - 3} + 1$ points.

4 Determining the MR geometries on at most 13 points

Proposition 3. *There is a unique MR geometry on 12 points.*

Proof. Let (S, χ) be an MR geometry on 12 points. By Lemma 6, $g(S) = r(S) = 6$ and S is minimal. By Lemmas 3 and 5 we then have that $4 \leq \deg(P) \leq 5$ for all $P \in S$. It follows that each green point G has one of the following ntypes:

Type 1 $\left[\begin{smallmatrix} * & * & * & * & 1 \\ 1 & 1 & 1 & 1 & 2 \end{smallmatrix}\right]$: $w(G) = 1$.

Type 2 $\left[\begin{smallmatrix} * & * & * & 1 \\ 1 & 1 & 1 & 3 \end{smallmatrix}\right]$: $w(G) = 3$.

Type 3 $\left[\begin{smallmatrix} * & * & a & 1 \\ 1 & 1 & 2 & 2 \end{smallmatrix}\right]$ for some $a \geq 1$: By Lemma 6 there are at least 3 green points not on the $\left[\begin{smallmatrix} a \\ 2 \end{smallmatrix}\right]$-line, hence $1 \leq a \leq 3$. There must be a green point on the line through the red points on the two $\left[\begin{smallmatrix} * \\ 1 \end{smallmatrix}\right]$-lines, giving $a \geq 2$. By Lemma 2, $a = 3$ is impossible. Thus $a = 2$ and $w(G) = 3/2$.

By Lemma 7 we have $\sum_{G \text{ green}} w(G) = \binom{6}{2} = 15$. By letting x_i be the number of green points of type i, this equation becomes $1x_1 + 3x_2 + \frac{3}{2}x_3 = 15$ where we also have $x_1 + x_2 + x_3 = 6, x_i \geq 0$. The only integral solution is $x_1 = 0, x_2 = 4, x_3 = 2$. Thus four of the green points are of type 2, and two of type 3. In particular, each green point is a 4-point. In the same way, each red point is a 4-point. By Vanstone's lemma [3, Proposition 2.2.3] S can be embedded into $PG(2,3)$, i.e., S is the punctured projective plane of order 3.

It remains to show uniqueness of the colouring up to isomorphism. Since each line of the punctured projective plane passes through at least three points, and each point is of type 2 or 3, each line must be a $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$-, $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$-, $\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right]$-, $\left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$- or $\left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]$-line. Consider the two green points G_1 and G_2 of type 3: Each has ntype $\left[\begin{smallmatrix} 3 & 3 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{smallmatrix}\right]$, $G_1 G_2$ is a $\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right]$-line, and we may label the remaining points as in Figure 6(a). Since G_3 is of type 2, it is on a $\left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$-line. Thus we have the collinearity $G_3 R_4 R_5$, and similarly, $G_4 R_4 R_6$, $G_5 R_3 R_5$ and $G_6 R_3 R_6$. Also we have either $G_3 R_4 R_1$ or $G_3 R_4 R_2$. Without loss of generality we may assume $G_3 R_4 R_1$ (otherwise relabel $R_1 \leftrightarrow R_2$). Then $G_4 R_2 R_4$ is also fixed.

In a similar way we consider the red points. Since there is only one $\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right]$-line, we know that R_1 and R_2 are the two red points playing a similar role to G_1 and G_2. We then obtain the two further collinearities $G_5 R_2 R_3 R_5$ and $G_6 R_1 R_3 R_6$ (Figure 6(b)). Now we have determined all lines containing at least two green points, and all lines containing at least two red points, hence all lines. Therefore, we have uniqueness up to isomorphism of MR geometries. $\square$

Proposition 4. *There is a unique MR geometry on 13 points.*

Proof. If (S, χ) is not minimal, it is an extension of the 12-point MR geometry, which can only be $PG(2,3)$ (e.g. by Vanstone's lemma), and then there are six points of one colour and seven of the other.
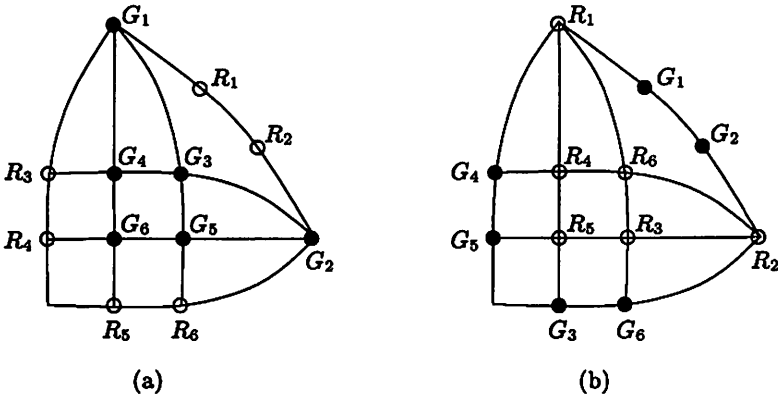
Figure 6: Proof of Proposition 3

We now show that there is no minimal MR geometry on 13 points. Suppose there is. By Lemma 6, $g(S) = 6$ and $r(S) = 7$ without loss of generality.

Claim 1. *The weight of a green point G is $\leq 7/2$, with equality iff the ntype of G is $\left[\begin{smallmatrix} 3 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 \end{smallmatrix}\right]$.*

Proof of Claim 1. By Lemma 3, $\deg(G) \leq 6$. If $\deg(G) = 6$ then clearly $w(G) = 1$, and if $\deg(G) = 5$ then clearly $w(G) \leq 3$. By Lemma 5, the only remaining case is $\deg(G) = 4$. Let ℓ be a $\left[\begin{smallmatrix}1\\y\end{smallmatrix}\right]$-line through G where $y \geq 2$ (Lemma 3). By Lemma 2, $y \leq 3$.

If $y = 2$, then G has ntype $\left[\begin{smallmatrix} * & * & a & 1 \\ 1 & 1 & 3 & 2 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} * & * & a & 1 \\ 1 & 2 & 2 & 2 \end{smallmatrix}\right]$ for some $a \geq 1$. In the first case, since there must be a green point on the line through the red points on the two $\left[\begin{smallmatrix}*\\1\end{smallmatrix}\right]$-lines, we must have $a \geq 2$, hence $w(G) \leq 5/2$. In the second case we already have $w(G) \leq 3$.

If $y = 3$, then the ntype of G must be $\left[\begin{smallmatrix} * & * & a & 1 \\ 1 & 1 & 2 & 3 \end{smallmatrix}\right]$, where $a \geq 2$ as before. Thus $w(G) \leq 7/2$. The case of equality implies $a = 2$. By considering the lines through the red point on one $\left[\begin{smallmatrix}*\\1\end{smallmatrix}\right]$-line and the two red points on the $\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$-line, we obtain that the other $\left[\begin{smallmatrix}*\\1\end{smallmatrix}\right]$-line has at least three green points. Hence each $\left[\begin{smallmatrix}*\\1\end{smallmatrix}\right]$-line has exactly three green points. □

By Lemma 7 and Claim 1 we have $21 = 6 \times \frac{7}{2} \geq \sum_{G \text{ green}} w(G) = \binom{r(S)}{2} = 21$. Thus we obtain that the ntype of each green point is $\left[\begin{smallmatrix} 3 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 \end{smallmatrix}\right]$. Therefore, each line is a $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$-, $\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$-, or $\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$-line. By minimality (Lemma 3), each red point R is on an $\left[\begin{smallmatrix}a\\1\end{smallmatrix}\right]$-line where $a \geq 2$. Thus $a = 3$. Since $\deg(R) \geq 4$ (Lemma 5), it follows that R must have ntype $\left[\begin{smallmatrix} 1 & 1 & 1 & 3 \\ * & * & * & 1 \end{smallmatrix}\right]$, hence $\left[\begin{smallmatrix} 1 & 1 & 1 & 3 \\ 3 & 3 & 3 & 1 \end{smallmatrix}\right]$, giving $g(S) = 6$, a contradiction.

The proof of the uniqueness of the 12-point MR geometry above can easily be adapted to show that the colouring of PG(2, 3) is unique. $\quad\square$

5 Determining the MR geometries on 14 points

Lemma 8. *An MR geometry* (S, χ) *on* 14 *points is minimal.*

Proof. Suppose (S, χ) is not minimal. Then there exists a $P \in S$ such that $(S, \chi) \setminus \{P\}$ is an MR geometry on 13 points. By Proposition 4 and Example 1, $S \setminus \{P\}$ is PG(2, 3). Without loss of generality P is green.

