The number of repeated blocks in twofold extended triple systems

Wen-Chung Huang*and Fu-Chang Ke
Department of Mathematics
Soochow University,
Taipei, Taiwan,
Republic of China.

Abstract

A twofold extended triple system with two idempotent elements, $TETS(v)$, is a pair (V, B), where V is a v-set and B is a collection of triples, called blocks, of type $\{x, y, z\}$, $\{x, x, y\}$ or $\{x, x, x\}$ such that every pair of elements of V, not necessarily distinct, belongs to exactly two triples and there are only two triples of the type $\{x, x, x\}$.

This paper shows that an indecomposable $TETS(v)$ exists which contains exactly k pairs of repeated blocks if and only if $v \not\equiv 0$ mod 3, $v \geq 5$ and $0 \leq k \leq b_v - 2$, where $b_v = (v+2)(v+1)/6$

1 Introduction

An extended triple system (a twofold extended triple system) of order v is a pair (V, B), where V is a v-set and B is a collection of triples of elements in V, where each triple may have repeated elements, such that every pair of elements of V, not necessarily distinct, belongs to exactly one (exactly two) triples. The elements of B are called blocks. There are three types of blocks: (1) $\{x, y, z\}$ (2) $\{x, x, y\}$ (3) $\{x, x, x\}$. For brevity we

*This research is supported by NSC 94-2115-M-031-004

write these blocks as xyz, xxy and xxx and call them triangle, lollipop and idempotent, respectively. If (V, B) is an extended triple system of order v with a idempotents, we say B is an ETS(v, a).

The concept of an extended triple system was first introduced by D. M. Johnson and N. S. Mendelsohn[4]. They established the necessary conditions for the existence of ETS(v, a), and F. E. Bennett and N. S. Mendelsohn[1] showed that the necessary conditions were also sufficient.

Theorem 1.1 *[1, 4] There exists an ETS(v, a), if and only if,* $0 \le a \le v$ *and*
(i) if $v \equiv 0$ *(mod 3) then* $a \equiv 0$ *(mod 3);*
(ii) if $v \not\equiv 0$ *(mod 3) then* $a \equiv 1$ *(mod 3);*
(iii) if v *is even then* $a \le v/2$*;*
(iv) if $a = v - 1$ *then* $v = 2$.

From now on, we restrict our attention to extended triple systems (twofold extended triple systems) with one idempotent (two idempotents). We shall denote such a design, based on a v-set, by ETS(v) (TETS(v)). An ETS(v) has $b_v = (v+2)(v+1)/6$ blocks and a TETS(v) has $(v+2)(v+1)/3$ blocks. From Theorem 1.1, an ETS(v) exists if and only if $v \not\equiv 0$ (mod 3). A necessary and sufficient condition for the existence of a TETS(v) is that $v \not\equiv 0$ (mod 3). If a TETS(v) contains two blocks b_1 and b_2 that are identical as subsets of V, then the block is called a repeated block of the design. In 1998, E. J. Billington and G. Lo Faro [2] studied the repeated blocks in indecomposable twofold extended triple systems without idempotent. We can find the repeated blocks in an indecomposable twofold extended triple system with one idempotent. But a small calculation proves that the system cannot exist. Therefore, we are interested in the following question: Given $v \not\equiv 0$ (mod 3) and a nonnegative integer k, does there exist a TETS(v) with exactly k repeated blocks? This question is related to the intersection problem for extended triple systems, solved by Huang [3] in 2000. The same question with the additional condition that the twofold extended triple system is indecomposable (that is, cannot have its blocks partitioned into two ETS) is also of interest.

Let $J[v]$ be the set of non-negative integers k such that there exists an indecomposable TETS(v) with k repeated blocks and let $I[v] = \{0, 1, 2, \ldots, b_v - 3, b_v - 2\}$. Clearly $J[v] \subseteq I[v]$, since a TETS(v) having b_v repeated blocks yield a decomposable TETS(v). In the following, we denote the k-tuple $\langle v_1, v_2, \ldots, v_k\rangle$ by $\{v_1v_1v_2,\ v_2v_2v_3,\ \ldots,\ v_{k-1}v_{k-1}v_k,\ v_kv_kv_1\}$, where $v_i \ne v_j$ for all $i \ne j$.

Main Theorem $J[v] = I[v]$, for all $v \not\equiv 0 \pmod 3$.