If all lines of S through P are 2-lines, then all the points of $S \setminus \{P\}$ must be red. Thus there is a monochromatic line in $(S, \chi) \setminus \{P\}$, a contradiction. Otherwise there exists a line ℓ of S through P with at least two more points of S on ℓ. By Lemma 6 there must be a green point $Q \notin \ell$. Then PQ is a line with at least three points (since it also passes through a red point). But now $\ell \setminus \{P\}$ and $PQ \setminus \{P\}$ are two non-intersecting lines of $S \setminus \{P\}$, contradicting the properties of PG(2, 3). $\quad\square$

Proposition 5. *The only MR geometries on* 14 *points are* MR_{14}^1 *and* MR_{14}^2.

Proof. Let (S, χ) be an MR geometry on 14 points. By Lemma 8 it is minimal. By Lemma 6 and Proposition 2, $g(S) = r(S) = 7$. Without loss of generality we now have two cases: Either each line of (S, χ) passes through at most three points of either colour, or (S, χ) has a line passing through at least four red points.

Case I. $r(\ell), g(\ell) \leq 3$ for each line ℓ of S.

Claim 2. *Any line ℓ with $r(\ell) = 3$ is a $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$-line, and any line ℓ with $g(\ell) = 3$ is a $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$-line.*

Proof of Claim 2. Suppose that $r(\ell) = 3$ and $g(\ell) = a \geq 2$. Let G be a green point on ℓ. By Lemmas 3 and 5 it follows that G has ntype $\begin{bmatrix} * & * & a & 1 \\ 1 & 1 & 3 & 2 \end{bmatrix}$. Let R_1, R_2, R_3 be the red points on ℓ, and R the red point on one of the $\begin{bmatrix} * \\ 1 \end{bmatrix}$-lines. Then each RR_i has to intersect the other $\begin{bmatrix} * \\ 1 \end{bmatrix}$-line in a green point, giving at least four green point on each $\begin{bmatrix} * \\ 1 \end{bmatrix}$-line, a contradiction. $\quad\square$

Claim 3. *The green point G on any $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$-line has ntype $\begin{bmatrix} * & * & * & * & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix}$ and weight 3.*

Proof of Claim 3. Clearly $\deg(G) \leq 5$.

If $\deg(G) = 4$, then the ntype is $\begin{bmatrix} * & * & * & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix}$. Since each line has at most three green points, the two $\begin{bmatrix} * \\ 1 \end{bmatrix}$-lines each contains at most two green points apart from G. Then the $\begin{bmatrix} * \\ 2 \end{bmatrix}$-line passes through a least three green points,
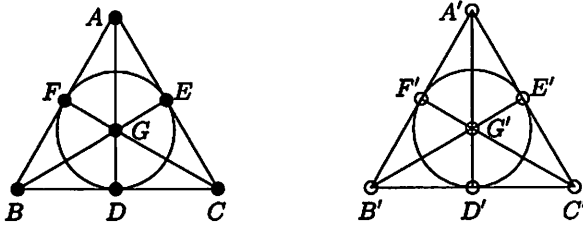
Figure 7: The Fano planes in MR_{14}^1 (Proposition 5)

hence exactly three by assumption, contradicting Claim 2. Thus there are at most six green points, a contradiction.

Thus $\deg(G) = 5$ from which the remainder of the claim follows. $\square$

Claim 4. *For any green point G, $w(G) \leq 3$ with equality iff G is on a $[\begin{smallmatrix}1\\3\end{smallmatrix}]$-line.*

Proof of Claim 4. Suppose G is not on a $[\begin{smallmatrix}1\\3\end{smallmatrix}]$-line (otherwise use Claim 3). Then by Claim 2 there are at most two red points on each line through G. If $w(G) \geq 3$, then there must be at least three $[\begin{smallmatrix}1\\2\end{smallmatrix}]$-lines, and thus only one further line with a single red point. But then all green points must be on this fourth line, a contradiction. Thus $w(G) < 3$. $\square$

By Claim 4 and Lemma 7 we have $7 \times 3 \leq \sum_{G \text{ green}} w(G) = \binom{7}{2} = 21$. Thus $w(G) = 3$ for all green G, and by Claims 4 and 3 they all have the same ntype $[\begin{smallmatrix} * & * & * & * & 1 \\ 1 & 1 & 1 & 1 & 3 \end{smallmatrix}]$. It follows that the red points form a Steiner triple system, hence a Fano plane. Similarly, the green points form a Fano plane. It follows that the ntype of each green point is $[\begin{smallmatrix} 3 & 3 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{smallmatrix}]$, and of each red point is $[\begin{smallmatrix} 1 & 1 & 1 & 1 & 3 \\ 3 & 3 & 3 & 1 & 1 \end{smallmatrix}]$. Label the green points $A, B, \ldots, G$ and the red points $A', B', \ldots, G'$ as in Figure 7. Since the Fano plane has enough symmetries, without loss of generality AA' is the unique 2-line passing through A, $A'BEG$ the unique $[\begin{smallmatrix}3\\1\end{smallmatrix}]$-line passing through A', and $AB'E'G'$ the unique $[\begin{smallmatrix}1\\3\end{smallmatrix}]$-line passing through A. By permuting B, E, G as well as B', E', G' we obtain without loss of generality the situation in Figure 8. At this stage green and red are symmetric. We already have the $[\begin{smallmatrix}1\\3\end{smallmatrix}]$-line and one of the $[\begin{smallmatrix}3\\1\end{smallmatrix}]$-lines through D. The remaining two $[\begin{smallmatrix}3\\1\end{smallmatrix}]$-lines through D must then be DB and DE. The only possibility for DB is $BCDE'$ and for DE is $DEFG'$. It follows that the $[\begin{smallmatrix}1\\1\end{smallmatrix}]$-line through D must be DD'. In the same way the lines through each of C, F, and by symmetry, through C', D', F', may be determined. Finally, to determine the lines through each of B, E, G, B', E', G' is then straightforward. Thus there is a unique MR geometry in Case I, which is clearly isomorphic to MR_{14}^1.
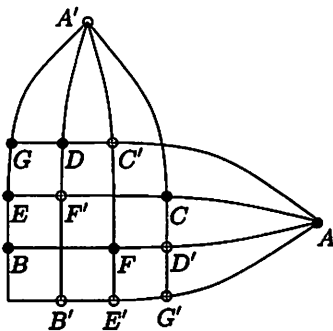
Figure 8: Proof of Proposition 5

Case II.   Line ℓ_0 passes through at least 4 red points.

Claim 5. ℓ_0 is a $\left[\begin{smallmatrix}1\\4\end{smallmatrix}\right]$-line, and the ntype of the green point G on ℓ_0 is $\left[\begin{smallmatrix}3&3&3&1\\1&1&1&4\end{smallmatrix}\right]$.

Proof of Claim 5. By Lemma 6, $r(\ell_0) = 4$. Fix a green point $G \in \ell_0$. Then by Lemma 5 $\deg(G) = 4$, and by Lemma 3 there is no other green point on ℓ_0.

Let R_1, R_2, R_3, R_4 be the red points on ℓ_0 and S_1, S_2, S_3 the red points $\notin \ell_0$. There is a green point on each of $S_1 R_i$ $(i = 1, 2, 3, 4)$ hence at least 4 green points $\neq G$ on $GS_2 \cup GS_3$. Similarly, there are at least 4 green points $\neq G$ on each $GS_i \cup GS_j$ $(1 \le i < j \le 3)$. Since there are 6 green points $\neq G$, it follows that each GS_i is a $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$-line. □

Label the green points $\neq G$ as $G_1, G_2, G_3, H_1, H_2, H_3$ such that we have the lines $GG_iH_iS_i$ $(i = 1, 2, 3)$.

Claim 6. The ntype of each S_i is $\left[\begin{smallmatrix}1&1&1&1&3\\2&2&3&3&1\end{smallmatrix}\right]$ (in particular, $w(S_i) = 3$ and $\deg(S_i) = 5$). Also, each $S_i S_j$ passes through some R_k.

Proof of Claim 6. Since S_i is not on the 5-line ℓ_0, $\deg(S_i) \ge 5$. Since there are four green points not on GS_i, $\deg(S_i) \le 5$.

Since there must be a green point on each $S_i R_j$ $(j = 1, 2, 3, 4)$, we must have the situation in Figure 9. □

We now apply Claim 6 to each S_i. For S_1 we have without loss of generality the situation in Figure 10 (after possibly permuting R_1, R_2, R_3, R_4 and interchanging $S_2 \leftrightarrow S_3$, $G_2 \leftrightarrow H_2$, $G_3 \leftrightarrow H_3$). Thus we have determined the lines $S_1 R_1 G_2$, $S_1 R_4 G_3$, $S_1 S_2 R_3 H_3$, $S_1 S_3 R_2 H_2$. By Claim 6, $S_2 S_3$ passes through some R_i. We must have $i \neq 2, 3$. Without loss of
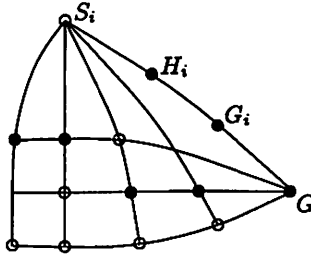
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Figure 9: Neighbourhood of S_i (Claim 6)

generality S_2S_3 passes through R_1 (if R_4, interchange $G_2 \leftrightarrow G_3$, $H_2 \leftrightarrow H_3$, $R_1 \leftrightarrow R_4$, $R_2 \leftrightarrow R_3$, $S_2 \leftrightarrow S_3$).

Applying Claim 6 to S_2 and using the collinearities obtained so far we obtain the situation in Figure 11. The two green points $GS_1 \cap S_2R_4$ and $GS_1 \cap S_2R_1$ are G_1 and H_1. Without loss of generality, $GS_1 \cap S_2R_4 = G_1$, $GS_1 \cap S_2R_1 = H_1$, since we may still interchange $G_1 \leftrightarrow H_1$. Thus we have the additional lines $S_2G_1R_4$, $S_2S_3R_1H_1$, $S_2G_3R_2$.

Applying Claim 6 and the collinearities obtained so far to S_3 we obtain the situation in Figure 12. In particular we have the lines $S_3R_3G_1$ and $S_3R_4G_2$.

Thus R_1, R_2, R_3 each lies on a $[{1 \atop 4}]$-, $[{1 \atop 3}]$- and $[{1 \atop 2}]$-line, and R_4 lies on a $[{1 \atop 4}]$-line and three $[{1 \atop 2}]$-lines. By Lemma 3, each R_i is furthermore on an $[{a \atop 1}]$-line, where $a \geq 2$.

Claim 7. *For each $i = 1, 2, 3, 4$, $w(R_i) \leq 3$ with equality iff the ntype of R_i is $[{1\ 1\ 1\ 1\ 3 \atop *\ *\ *\ *\ 1}]$.*
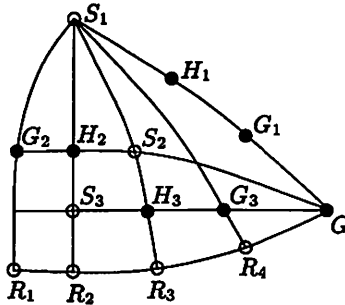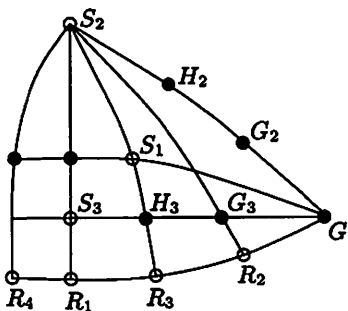

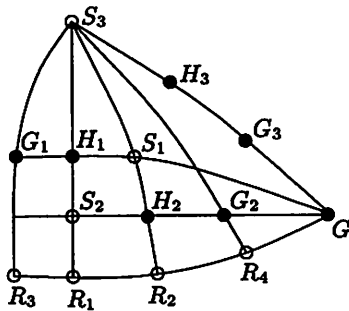Figure 10: Neighbourhood of S_1

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Figure 11: Neighbourhood of S_2 Figure 12: Neighbourhood of S_3

Proof of Claim 7. If $\deg(R_i) = 4$, then $a = 4$, hence there is a $\begin{bmatrix}4\\1\end{bmatrix}$-line not passing through G. Thus $\deg(G) \geq 5$, a contradiction. Thus $\deg(R_i) \geq 5$, and the claim is now clear. ☐

We now have $7 \times 3 \geq \sum_{R \text{ red}} w(R) = \binom{g(S)}{2} = 21$, and it follows that each R_i has ntype $\begin{bmatrix}1&1&1&1&3\\ *&*&*&*&1\end{bmatrix}$. By also considering the ntypes of the S_i's (see Claim 6), we see that each line passes through exactly 1 or 3 green points. Thus the green points form a Steiner triple system, hence a Fano plane. We already have $R_1R_2R_3R_4G$, $R_4S_1G_3$, $R_4S_2G_1$, $R_4S_3G_2$ as lines through R_4. Thus the fifth line must be $R_4H_1H_2H_3$. We now have GG_iH_i ($i = 1, 2, 3$) and $H_1H_2H_3$ as collinear green triples. This determines the Fano plane: the remaining green collinear triples must be $H_iG_{i+1}G_{i+2}$ ($i = 1, 2, 3$) (subscripts modulo 3).

Considering the neighbourhoods of R_1, R_2, R_3 and the green collinearities determined above we obtain the remaining lines of (S, χ): H_iR_{i+1}, $R_iG_iG_{i+2}H_{i+1}$ ($i = 1, 2, 3$) (subscripts mod 3). Thus we have found a unique linear space with a unique colouring in Case II, which is clearly MR_{14}^2. ☐

6 Determining the MR geometries on 15 points

The proof of the following lemma is routine, for example, by checking the list of finite linear spaces on 7 points in [3, Appendix].

Lemma 9. *There is only one finite linear space on 7 points with six 3-lines and three 2-lines, namely the Fano plane with one line ABC replaced by the three lines AB, BC, AC. See Figure 13.*
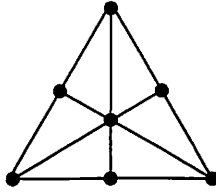
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Figure 13: The unique f.l.s. on 7 points with six 3-lines and three 2-lines

The non-minimal MR geometries on 15 points are enumerated in Examples 5 to 8 and the Appendix.

Proposition 6. *There is exactly one minimal MR geometry on 15 points.*

Proof. By Lemmas 6 and 2 there are at least 7 green points and at least 7 red points. Without loss of generality assume $g(S) = 7$ and $r(S) = 8$.

Claim 8. *There is no $\begin{bmatrix} a \\ b \end{bmatrix}$-line if $a \geq 3, b \geq 2$.*

Proof of Claim 8. Let two of the red points on an $\begin{bmatrix} a \\ b \end{bmatrix}$-line be R_1 and R_2. By Lemma 3 there is also a $\begin{bmatrix} c \\ 1 \end{bmatrix}$-line through R_1 for some $c \geq 2$. Since there are $a + c \geq 5$ green points on these two lines, there are at most two further green points, hence at most two further lines through R_1. Thus $\deg(R_1) \leq 4$, and by Lemma 5, $\deg(R_1) = 4$, and $a = 3, c = 2$. It follows that the ntype of R_1, and similarly, of R_2, must be $\begin{bmatrix} 1 & 1 & 3 & 2 \\ * & * & b & 1 \end{bmatrix}$. By considering the (R_1, R_2)-grid, we see that there are in total at most $3^2 - 1 + 3 + b$ points, hence $b \geq 4$. Thus there are at least seven points on $R_1 R_2$, contradicting Lemma 3. □

Claim 9. *There is no green 4-point.*

Proof of Claim 9. Suppose G is a green 4-point. Let the lines through G be GR_i, $i = 1, 2, 3, 4$, and let GR_i be an $\begin{bmatrix} x_i \\ y_i \end{bmatrix}$-line. By Lemma 3 we may assume without loss that $x_4 = 1, y_4 \geq 2$.

Suppose that $x_1 \geq 4$. Then the number of green points not on GR_1 is at most three. Since for each red $R \in GR_4$ there is a different green point on $R_1 R$, we then must have $y_4 \leq 3$. Also, by Claim 8, $y_1 = 1$. Then $y_2 + y_3 \geq 4$. By Lemma 6 there are at least three green points not on GR_1, hence $x_2 \geq 3$ or $x_3 \geq 3$. Without loss we consider only the case $x_2 \geq 3$. By Claim 8, $y_2 = 1$. Thus $y_3 \geq 3$. For each red $R \in GR_3$ there is a distinct green point on $R_1 R$. These green points must all lie on GR_2, thus $x_2 \geq 4$. Since $x_1 \geq 4$ as well, all green points now lie on $GR_1 \cup GR_2$. Thus there is no green point on $R_1 R_2$, a contradiction.