Let A and B be two sets of integers and k a positive integer. We define $A+B = \{a+b \mid a \in A, b \in B\}$, $k+A = \{k\}+A$, and $kA = \{k \cdot a \mid a \in A\}$.

2 Recursive constructions of TETS(v)

In order to count the repeated blocks, we use some special embedding constructions. Let (V_1, B_1) be an indecomposable TETS(v) and $(V_1 \cup V_2, B_2)$ be an extended triple system of order u with a hole v and without idempotent, then $(V_1 \cup V_2, B_1 \cup 2B_2)$ is an indecomposable TETS(u), where $2B_2$ consists of blocks of B_2, each block occurring twice. For convenience, we write $V_1 = \{a_1, a_2, \ldots, a_v\}$ and $V_2 = \{x_1, x_2, \ldots, x_{u-v}\}$. Below we describe the structure of an extended triple system of order u with a hole v, for certain values of u.

1) $u = 2v$, v is even

Let $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, v-1\}$ be a 1-factorization of K_v on V_2. Let $V = V_1 \cup V_2$ and $B = T \cup L$, where $T = \{a_i xy \mid xy \in F_i,\ i = 1, 2, \ldots, v-1\}$ and $L = \{a_v xx \mid x \in V_2\}$. Then (V, B) is an ETS($2v$) with a hole v.

2) $u = 2v$, v is odd

Let $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, v\}$ be a near 1-factorization of K_v on $V_2 = \{x_1, x_2, \ldots, x_v\}$, where $x_i \notin F_i$. Let $V = V_1 \cup V_2$ and $B = T \cup L$, where $T = \{a_i xy \mid xy \in F_i,\ i = 1, 2, \ldots, v\}$ and $L = \{a_i x_i x_i \mid i = 1, 2, \ldots, v\}$. Then (V, B) is an ETS($2v$) with a hole v.

Let $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, 2v-1\}$ be a 1-factorization of K_{2v} on $N = \{1, 2, \ldots, 2v\}$. If $F_a, F_b \in \mathcal{F}$, the notation $F_a \cdot F_b$ [6] will denote the following set of blocks: $\langle 1, x_{i_2}, x_{i_3}, \ldots, x_{i_r} \rangle \cup \langle x_{j_1}, x_{j_2}, x_{j_3}, \ldots, x_{j_s} \rangle \cup \ldots \cup \langle x_{p_1}, x_{p_2}, x_{p_3}, \ldots, x_{p_t} \rangle \cup \langle x_{q_1}, x_{q_2}, x_{q_3}, \ldots, x_{q_m} \rangle$, where $x_{j_1} = min(N \setminus \{1, x_{i_2}, x_{i_3}, \ldots, x_{i_r}\})$, $\ldots$, $x_{q_1} = min(N \setminus \{1, x_{i_2}, x_{i_3}, \ldots, x_{i_r}, x_{j_1}, x_{j_2}, x_{j_3}, \ldots, x_{j_s}, \ldots, x_{t_1}, x_{t_2}, x_{t_3}, \ldots, x_{t_m}\})$; $F_a = \{1x_{i_2}, x_{i_3}x_{i_4}, \ldots, x_{i_{r-1}} x_{i_r}, x_{j_1}x_{j_2}, x_{j_3}x_{j_4}, \ldots, x_{j_{s-1}}x_{j_s}, \ldots, x_{p_1}x_{p_2}, x_{p_3}x_{p_4}, \ldots, x_{p_{t-1}}x_{p_t}, x_{q_1}x_{q_2}, x_{q_3}x_{q_4}, \ldots, x_{q_{m-1}}x_{q_m}\}$ and $F_b = \{x_{i_2}x_{i_3}, x_{i_4}x_{i_5}, \ldots, x_{i_r}1, x_{j_2}x_{j_3}, x_{j_4}x_{j_5}, \ldots, x_{j_s}x_{j_1}, \ldots, x_{p_2}x_{p_3}, x_{p_4}x_{p_5}, \ldots, x_{p_t}x_{p_1}, x_{q_2}x_{q_3}, x_{q_4}x_{q_5}, \ldots, x_{q_m}x_{q_1}\}$.

3) $u = 2v + 3$, v is odd

Let $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, v+2\}$ be a 1-factorization of K_{v+3} on V_2. Let $V = V_1 \cup V_2$ and $B = T \cup F_{v+1} \cdot F_{v+2}$, where $T = \{a_i xy \mid xy \in F_i,\ i = 1, 2, \ldots, v\}$. Then (V, B) is an ETS$(2v+3)$ with a hole v.

Let K_{2v} be a complete graph on $2v$ vertices($2v \geq 8$). The edges of K_{2v} fall into v disjoint classes P_1, P_2, $\ldots$, P_v with $\{i, k\} \in P_j$ if and only if $i - k \equiv j \pmod{2v}$. R. G. Stanton and I. P. Goulden [7] prove that

P1. If $2x + 1 < v$ then $P_{2x} \cup P_{2x+1}$ splits into four 1-factors;

P2. The graph K_{2v} may be factored into a set of $2v$ triangles covering P_1, P_{2j}, P_{2j+1} $(2j+1 < v)$ and a set of $2v - 7$ 1-factors covering the other P_i.

4) $u = 2v + 9$, v is odd

Factor the complete graph K_{v+9} on V_2 by the above description. Let T_1 be the set of $v + 9$ triangles and $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, v+2\}$ the set of 1-factors. Put $V = V_1 \cup V_2$ and $B = T_1 \cup T \cup F_{v+1} \cdot F_{v+2}$, where $T = \{a_i xy \mid xy \in F_i,\ i = 1, 2, \ldots, v\}$. Then (V, B) is an ETS$(2v+9)$ with a hole v.

3 Basic Lemmas

For even v, let $\mathcal{F}$ and $\mathcal{G}$ be two 1-factorizations of K_v, where $\mathcal{F} = \{F_1, F_2, \ldots, F_{v-1}\}$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_{v-1}\}$. We will say that $\mathcal{F}$ and $\mathcal{G}$ have k edges in common if $k = \sum_{i=1}^{v-1} |F_i \cap G_i|$. Let $J_F(v)$ be the set of k such that there exist a pair of 1-factorizations of K_v having k common edges. In [5], C. C. Lindner and W. D. Wallis showed that $J_F(2) = \{1\}$, $J_F(6) = \{0, 1, 2, 3, 5, 6, 7, 9, 15\}$ and $J_F(v) = \{0, 1, 2, \ldots, \binom{v}{2} = t\} \setminus \{t-1, t-2, t-3, t-5\}$ for $v = 4$ or $v \geq 8$.

Lemma 3.1 *Let v be even, $v \not\equiv 0 \pmod 3$ and $v \geq 8$. If $J[v] = I[v]$ then $J[2v] = I[2v]$.*

Proof. Let (V_1, B) be a TETS(v) with k repeated blocks. Let $\mathcal{F}$ and $\mathcal{G}$ be two 1-factorizations of K_v on $V_2 = \{x_1, x_2, \ldots, x_v\}$ such that $h = \sum_{i=1}^{v-1} |F_i \cap G_i|$. Put $T_1 = \{a_i xy \mid xy \in F_i,\ i = 1, 2, \cdots, v-1\}$, $T_2 = \{a_i xy \mid xy \in G_i,\ i = 1, 2, \cdots, v-1\}$ and $L = \{a_v xx \mid x \in V_2\}$. The system $B \cup T_1 \cup T_2 \cup 2L$ has $k + v + h$ repeated blocks. Therefore

$$J[2v] \supseteq J[v] + v + J_F(v).$$

Since $J[v] = I[v]$,

$$J[2v] \supseteq I[v] + v + J_F(v) = I[2v] \setminus \{0, 1, \cdots, v-1\}.$$

For the remaining values, let (V_1, B) be a TETS(v) with k repeated blocks, $k \in \{0, 1, \cdots, v-1\}$. Let $T_1 = \{a_i xy \mid xy \in F_i,\ i = 1, 2, \cdots, v-1\}$, $T_2 = \{a_{i+1} xy \mid xy \in F_i,\ i = 1, 2, \cdots, v-1\}$, $L_1 = \{a_v xx \mid x \in V_2\}$ and $L_2 = \{a_1 xx \mid x \in V_2\}$, then $B \cup T_1 \cup T_2 \cup L_1 \cup L_2$ has k repeated blocks. This implies that $J[2v] = I[2v]$. ∎

Lemma 3.2 *Let v be odd, $v \not\equiv 0$ (mod 3) and $v \geq 5$. If $J[v] = I[v]$ then $J[2v] = I[2v]$.*