Therefore, $x_1 \leq 3$, and similarly, $x_2, x_3 \leq 3$. Since there are only seven green points, we must have $x_1 = x_2 = x_3 = 3$. By Claim 8, $y_1 = y_2 = y_3 = 1$. Thus $y_4 = 5$. For each red $R \in GR_4$ there is a different green point on $R_1 R$. These five green points must all lie on $GR_2 \cup GR_3$. Therefore, $x_2 + x_3 \geq 7$, a contradiction. $\qquad \square$

Claim 10. *There is at most one red 4-point.*

Proof of Claim 10. Suppose R_1 and R_2 are two red 4-points. Let $R_1 R_2$ be an $\begin{bmatrix} x \\ y \end{bmatrix}$-line. By considering the (R_1, R_2)-grid, we see that there are at least $15 - 9 = 6$ points on $R_1 R_2$. Thus $x + y \geq 6$. By Lemma 3, $x + y = 6$ and the 3×3 grid is full. Again by Lemma 3 both R_1 and R_2 lie on $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$-lines. Thus there are at least five green points not on $R_1 R_2$, and $x \leq 2$.

If $x = 1$, then $y = 5$, and there are three red points not on $R_1 R_2$. Thus the degree of the green point on $R_1 R_2$ is at most 4, contradicting Claim 9.

If $x = 2$, then $y = 4$, and by Lemma 3, through any green $G \in R_1 R_2$ there must be some $\begin{bmatrix} 1 \\ a \end{bmatrix}$-line, $a \geq 2$. Then $\deg(G) \leq 4$, again a contradiction. $\qquad \square$

Claim 11. *For any green point G exactly one of the following cases holds:*

1. $w(G) = 6$ and the ntype of G is $\begin{bmatrix} * & * & * & * & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}$.

2. $w(G) = 4$ and the ntype of G is $\begin{bmatrix} * & * & * & 1 & 1 \\ 1 & 1 & 1 & 2 & 3 \end{bmatrix}$.

3. $w(G) < 4$.

Proof of Claim 11. By Lemma 3, $\deg(G) \leq 7$, and by Claim 9, $\deg(G) \geq 5$.

If $\deg(G) = 7$, then by Lemma 3 its ntype must be $\begin{bmatrix} * & * & * & * & * & * & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$, and $w(G) = 1$.

If $\deg(G) = 6$, then its ntype must be either $\begin{bmatrix} * & * & * & * & * & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 \end{bmatrix}$ with $w(G) = 3$, or $\begin{bmatrix} * & * & * & * & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 \end{bmatrix}$ with $w(G) \leq 2$.

If $\deg(G) = 5$, its ntype must be one of:

- $\begin{bmatrix} * & * & * & * & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}$ with $w(G) = 6$,

- $\begin{bmatrix} * & * & * & * & * \\ 1 & 1 & 1 & 2 & 3 \end{bmatrix}$ with $w(G) \leq 4$, and equality implies that the ntype must be $\begin{bmatrix} * & * & * & 1 & 1 \\ 1 & 1 & 1 & 2 & 3 \end{bmatrix}$,

- $\begin{bmatrix} * & * & * & * & 1 \\ 1 & 1 & 2 & 2 & 2 \end{bmatrix}$ with $w(G) \leq 3$. $\qquad \square$

Claim 12. *Some green point has weight 6.*

Proof of Claim 12. Suppose no green point has weight 6. Then by Claim 11 the sum of the green weights is

$$28 = \binom{r(S)}{2} = \sum_{G \text{ green}} w(G) \leq 4 \times 7.$$

Thus each green point has weight 4 and ntype $\left[\begin{smallmatrix} * & * & * & 1 & 1 \\ 1 & 1 & 1 & 2 & 3 \end{smallmatrix}\right]$ by Claim 11. In particular, each line has at most 3 red points, and a line passing through at least 2 red points passes through only one green point.

We now consider the weights of the red points. Let all the $\left[\begin{smallmatrix} 1 \\ * \end{smallmatrix}\right]$-lines through a red point R be $RG_1, \ldots, RG_n$. Then by the above, RG_i is a $\left[\begin{smallmatrix} 1 \\ b_i \end{smallmatrix}\right]$-line for some $b_i \leq 3$. Also, all other lines through R must be $\left[\begin{smallmatrix} * \\ 1 \end{smallmatrix}\right]$-lines. Adding up all the red points on the lines through R we find $7 = \sum_{i=1}^{n}(b_i - 1) \leq 2n$, hence $n \geq 4$. By Lemma 3, there is an $\left[\begin{smallmatrix} a \\ 1 \end{smallmatrix}\right]$-line through R, with $a \geq 2$. Let H_1 and H_2 be two green points on this line. We now already have six of the green points: $G_1, \ldots, G_4, H_1, H_2$. The seventh green point can be either on $H_1 H_2$, giving $w(R) = 3$, or on a sixth line through R, giving $w(R) = 1$.

Thus the sum of the red weights is a sum of eight odd numbers (1 or 3), which is even. By Lemma 7, this sum must equal $\binom{\vartheta(S)}{2} = 21$, a contradiction. □

Fix a green point G of weight 6. By Claim 11, G has ntype $\left[\begin{smallmatrix} * & * & * & * & 1 \\ 1 & 1 & 1 & 1 & 4 \end{smallmatrix}\right]$. Denote the red points on the $\left[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}\right]$-line through G by R_1, R_2, R_3, R_4. Thus we have the full line

$$GR_1R_2R_3R_4. \tag{1}$$

Claim 13. *G has ntype either*

Case α: $\left[\begin{smallmatrix} 1 & 3 & 3 & 3 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{smallmatrix}\right]$ *or*

Case β: $\left[\begin{smallmatrix} 2 & 2 & 3 & 3 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{smallmatrix}\right]$.

Proof of Claim 13. Without loss of generality G has ntype $\left[\begin{smallmatrix} x_1 & x_2 & x_3 & x_4 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{smallmatrix}\right]$ with

$$1 \leq x_1 \leq x_2 \leq x_3 \leq x_4.$$

Let S_i be the red point on the $\left[\begin{smallmatrix} x_i \\ 1 \end{smallmatrix}\right]$-line. Since there is a different green point on each line $S_4 R_i$, $i = 1, 2, 3, 4$, and these 4 green points must be in $GS_1 \cup GS_2 \cup GS_3 \setminus \{G\}$, we obtain

$$x_1 + x_2 + x_3 \geq 7.$$

We also have

$$x_1 + x_2 + x_3 + x_4 = 10.$$

There are only two solutions to this equation under the above two constraints:

$$(x_1, x_2, x_3, x_4) = (2, 2, 3, 3), (1, 3, 3, 3). \qquad \square$$

We now show that Case α in Claim 13 leads to a contradiction, while Case β leads to exactly one MR geometry.

Case α: G has ntype $\left[\begin{smallmatrix}1&3&3&3&1\\1&1&1&1&4\end{smallmatrix}\right]$

Let S be the red point on the $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$-line through G, and let S_1, S_2, S_3 be the red points on the three $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$-lines through G. We now consider the weights of the red points.

Claim 14.

- *S has weight 1 and ntype $\left[\begin{smallmatrix}1&1&1&1&1&2\\ *& *& *& *& *&1\end{smallmatrix}\right]$.*

- *Each S_i has weight 3 and ntype $\left[\begin{smallmatrix}1&1&1&1&3\\ *& *& *& *&1\end{smallmatrix}\right]$.*

Proof of Claim 14. The lines SG and SR_i, $i = 1, 2, 3, 4$, give $\deg(S) \geq 5$. By Lemma 3, S must also be on a sixth $\left[\begin{smallmatrix}a\\1\end{smallmatrix}\right]$-line, $a \geq 2$. It follows that $a = 2$, $\deg(S) = 6$, and then S must have the stated ntype and weight.

Without loss consider S_1. The line $S_1 G$ passes through three green points, and each $S_1 R_i$, $i = 1, 2, 3, 4$, passes through at least one green point. Since there are seven green points, it follows that each $S_1 R_i$ passes through exactly one green point, and $\deg(S_1) = 5$. Thus S_1 has the stated ntype and weight. $\qquad\square$

Claim 15. *For each R_i, $i = 1, 2, 3, 4$, we have one of the following:*

- $\deg(R_i) = 6$ *and* $w(R_i) = 1$,

- $\deg(R_i) = 5$ *and either* $w(R_i) < 3/2$ *or* $w(R_i) \in \{3/2, 2, 3\}$,

- $\deg(R_i) = 4$ *and* $3 < w(R_i) \leq 7/2$.