Proof. Let (V_1, B) be a TETS(v) with k repeated blocks and $\mathcal{F}$ a near 1-factorization of K_v on $V_2 = \{x_1, x_2, \ldots, x_v\}$, where $x_i \notin F_i$. Let T and L be the same as construction 2 in section 2, $T_\alpha = \{a_i xy \mid xy \in F_{\alpha(i)},\ i = 1, 2, \ldots, v\}$ and $L_\alpha = \{a_i x_{\alpha(i)} x_{\alpha(i)} \mid i = 1, 2, \ldots, v\}$, where α is a permutation of $\{1, 2, \ldots, v\}$ with exactly p elements fixed. Note that α exists for $p = 0, 1, 2, \ldots, v-2, v$. Then the system $B \cup T \cup L \cup T_\alpha \cup L_\alpha$ has $k + p(v+1)/2$ repeated blocks. Therefore

$$J[2v] \supseteq J[v] + \frac{v+1}{2}\{0, 1, 2, \ldots, v-2, v\}.$$

Since $J[v] = I[v]$,

$$J[2v] \supseteq I[v] + \frac{v+1}{2}\{0, 1, 2, \ldots, v-2, v\} = I[2v].$$

∎

Lemma 3.3 *Let v be odd, $v \not\equiv 0$ (mod 3) and $v \geq 7$. If $J[v] = I[v]$ then $J[2v+3] = I[2v+3]$.*

Proof. Let (V_1, B) be a TETS(v) with k repeated blocks and $\mathcal{F}$ a 1-factorization of K_{v+3} on $V_2 = \{x_1, x_2, \ldots, x_{v+3}\}$. Let $T = \{a_i xy \mid xy \in$

$F_i,\ i = 1,2,\ldots,v\}$ and $T_\alpha = \{a_i xy \mid xy \in F_{\alpha(i)},\ i = 1,2,\ldots,v\}$, where α is a permutation of $\{1,2,\ldots,v\}$ with exactly p elements fixed. Set $\mathcal{L} = 2F_{v+1}\cdot F_{v+2}$ or $F_{v+1}\cdot F_{v+2} \cup F_{v+2}\cdot F_{v+1}$. Then the system $B \cup T \cup T_\alpha \cup \mathcal{L}$ has $k+p(v+3)/2+q(v+3)$ repeated blocks, where $p \in \{0,1,2,\ldots,v-2,v\}$ and $q \in \{0,1\}$. Therefore

$$J[2v+3] \supseteq J[v] + \frac{v+3}{2}\{0,1,2,\ldots,v-2,v\} + (v+3)\{0,1\}.$$

Since $J[v] = I[v]$,

$$J[2v+3] \supseteq I[v] + \frac{v+3}{2}\{0,1,2,\ldots,v-2,v\} + (v+3)\{0,1\} = I[2v+3].$$

This implies that $J[2v+3] = I[2v+3]$. ∎

Lemma 3.4 *Let v be odd, $v \not\equiv 0$ (mod 3) and $v \geq 11$. If $J[v] = I[v]$ then $J[2v+9] = I[2v+9]$.*

Proof. Let (V_1, B) be a TETS(v) with k repeated blocks and $\mathcal{F}$ a 1-factorization of K_{v+9} on $V_2 = \{x_1, x_2, \ldots, x_{v+9}\}$. Let $\{P_i \mid i = 1,2,\cdots,(v+9)/2\}$ be the set of disjoint classes which is a partition of the edges of K_{v+9} [7]. From the property **P2** in the page 4, we have the set of triangles $T_1 = \{x_i x_{i+1} x_{i+3} \mid i = 1,2,\cdots,v+9\}$ covering P_1, P_2, P_3, and the set of 1-factors $\{F_i \mid i = 1,2,\cdots,v+2\}$ covering other P_i. Let α be a permutation of $\{1,2,\ldots,v\}$ with p elements fixed, $T = \{a_i xy \mid xy \in F_i,\ i = 1,2,\ldots,v\}$ and $T_\alpha = \{a_i xy \mid xy \in F_{\alpha(i)},\ i = 1,2,\ldots,v\}$. Set $\mathcal{L} = 2F_{v+1}\cdot F_{v+2}$ or $F_{v+1}\cdot F_{v+2} \cup F_{v+2}\cdot F_{v+1}$. Then $B \cup T \cup T_\alpha \cup 2T_1 \cup \mathcal{L}$ has $k + (v+9) + p(v+9)/2 + q(v+9)$ repeated blocks, where $p \in \{0,1,2,\ldots,v-2,v\}$ and $q \in \{0,1\}$. Therefore

$$J[2v+9] \supseteq J[v] + \frac{v+9}{2}\{0,1,2,\ldots,v-2,v\} + (v+9)\{1,2\}.$$

Since $J[v] = I[v]$,

$$\begin{aligned} J[2v+9] &\supseteq I[v] + \frac{v+9}{2}\{0,1,2,\ldots,v-2,v\} + (v+9)\{1,2\} \\ &= I[2v+9] \setminus \{0,1,2,\cdots,v+8\}. \end{aligned}$$

For the missing values $\{0,1,2,\cdots,v+8\}$, we can take a TETS(v), (V_1, B), with k repeated blocks, where $k \in \{0,1,2,\cdots,v+8\}$. From the property **P2** in the page 4, we can choose a set of triangles $T_2 = \{x_i x_{i+4} x_{i+5} \mid i = 1,2,\cdots,v+9\}$ covering P_1, P_4, P_5, and a set of 1-factors

$\{G_i \mid i = 1, 2, \cdots, v+2\}$ covering the other P_i. (We can assume that $F_i = G_i$, for $i = 5, 6, \cdots, v+2$.) If α is a permutation of $\{1, 2, \ldots, v\}$ without a fixed element and $T_\alpha = \{a_i xy \mid xy \in G_{\alpha(i)}, i = 1, 2, \ldots, v\}$, then $B \cup T \cup T_\alpha \cup T_1 \cup T_2 \cup F_{v+1} \cdot F_{v+2} \cup F_{v+2} \cdot F_{v+1}$ has exactly k repeated blocks. Thus $\{0, 1, 2, \cdots, v+8\} \subset J[2v+9]$. ∎

4 J[v] for small v

For convenience, we write t_i for $10+i$, d_i for $20+i$, t for 10 and d for 20 from now on. Also we shall denote the sets $\{abc, abd, acd, bcd\}$ and $\{abc, abd, cca, ccb, dda, ddb\}$ by $E_1(abcd)$ and $E_2(abcd)$, respectively, which are indecomposable partial twofold extended triple systems. In the following cases, $B = B_N \cup B_R$, where B_N denotes single occurrences of blocks in the system B and B_R denotes blocks which are to be repeated.

J[5] = I[5]:

$$0 \in J[5] \text{ by } \begin{cases} B_R = \emptyset \\ B_N = \{111, 222, 123, 145, 234, 345, 331, 335, 441, 442, 552\} \cup \\ \qquad \langle 1, 2, 5 \rangle \end{cases}$$

$$1 \in J[5] \text{ by } \begin{cases} B_R = \{125\} \\ B_N = E_2(3425) \cup \langle 1, 3 \rangle \cup \langle 1, 4 \rangle \cup \{333, 444\} \end{cases}$$

$$2 \in J[5] \text{ by } \begin{cases} B_R = \{111, 553\} \\ B_N = \{123, 145, 125, 234, 225, 224, 331, 334, 441, 445\} \end{cases}$$

$$3 \in J[5] \text{ by } \begin{cases} B_R = \{115, 225, 335\} \\ B_N = E_1(1234) \cup \langle 4, 5 \rangle \cup \{444, 555\} \end{cases}$$

$$4 \in J[5] \text{ by } \begin{cases} B_R = \{111, 125, 331, 441\} \\ B_N = E_2(3425) \end{cases}$$

$$5 \in J[5] \text{ by } \begin{cases} B_R = \{115, 225, 335, 445, 555\} \\ B_N = E_1(1234) \end{cases}$$