For the following weights, the ntypes must be as in the table:

$w(R_i)$	ntype
3/2	$\left[\begin{smallmatrix}1&1&1&2&2\\ *& *& *&2&1\end{smallmatrix}\right]$
2	$\left[\begin{smallmatrix}1&1&1&2&2\\ *& *& *&1&1\end{smallmatrix}\right]$
3	$\left[\begin{smallmatrix}1&1&1&1&3\\ *& *& *& *&1\end{smallmatrix}\right]$
7/2	$\left[\begin{smallmatrix}1&1&2&3\\ *& *&2&1\end{smallmatrix}\right]$

Proof of Claim 15. By Lemmas 3 and 5 we have $4 \leq \deg(R_i) \leq 6$. The cases $\deg(R_i) = 6$ and $\deg(R_i) = 5$ are clear. In the case $\deg(R_i) = 4$ we consider the (R_i, G)-grid. Note that GR_i is a $\left[\begin{smallmatrix}1\\4\end{smallmatrix}\right]$-line and GS is a $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$-line. Thus the complement of $GR_i \cup GS$ must consist of six green points and three red points. See Figure 14. By this and Lemma 3, R_i must lie on some $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$-line. Thus $w(R_i) \geq 3$. By placing the remaining three green points and three red points we see that the remainder of the claim follows. $\qquad\square$
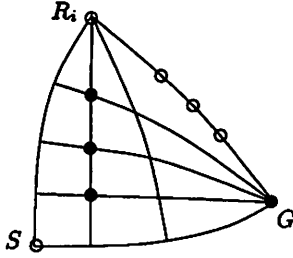
Figure 14: $\deg(R_i) = 4$ in Claim 15

Since by Lemma 7 the sum of the red weights must be $\binom{7}{2}$, by Claim 14 we have $\sum_{i=1}^{4} w(R_i) = 11$. Suppose that at most two R_i's satisfy $w(R_i) \geq 3$. Then by Claim 15, the remaining R_i's satisfy $w(R_i) \leq 2$, and recalling that there is at most one red 4-point (Claim 10), we obtain

$$11 = \sum_{i=1}^{4} w(R_i) \leq 7/2 + 3 + 2 + 2,$$

a contradiction.

Thus at least three R_i's have $w(R_i) \geq 3$. We then have by Claim 10 one of the following two cases without loss of generality:

Case $\alpha 1$: $w(R_1) = w(R_2) = w(R_3) = 3$.

Case $\alpha 2$: $w(R_1) > 3, w(R_2) = w(R_3) = 3$.

Case $\alpha 1$

We must have $w(R_4) = 2$. We now consider the subgeometry of the seven green points. By examining the ntypes of the red points (see Claims 14 and 15), we see that there are in total six $[^3_*]$-lines and three $[^2_*]$-lines. By Lemma 9 we see that any two $[^2_*]$-lines must intersect in a green point. However, from the ntype of R_4 we see that there are two $[^2_*]$-lines not intersecting in a green point, a contradiction.

Case $\alpha 2$

We must have $w(R_4) < 2$. By Claim 15, $w(R_4) \leq 3/2$. Thus $w(R_1) \geq 7/2$. Again by Claim 15, $w(R_1) = 7/2$ and R_1 has ntype $[^{1\ 1\ 2\ 3}_{4\ 4\ 2\ 1}]$, $w(R_4) = 3/2$ and R_4 has ntype $[^{1\ 1\ 1\ 2\ 2}_{*\ *\ *\ 2\ 1}]$. By considering the subgeometry of the seven green points we obtain the same contradiction as in Case $\alpha 1$.

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Case β: G has ntype $\left[\begin{smallmatrix} 2 & 2 & 3 & 3 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{smallmatrix}\right]$

Let S_1 and S_2 be the red points on the $\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$-lines through G, and let T_1 and T_2 be the red points on the $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$-lines through G. We again consider the weights of the red points.

Claim 16.

- $w(S_i) \in \{1, 4/3, 3/2\}$.

- $w(T_i) = 3$.

- If $\deg(R_i) \geq 5$ then $w(R_i) = 3$ or $w(R_i) \leq 2$.

- If $\deg(R_i) = 4$ then $w(R_i) \in \{10/3, 7/2, 6\}$.

For the following weights the ntypes are as in the table:

$w(S_i)$	ntype	$w(T_i)$	ntype	$w(R_i)$	ntype
1	$\left[\begin{smallmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ * & * & * & * & * & 1 \end{smallmatrix}\right]$	3	$\left[\begin{smallmatrix} 1 & 1 & 1 & 1 & 3 \\ * & * & * & * & 1 \end{smallmatrix}\right]$	3	$\left[\begin{smallmatrix} 1 & 1 & 1 & 3 \\ * & * & * & 1 \end{smallmatrix}\right]$
4/3	$\left[\begin{smallmatrix} 1 & 1 & 1 & 2 & 2 \\ * & * & * & 3 & 1 \end{smallmatrix}\right]$			10/3	$\left[\begin{smallmatrix} 1 & 1 & 2 & 3 \\ * & * & 3 & 1 \end{smallmatrix}\right]$
3/2	$\left[\begin{smallmatrix} 1 & 1 & 1 & 2 & 2 \\ * & * & * & 2 & 1 \end{smallmatrix}\right]$			7/2	$\left[\begin{smallmatrix} 1 & 1 & 2 & 3 \\ * & * & 2 & 1 \end{smallmatrix}\right]$
				6	$\left[\begin{smallmatrix} 1 & 1 & 1 & 4 \\ * & * & * & 1 \end{smallmatrix}\right]$

Proof of Claim 16. Since GR_1 is a 5-line not passing through S_i, $\deg(S_i) \geq 5$. Thus we have either $\deg(S_i) = 6$ and $w(S_i) = 1$, or S_i has degree 5 and ntype $\left[\begin{smallmatrix} 1 & 1 & 1 & 2 & 2 \\ * & * & * & a & 1 \end{smallmatrix}\right]$ for some $a \geq 2$ (consider the lines S_iG and S_iR_j), and weight $1 + 1/a$. Suppose $a \geq 4$. Let H be a green point on the $\left[\begin{smallmatrix}2\\a\end{smallmatrix}\right]$-line through S_i. By Lemma 3 it follows that $\deg(H) = 4$, contradicting Claim 9. Thus $a \leq 3$, giving $w(S_i) \in \{4/3, 3/2\}$.

It is easily seen that $w(T_i) = 3$ and if $\deg(R_i) \geq 5$, then $w(R_i) = 3$ or $w(R_i) \leq 2$.

Suppose now $\deg(R_i) = 4$. Consider the (R_i, T_1)-grid (Figure 15). Let the green points on GT_1 be G, G_1, G_2. Let G_3 be the green point on R_iT_1. Let R be a red point on GG_3. Let G_4 be the green point on RT_1. By Claim 8, there is at most one green point $\neq G_1$ on G_1R_i. Thus there must be a third green point G_5 on R_iG_2. Let R_j be on the line through T_1 other than $T_1R_i, T_1R, T_1G_5, T_1G$. The final green point G_6 must be on T_1R_j. If $G_6 \in R_iG_2$, then clearly R_i has ntype $\left[\begin{smallmatrix} 1 & 1 & 1 & 4 \\ * & * & * & 1 \end{smallmatrix}\right]$ and weight 6. If on the other hand $G_6 \in R_iG_1$, then by Lemma 3 R_iG_2 is a $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$-line; R_i then has ntype $\left[\begin{smallmatrix} 1 & 1 & 2 & 3 \\ 4 & b & c & 1 \end{smallmatrix}\right]$, where $(b, c) = (4, 2)$ or $(3, 3)$, which gives $w(R_i) \in \{7/2, 10/3\}$. $\square$
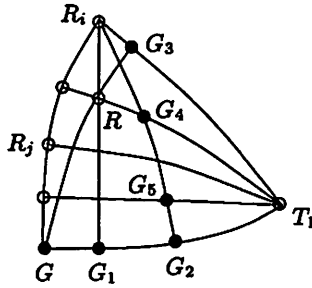
Claim 17. *Some R_i has weight 6.*

Figure 15: $\deg(R_i) = 4$ in Claim 16

Proof of Claim 17. Suppose not. Since by Lemma 7 the sum of the red weights must be $\binom{7}{2}$, we obtain by Claim 16 that $\sum_{i=1}^{4} w(R_i) \geq 12$.