J[7] = I[7]:

$$0 \in J[7] \text{ by } \begin{cases} B_R = \emptyset \\ B_N = E_1(1234) \cup \{167, 257, 356, 456, 357, 267, 111, 444, 225, \\ \qquad 226, 336, 337, 661, 664, 771, 554\} \cup \langle 1, 5 \rangle \cup \langle 4, 7 \rangle \end{cases}$$

$$1 \in J[7] \text{ by } \begin{cases} B_R = \{444\} \\ B_N = E_1(1234) \cup \{456, 457, 467, 167, 256, 357\} \cup \langle 2, 5, 3, 7\rangle \\ \qquad \cup \langle 1, 6, 2, 7\rangle \cup \langle 3, 6\rangle \cup \langle 1, 5\rangle \end{cases}$$

$$2 \in J[7] \text{ by } \begin{cases} B_R = \{115, 447\} \\ B_N = E_1(1234) \cup \{167, 257, 356, 456, 357, 267, 555, 777, 771, \\ \qquad 554, 661, 664, 225, 226, 336, 337\} \end{cases}$$

$$3 \in J[7] \text{ by } \begin{cases} B_R = \{666, 551, 774\} \\ B_N = E_1(1234) \cup \{167, 257, 356, 456, 357, 267, 116, 117, 445, \\ \qquad 446, 225, 226, 336, 337\} \end{cases}$$

$$4 \in J[7] \text{ by } \begin{cases} B_R = \{444, 554, 664, 774\} \\ B_N = E_1(1234) \cup \{156, 167, 257, 267, 356, 357, 115, 117, 225, \\ \qquad 226, 336, 337\} \end{cases}$$

$$5 \in J[7] \text{ by } \begin{cases} B_R = \{115, 336, 447, 167, 456\} \\ B_N = E_1(1234) \cup \{257, 357, 222, 666, 772, 773, 552, 553\} \cup \\ \qquad \langle 2, 6\rangle \end{cases}$$

$$6 \in J[7] \text{ by } \begin{cases} B_R = \{167, 456, 115, 336, 447, 662\} \\ B_N = E_1(1234) \cup \{257, 357, 555, 777, 225, 227, 773, 553\} \end{cases}$$

$$7 \in J[7] \text{ by } \begin{cases} B_R = \{167, 456, 666, 115, 226, 336, 447\} \\ B_N = E_1(1234) \cup \{257, 357, 772, 773, 552, 553\} \end{cases}$$

$$8 \in J[7] \text{ by } \begin{cases} B_R = \{167, 257, 356, 115, 554, 446, 774, 337\} \\ B_N = E_1(1234) \cup \{222, 666\} \cup \langle 2, 6\rangle \end{cases}$$

$$9 \in J[7] \text{ by } \begin{cases} B_R = \{167, 145, 256, 357, 227, 336, 664, 774, 555\} \\ B_N = E_2(2314) \end{cases}$$

$$10 \in J[7] \text{ by } \begin{cases} B_R = \{167, 256, 357, 444, 115, 227, 336, 554, 664, 774\} \\ B_N = E_1(1234) \end{cases}$$

J(8) = I[8]:

$$0 \in J[8] \text{ by } \begin{cases} B_R = \emptyset \\ B_N = E_1(1234) \cup \{178, 156, 478, 456, 267, 258, 357, 368, 467, \\ \qquad 158, 444, 555, 116, 117, 225, 662, 775\} \cup \langle 3, 5, 4, 8, 6\rangle \cup \\ \qquad \langle 2, 7, 3, 8\rangle \end{cases}$$

$$1 \in J[8] \text{ by } \begin{cases} B_R = \{333\} \\ B_N = E_1(1234) \cup \{178, 156, 478, 456, 267, 258, 367, 358, 357, \\ \qquad 368\} \cup \langle 1, 5, 4, 7\rangle \cup \langle 1, 6, 4, 8\rangle \cup \langle 2, 5, 7\rangle \cup \langle 2, 6, 8\rangle \end{cases}$$

$$2 \in J[8] \text{ by } \begin{cases} B_R = \{111, 468\} \\ B_N = E_1(1234) \cup \{156, 167, 178, 158, 256, 278, 367, 358, 225, \\ \qquad 227, 335, 337, 662, 663, 882, 883\} \cup \langle 4, 5, 7\rangle \cup \langle 4, 7, 5\rangle \end{cases}$$

$$3 \in J[8] \text{ by } \begin{cases} B_R = \{111, 156, 178\} \\ B_N = E_1(1234) \cup \{267, 258, 357, 457, 468\} \cup \langle 2, 5, 3, 7\rangle \cup \langle \\ \qquad 2, 6, 3, 8\rangle \cup \langle 4, 7, 5\rangle \cup \langle 4, 8, 6\rangle \end{cases}$$