Suppose $w(R_i) < 3$ for some i, without loss for $i = 4$. Then $w(R_4) \leq 2$ and $\deg(R_4) \neq 4$ by Claim 16. Since at most one red point has degree 4 (Claim 10), we have by Claim 16

$$\sum_{i=1}^{4} w(R_i) \leq 7/2 + 3 + 3 + 2 < 12,$$

a contradiction.

Thus $w(R_i) \geq 3$ for all $i = 1, 2, 3, 4$. By Claim 10 we have without loss one of the following two cases:

Case $\beta 1$: $w(R_i) = 3$ for all $i = 1, 2, 3, 4$.

Case $\beta 2$: $w(R_1) = w(R_2) = w(R_3) = 3$, $w(R_4) \in \{10/3, 7/2\}$.

Case $\beta 1$

Since $\sum_{i=1}^{4} w(R_i) = 12$ in this case, we must have $w(S_1) = w(S_2) = 3/2$ by Claim 16. If we now consider the subgeometry consisting of the green points, we find as in Case α that there are six $[^3_*]$-lines and three $[^2_*]$-lines, with some two of the $[^2_*]$-lines not intersecting in a green point, contradicting Lemma 9 as before.

Case $\beta 2$

In this case $\sum_{i=1}^{4} w(R_i) \in \{37/3, 25/2\}$, giving $w(S_1) + w(S_2) \in \{8/3, 5/2\}$. Without loss of generality we have one of

- $w(S_1) = w(S_2) = 4/3, w(R_4) = 10/3$,
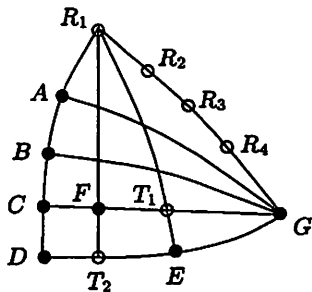- $w(S_1) = 1, w(S_2) = 3/2, w(R_4) = 7/2$.

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Figure 16: The (R_1, G)-grid in Case β

Considering the green subgeometry we obtain in both cases a contradiction as before. $\qquad\square$

Thus without loss of generality R_1 has weight 6 and ntype $\left[\begin{smallmatrix} 1 & 1 & 1 & 4 \\ * & * & * & 1 \end{smallmatrix}\right]$. Denote the $\left[\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}\right]$-line through R_1 by ℓ. Since $\deg(G) = 5$, GT_1 must intersect ℓ in a green point, which we call C. Let the other green point on GT_1 be F. Choose a green point on $R_1 T_1$ and call it E. Then GE intersects ℓ in a green point, say D. Since GE is then a $\left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]$-line, we must have $T_2 \in GE$ (see Figure 16). The only possible green point on $T_1 T_2$ must be on ℓ, say A. Finally, we let B be the fourth green point on ℓ. We thus have determined three full lines:

$$CFGT_1, \quad DEGT_2, \quad R_1ABCD. \tag{2--4}$$

Without loss we may have $S_1 \in AG$ and $S_2 \in BG$. There are now essentially two possible cases for the positions of S_1 and S_2.

Case $\beta 3$ R_1, S_1, S_2 are not collinear.

Case $\beta 4$ R_1, S_1, S_2 are collinear.

We now show that $\beta 3$ leads to a contradiction, while $\beta 4$ leads to a unique MR geometry.

Case $\beta 3$

Without loss of generality $S_1 \in R_1 T_1$ and $S_2 \in R_1 T_2$ (otherwise interchange $T_1 \leftrightarrow T_2, C \leftrightarrow D, E \leftrightarrow F$). It follows immediately that

$$FR_1 S_2 T_2 \text{ and } ER_1 S_1 T_1$$

are full lines. Since T_1, T_2, A are collinear, $A \notin T_1 S_2$. Also, $B, C, E, F, G \notin T_1 S_2$, and it follows that we must have $D \in T_1 S_2$. Similarly, since then $D \notin S_1 S_2$, we obtain $C \in S_1 S_2$. Also, since then $C \notin S_1 T_2$, we must have $B \in S_1 T_2$.
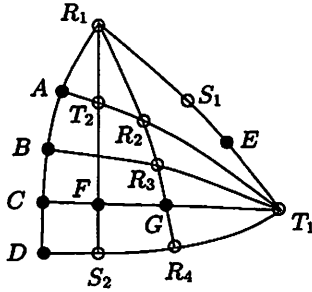
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Figure 17: The (R_1, T_1)-grid in Case $\beta 3$

We now consider the (R_1, T_1)-grid, recalling that the ntype of T_1 is known from Claim 16 (Figure 17). We may assume without loss that $R_2 \in AT_1, R_3 \in BT_1, R_4 \in DT_1$. We now have determined three full lines:

$$AR_2T_1T_2, \quad BR_3T_1, \quad DR_4S_2T_1.$$

Considering the (R_1, T_2)-grid (Figure 18), we see that we must have $R_3 \in CT_2$ (since $T_2 \notin BR_3$) and then $R_4 \in BT_2$. Since $B \in S_1T_2$, we have determined the full lines

$$BR_4S_1T_2 \text{ and } CR_3T_2.$$

By now we have the following full lines through B: $ABCDR_1$, BGS_2, $BS_1R_4T_2$, BR_3T_1. Thus there must be exactly one more line through B, namely BR_2, which must also contain the remaining green points E and F. Thus
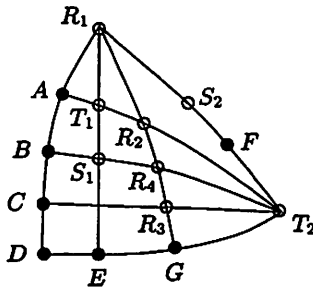
$$BEFR_2$$

is a full line.

Figure 18: The (R_1, T_2)-grid in Case $\beta 3$

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Since $S_2 \notin R_2A, R_2B, R_2G$, we obtain that $A, B, E, F, G \notin S_2R_2$. Also, since $R_2 \notin DS_2$, we have $D \notin S_2R_2$. It follows that $C \in S_2R_2$. We already have that C, S_1, S_2 are collinear, hence C, S_1, S_2, R_2 are collinear. All other points are on the other lines through R_2 (R_2A, R_2B, R_2G). Thus we have the full line

$$CS_1S_2R_2,$$

and $w(R_2) = 3$.

We now have all the lines through C except for CR_4, which must then be the full line

$$CER_4.$$

We already have the following full lines through R_3: $GR_1R_2R_3R_4$, BR_3T_1, CR_3T_2. Because of the full line $CS_1S_2R_2$ the points S_1, S_2, R_3 are not collinear. Thus R_3S_1 and R_3S_2 are two more lines through R_3. By Lemma 3 there must be a sixth line through R_3 as well. Thus R_3 must have ntype $\left[\begin{smallmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 4 & 2 & 2 & 2 & 2 & 1 \end{smallmatrix}\right]$ and $w(R_3) = 1$.

Let X_i be the green point on R_3S_i for $i = 1, 2$, and let X_3, X_4 be the two green points on the $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$-line through R_3. Because of AGS_1 and $ER_1S_1T_1$, we have $X_1 \neq A, E$, hence $X_1 \in \{D, F\}$. Because of $DR_4S_2T_1$ and $FR_1S_2T_2$, we have $X_2 \neq D, F$, hence $X_2 \in \{A, E\}$.

Because of $BEFR_2$, $\{X_3, X_4\} \neq \{E, F\}$, hence $(X_1, X_2) \neq (D, A)$. Because of $DEGT_2$, $\{X_3, X_4\} \neq \{D, E\}$, hence $(X_1, X_2) \neq (F, A)$. Because of $ABCDR_1$, $\{X_3, X_4\} \neq \{A, D\}$, hence $(X_1, X_2) \neq (F, E)$. Therefore we must have $(X_1, X_2) = (D, E)$, hence $\{X_3, X_4\} = \{A, F\}$, and we have the full line

$$AFR_3.$$

We already have the following full lines through R_4: $GR_1R_2R_3R_4$, $BR_4S_1T_2$, $DR_4S_2T_1$, CER_4. Because of $ABCDR_1$, R_4A is a fifth line. Since $F \in AR_3$, R_4F is a sixth line. Thus $w(R_4) = 1$.

If we now add up the weights of all the red points we obtain

$$
\begin{aligned}
21 &= w(S_1) + w(S_2) + w(T_1) + w(T_2) \\
&\quad + w(R_1) + w(R_2) + w(R_3) + w(R_4) \\
&\leq \frac{3}{2} + \frac{3}{2} + 3 + 3 + 6 + 3 + 1 + 1 = 20,
\end{aligned}
$$

a contradiction.

Case $\beta 4$

Without loss of generality $S_1, S_2 \in R_1T_1$ (otherwise interchange $T_1 \leftrightarrow T_2, C \leftrightarrow D, E \leftrightarrow F$). It follows immediately that the following are full lines:

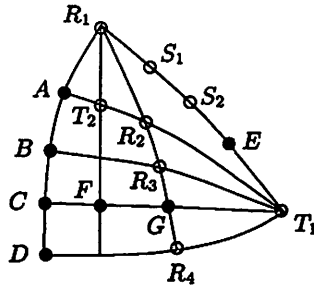$$FR_1T_2, \quad ER_1S_1S_2T_1, \quad AGS_1, \quad BGS_2. \tag{5--8}$$

Figure 19: The (R_1, T_1)-grid in Case $\beta 4$

Since T_1, T_2, A are collinear, $A \notin T_2 S_2$. Also, $B, D, E, F, G \notin T_2 S_2$, and it follows that we must have $C \in T_2 S_2$. Similarly, since then $C \notin T_2 S_1$, we obtain $B \in T_2 S_1$.

As in Case $\beta 3$ we consider the (R_1, T_1)-grid (Figure 19). We may assume without loss that $R_2 \in AT_1, R_3 \in BT_1, R_4 \in DT_1$. We now have determined three full lines:

$$AR_2T_1T_2, \quad BR_3T_1, \quad DR_4T_1. \qquad (9\text{--}11)$$

From the (R_1, T_2)-grid (Figure 20) we see that we must have $R_3 \in CT_2$ (since $T_2 \notin BR_3$), and then $R_4 \in BT_2$, and also $S_2 \in CT_2$ (since B, S_1, T_2 are collinear). We have determined the full lines

$$BR_4S_1T_2, \quad CR_3S_2T_2. \qquad (12\text{--}13)$$

We already have the following full lines through B: $ABCDR_1$, BGS_2, $BR_4S_1T_2$, BR_3T_1. Thus there must be exactly one more line through B, namely BR_2, which must also contain the remaining green points E and F. Thus
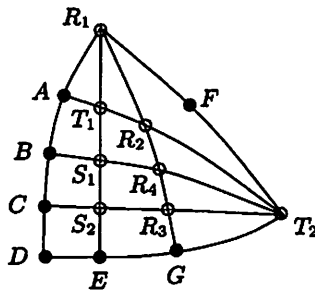
$$BEFR_2 \qquad (14)$$
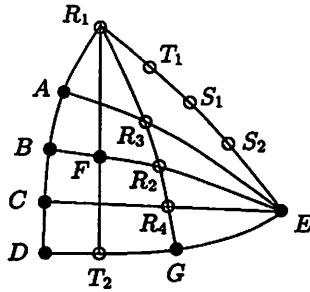

Figure 20: The (R_1, T_2)-grid in Case $\beta 4$

Figure 21: The (R_1, E)-grid in Case $\beta 4$

is a full line.

Now consider the (R_1, E)-grid (Figure 21). There is a red point on each of AE, BE, CE, DE, and 4 red points on $R_1 E$. Thus $\deg(E) = 5$. Since $E \notin CR_3, R_3 \in AE$, and then $R_4 \in CE$. Thus we have the full lines

$$AER_3, \quad CER_4. \tag{15--16}$$

Consider the (R_1, R_2)-grid (Figure 22). Considering the green points we see that $\deg(R_2) = 5$ and $w(R_2) = 3$. Since $R_2 \notin CS_2$, $S_2 \notin CR_2$, hence $S_2 \in DR_2$ and $S_1 \in CR_2$. Thus we have the full lines

$$CR_2S_1, \quad DR_2S_2. \tag{17--18}$$

Consider the (R_1, R_3)-grid (Figure 23). If $S_1 \in DR_3$, then either $F \in DR_3$ and $w(R_3) = 3/2$, or $F \notin DR_3$ and $w(R_3) = 1$. If on the other hand $S_1 \notin DR_3$, then $F \in S_1 R_3$ and $w(R_3) = 1$. Summarizing: $w(R_3) \leq 3/2$, with equality iff DFR_3S_1 is a line.

Consider the (R_1, R_4)-grid (Figure 24). If $S_2 \in AR_4$, then either $F \in AR_4$ and $w(R_4) = 3/2$, or $F \notin AR_4$ and $w(R_4) = 1$. If on the other hand $S_2 \notin AR_4$, then $F \in S_2 R_4$ and $w(R_4) = 1$. Summarizing: $w(R_4) \leq 3/2$, with equality iff AFR_4S_2 is a line.
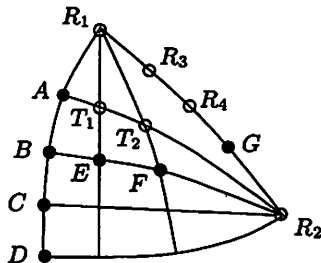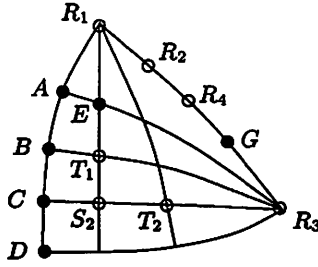

Figure 22: The (R_1, R_2)-grid in Case $\beta 4$

Figure 23: The (R_1, R_3)-grid in Case $\beta 4$

We now add up the weights of the red points:

$$
\begin{aligned}
21 \;=\; & w(S_1) + w(S_2) + w(T_1) + w(T_2) \\
& + w(R_1) + w(R_2) + w(R_3) + w(R_4) \\
\leq \; & \frac{3}{2} + \frac{3}{2} + 3 + 3 + 6 + 3 + \frac{3}{2} + \frac{3}{2} = 21.
\end{aligned}
$$

It follows that $w(R_3) = w(R_4) = 3/2$, giving the lines

$$DFR_3S_1, \quad AFR_4S_2. \tag{19-20}$$

We have now determined all the lines (1–20). We have found a unique MR geometry, which is isomorphic to MR_{15}^5 by the following isomorphism:

A	B	C	D	E	F	G	R_1	R_2	R_3	R_4	S_1	S_2	T_1	T_2
5_2	4_0	5_1	3_0	0_0	1_2	1_0	6_0	5_0	2_2	2_1	4_2	2_0	4_1	6_1

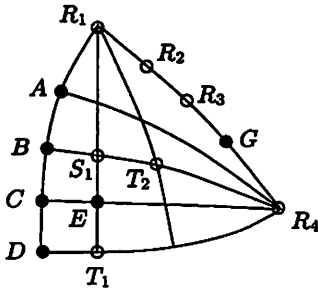We have finished the proof of Proposition 6 and thus of Theorem 1. $\quad\square$

Figure 24: The (R_1, R_4)-grid in Case $\beta 4$

Appendix: Finding all one-point extensions of the MR geometries on 14 points

A *partial parallel class* (ppc) of a finite linear space S is a partition of the points of S such that each set in the partition is either a line of S or a singleton. An *extendable* ppc (eppc) of an MR geometry is a ppc in which all singletons (if any) are of the same colour. Clearly each one-point extension of an f.l.s. S determines a ppc: If $S \cup \{\infty\}$ is an extension of S then

$$\{\ell \setminus \{\infty\} : \infty \in \ell\}$$

is a ppc of S. Furthermore, for each MR geometry, this ppc is an eppc. Conversely, each eppc of an MR geometry determines at least one one-point extension:

Lemma 10. *Let (S, χ) be an MR geometry.*

- *For each ppc of (S, χ) not containing singletons there exist exactly two one-point extensions of (S, χ).*

- *For each ppc of (S, χ) containing at least one singleton, and with all singletons of the same colour, there exits exactly one one-point extension of (S, χ).*

Proof. If $\{B_i : i \in I\}$ is a ppc then the extension we are looking for is $S \cup \{\infty\}$ with the original lines of S together with all $B_i \cup \{\infty\}$. In the first case we are free to give ∞ any colour, while in the second case the colour of ∞ is forced. $\square$

Extending MR_{14}^1

By Lemma 10 it is sufficient to enumerate all eppc's of MR_{14}^1.