$$4 \in J[8] \text{ by } \begin{cases} B_R = \{111, 156, 178, 468\} \\ B_N = E_1(1234) \cup \{267, 258, 337, 368, 225, 227, 335, 337, 662, \\ \qquad 663, 882, 883\} \cup \langle 4, 5, 7\rangle \cup \langle 4, 7, 5\rangle \end{cases}$$

$$6 \in J[8] \text{ by } \begin{cases} B_R = \{178, 267, 357, 458, 228, 552\} \\ B_N = E_1(1234) \cup \{156, 368, 115, 116, 665, 336, 338, 886, 777, \\ \qquad 888\} \cup \langle 4, 6\rangle \cup \langle 4, 7\rangle \end{cases}$$

$$8 \in J[8] \text{ by } \begin{cases} B_R = \{257, 358, 226, 336, 668, 773, 882, 555\} \\ B_N = E_1(1234) \cup \{178, 167, 156, 456, 467, 478, 115, 118, 445, \\ \qquad 448\} \end{cases}$$

$$9 \in J[8] \text{ by } \begin{cases} B_R = \{178, 267, 357, 458, 156, 228, 774, 446, 552\} \\ B_N = \{124, 134, 234, 368, 112, 113, 332, 666, 888\} \cup \langle 3, 6, 8\rangle \end{cases}$$

$$10 \in J[8] \text{ by } \begin{cases} B_R = \{156, 178, 258, 468, 367, 111, 227, 662, 335, 883\} \\ B_N = E_1(1234) \cup \langle 4, 5, 7\rangle \cup \langle 4, 7, 5\rangle \end{cases}$$

$$11 \in J[8] \text{ by } \begin{cases} B_R = \{356, 378, 157, 168, 267, 664, 774, 448, 228, 885, 111\} \\ B_N = \{123, 124, 134, 245, 332, 334, 552, 554\} \end{cases}$$

$$12 \in J[8] \text{ by } \begin{cases} B_R = \{467, 158, 863, 278, 137, 256, 661, 112, 884, 441, 775, 333\} \\ B_N = E_2(3425) \end{cases}$$

Let A, B and C be the systems constructed as above with repeated numbers 2, 4 and 10, repectively. Then $5, 7, 13 \in J[8]$ can be obtained by replacing the non-repeated blocks $\langle 4, 5, 7\rangle \cup \langle 4, 7, 5\rangle$ of the systems A, B, C with $\langle 4, 5, 7\rangle \cup \langle 4, 5, 7\rangle$, respectively.

J[11] = I[11]:

Let B_1 be a TETS(5) with k repeated blocks as in the above cases, where $k \in S_1 = \{0, 1, 2, 3, 4, 5\}$. Set $A_1 = \{167, 189, 1tt_1, 278, 29t, 26t_1\}$, $A_2 = \{368, 379, 38t, 39t_1, 3t6, 3t_17, 68t, 79t_1, 469, 47t, 48t_1, 569, 57t, 58t_1, 664, 774, 884, 994, tt4, t_1t_14, 665, 775, 885, 995, tt5, t_1t_15\}$ and $A_3 = A_1 \cup \{178, 19t, 16t_1, 267, 289, 2tt_1\}$. Let $B_2 = B_R \cup B_N$ be the twofold extended triple system of order 11 with a hole 5, where $B_R = \emptyset$ and $B_N = A_2 \cup A_3$ or

$B_R = A_1$ and $B_N = A_2$. Then $B_1 \cup B_2$ is an indecomposable TETS(11) with $k+l$ repeated blocks, where $l \in S_2 = \{0, 6\}$. Therefore

$$J[11] \supseteq S_1 + S_2 = \{0, 1, \ldots, 5\} + \{0, 6\} = \{0, 1, 2, \ldots, 11\}.$$