Case I: The eppc contains a 4-line of MR_{14}^1

By symmetry we may take the line to be $0_0 1_0 3_0 6_1$. The only lines parallel to $0_0 1_0 3_0 6_1$ are $0_1 1_1 3_1 6_0$, $5_0 5_1$, $2_0 2_1$, and $4_0 4_1$. These five lines already partition MR_{14}^1. If one of these lines, say ℓ, does not belong to the eppc then the eppc must contain singletons, one corresponding to each point of ℓ. Then there are singletons of different colours, a contradiction. Thus the only eppc is

$$\{0_0 1_0 3_0 6_1, 0_1 1_1 3_1 6_0, 5_0 5_1, 2_0 2_1, 4_0 4_1\}.$$

The point at infinity on these lines correspond to 6_2 in $PG(2, 4)$. Since the eppc has no singletons, 6_2 can be assigned any colour. Thus there are two extensions that are clearly isomorphic. If we colour 6_2 green we obtain MR_{15}^1.

G	R	G_1	G_2	G_3	S_1	S_2	S_3	H_1	H_2	H_3	R_1	R_2	R_3
6_0	6_1	4_1	2_1	5_1	2_0	5_0	4_0	0_0	1_0	3_0	1_2	3_2	0_2
6_0	6_1	2_0	4_0	5_0	4_1	5_1	2_1	0_0	3_0	1_0	0_2	3_2	1_2
6_0	6_1	5_0	2_0	4_0	2_1	4_1	5_1	1_0	0_0	3_0	1_2	0_2	3_2
6_0	6_1	2_1	5_1	4_1	5_0	4_0	2_0	1_0	3_0	0_0	3_2	0_2	1_2
6_0	6_1	5_1	4_1	2_1	4_0	2_0	5_0	3_0	0_0	1_0	0_2	1_2	3_2
6_0	6_1	4_0	5_0	2_0	5_1	2_1	4_1	3_0	1_0	0_0	3_2	1_2	0_2

Figure 25: All isomorphisms between the two representations of MR_{14}^2

Case II: There are no 4-lines in the eppc

The seven 2-lines of MR_{14}^1 already form a partition. If one of them does not belong to the eppc, then as before, there are singletons of different colours, a contradiction.

Thus there are two one-point extensions, depending on the colour of ∞ adjoined to each of the 2-lines. These two extensions are clearly isomorphic. If we colour ∞ green we obtain MR_{15}^2.

Extending MR_{14}^2

Since it is not clear from the given representation of MR_{14}^2 what its symmetries are, we use another representation:

> Green points: $G, G_1, G_2, G_3, H_1, H_2, H_3$
> Red points: $R, R_1, R_2, R_3,, S_1, S_2, S_3$
> Lines: $GR_1R_2R_3R$, $RH_1H_2H_3$
> $GG_iH_iS_i$, $G_iG_{i+1}H_{i-1}R_{i+1}$
> $H_{i-1}R_{i-1}S_iS_{i+1}$, RS_iG_{i-1}
> $G_{i+1}S_iR_i$, H_iR_{i+1},

with $i = 1, 2, 3$, and indices modulo 3.

The mappings in Figure 25 can easily be checked to be isomorphisms (between linear spaces). We now show that these are the only isomorphisms.

The only point occurring in only one 4-line is 6_1 in the first representation, and R in the second representation. Thus we must have $R \mapsto 6_1$ and $\{H_1, H_2, H_3\} \mapsto \{0_0, 1_0, 3_0\}$.

The only 2-lines are $1_2 3_0$, $3_2 0_0$, $0_2 1_0$, and H_1R_2, H_2R_3, H_3R_1. Thus we must have $\{R_1, R_2, R_3\} \mapsto \{0_2, 1_2, 3_2\}$. Furthermore, the restriction to $\{H_1, H_2, H_3\}$ determines the restriction to $\{R_1, R_2, R_3\}$.

The only 5-line is $0_2 1_2 3_2 6_0 6_1$, $GR_1R_2R_3R$. Thus we must have $G \mapsto 6_0$.

At this stage G and R are fixed. If we choose any of the 3! mappings $\{H_1, H_2, H_3\} \mapsto \{0_0, 1_0, 3_0\}$ then, as mentioned above, the images of R_1, R_2, R_3 are determined. In each case, by considering the three lines $H_i R_i$, we determine the images of the pairs $\{S_2, S_3\}$, $\{S_3, S_1\}$, $\{S_1, S_2\}$, from which the image of each S_i is determined. Then by considering the lines RS_i we determine the images of G_1, G_2, G_3, and the whole isomorphism is fixed. It then only remains to check that the remaining lines $GG_i H_i S_i$, $G_i G_{i+1} H_{i-1} R_{i+1}$, $G_{i+1} S_i R_i$ are mapped properly. In this way all six isomorphisms are obtained. (Note that the even automorphisms preserve the colours, i.e., they are also isomorphisms of MR geometries.)

Now we can write down the automorphisms of the second representation:

identity
$(G_1 S_1)(G_2 S_3)(G_3 S_2)(H_3 H_2)(R_3 R_1)$
$(G_1 S_2)(G_2 S_1)(G_3 S_3)(H_2 H_1)(R_2 R_3)$
$(G_1 S_3)(G_2 S_2)(G_3 S_1)(H_1 H_3)(R_1 R_2)$
$(G_1 G_2 G_3)(S_1 S_2 S_3)(H_1 H_2 H_3)(R_1 R_2 R_3)$
$(G_3 G_2 G_1)(S_3 S_2 S_1)(H_3 H_2 H_1)(R_3 R_2 R_1)$

The action of the automorphism group on the lines has seven orbits:

(i) $GR_1 R_2 R_3 R$
(ii) $RH_1 H_2 H_3$
(iii) $GG_i H_i S_i$
(iv) $G_i G_{i+1} H_{i-1} R_{i+1}, \ H_{i-1} R_{i-1} S_i S_{i+1}$
(v) $RS_i G_{i-1}$
(vi) $G_{i+1} S_i R_i$
(vii) $H_i R_{i+1}$

We now determine all eppc's of MR_{14}^2. We consider seven cases, depending on whether the eppc contains a line from one of the seven orbits. (We could also have considered 20 cases, one for each line, in which case it wouldn't have been necessary to calculate the automorphism group. However, such an approach would have been more tedious and less insightful.)

Case (i)

Since $GR_1 R_2 R_3 R$ intersects all other lines, all points not on this line must be singletons in a ppc. Thus there is no eppc containing $GR_1 R_2 R_3 R$.

Case (ii)

Suppose $RH_1 H_2 H_3$ is an eppc. The only parallel lines are $G_{i+1} S_i R_i$. All of them must be in the eppc. The only remaining point is G, which must

be a singleton in the eppc. Thus there is one extension, with ∞ coloured red. This MR geometry is isomorphic to MR_{15}^1:

G	R	G_1	G_2	G_3	S_1	S_2	S_3	H_1	H_2	H_3	R_1	R_2	R_3	∞
6_1	6_2	5_1	4_1	2_1	2_0	5_0	4_0	1_1	3_1	0_1	0_0	1_0	3_0	6_0

Case (iii)

Assume without loss of generality that $RH_1H_2H_3$ is not in the eppc, and $GG_1H_1S_1$ is. Lines parallel to $GG_1H_1S_1$ are RS_3G_2, $G_3S_2R_2$, H_2R_3, H_3R_1. They already partition MR_{14}^2. All of them must be in the eppc. The point ∞ may have either colour. In the first two isomorphisms in Figure 25 the point ∞ corresponds to 4_2. We have obtained MR_{15}^{3g} and MR_{15}^{3r}.

Case (iv)

Assume without loss of generality that Cases (i), (ii) and (iii) do not occur, and that $G_1G_2H_3R_2$ is in an eppc. Lines parallel to $G_1G_2H_3R_2$ are $H_1R_1S_2S_3$, RS_1G_3, H_2R_3. All these lines must be in the eppc, leaving G as a singleton. Thus there is a single extension, with ∞, which has to be red, corresponding to 3_1 in the second and fourth isomorphisms in Figure 25. We have obtained MR_{15}^4.

Cases (v), (vi), and (vii)

In each of the remaining cases we obtain that a ppc must contain singletons of different colours, giving no extensions.

We have shown that MR_{15}^1, MR_{15}^{3g}, MR_{15}^{3r}, and MR_{15}^4 are the only extensions of MR_{14}^2. $\square$

Acknowledgements

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