Let $B_3 = B_R \cup B_N$ be the indecomposable TETS(11) with 24 repeated blocks, where $B_R = \{1tt_1,\ 189,\ 167,\ 29t,\ 278,\ 256,\ 357,\ 38t,\ 39t_1,\ 479,\ 468,\ 45t,\ 58t_1,\ 888,\ 115,\ 22t_1,\ 336,\ 44t_1,\ 559,\ 996\} \cup \langle 6, t, 7, t_1\rangle$ and $B_N = E_1(1234)$. Let B_4 be obtained by replacing the repeated blocks $\langle 6, t, 7, t_1\rangle$ of B_3 with the non-repeated blocks $\langle 6, t, 7, t_1\rangle \cup \langle 6, t_1, 7, t\rangle$. Let B_5 be obtained by replacing the repeated blocks $\{888, 115, 559, 189\}$ of B_3 with the repeated block $\{111\}$ and the non-repeated blocks $\{881, 889, 551, 559, 189, 159\}$. Let B_6 be obtained by replacing the repeated blocks $\langle 6, t, 7, t_1\rangle$ of B_5 with the non-repeated blocks $\langle 6, t, 7, t_1\rangle \cup \langle 6, t_1, 7, t\rangle$. Let B_7 be obtained by replacing the repeated blocks $\{115, 559, 996, 66t, t_1t_16, 1tt_1\}$ of B_3 with the non-repeated blocks $\{1tt_1, 6tt_1, t_1t_11, t_1t_16, 11t, 66t\} \cup \langle 1, 5\rangle \cup \langle 5, 9\rangle \cup \langle 9, 6\rangle$. Let B_8 be obtained by replacing the repeated blocks $\{1tt_1,\ 189, 167, 115, 29t, 278, 256, 22t_1\}$ of B_3 with the non-repeated blocks $\{156,\ 167, 178, 189, 19t, 1tt_1, 256, 267, 278, 289, 29t, 2tt_1, 115, 11t_1, 225, 22t_1\}$. Let B_9 be obtained by replacing the repeated blocks $\langle 6, t, 7, t_1\rangle$ of B_8 with the non-repeated blocks $\langle 6, t, 7, t_1\rangle \cup \langle 6, t_1, 7, t\rangle$. Let B_{10} be obtained by replacing the repeated blocks $\{357, 38t, 39t_1, 336, 479,\ 468,\ 45t, 44t_1\}$ of B_5 with the non-repeated blocks $\{3t_19, 397, 375, 35t, 3t8, 386, 33t_1, 336, 4t_19, 497, 475, 45t, 4t8, 486, 44t_1, 446\}$. Then B_3, B_4, B_5, B_6, B_7, B_8, B_9 and B_{10} are indecomposable TETS(11) with exactly 24, 20, 21, 17, 18, 16, 12 and 13 repeated blocks, respectively. Therefore $\{12, 13, 16, 17, 18, 20, 21, 24\} \subseteq J[11]$.

$$11 \in J[11] \text{ by } \begin{cases} B_R = \{167, 189, 1tt_1, 278, 29t, 26t_1, 115, 225, 335, 445, 555\} \\ B_N = E_1(1234) \cup \{368, 379, 57t, 58t_1, 38t, 39t_1, 3t6, 3t_17, \\ \qquad 38t, 79t_1, 469, 47t, 48t_1, 569, 664, 774, 884, 994, tt4, \\ \qquad t_1t_14, 665, 775, 885, 995, tt5, t_1t_15\} \end{cases}$$

$$14 \in J[11] \text{ by } \begin{cases} B_R = \{469, 47t, 48t_1, 665, 775, 885, 995, tt5, t_1t_15, 115, 225, \\ \qquad 335, 445, 555\} \\ B_N = E_1(1234) \cup \{368, 379, 38t, 39t_1, 3t6, 3t_17, 68t, 79t_1, \\ \qquad 167, 178, 189, 19t, 1tt_1, 16t, 267, 278, 289, 29t, 2tt_1, \\ \qquad 26t_1\} \end{cases}$$

$$15 \in J[11] \text{ by } \begin{cases} B_R = \{167, 189, 1tt_1, 278, 29t,\ 26t_1, 469, 47t, 48t_1, 665,\ 775, \\ \qquad 885, 995, tt5, t_1t_15\} \\ B_N = \{111, 222, 123, 145, 234, 345, 334, 335, 441, 442, 552, \\ \qquad 368, 379, 38t, 39t_1, 3t6, 3t_17, 68t, 79t_1\} \cup \langle 1, 2, 5\rangle \end{cases}$$

$$19 \in J[11] \text{ by } \begin{cases} B_R = \{167,\ 189, 1tt_1,\ 278, 29t,\ 26t_1, 469, 47t, 48t_1, 665, 775, \\ \qquad\quad 885, 995, tt5, t_1t_15, 111, 125, 331, 441\} \\ B_N = E_2(3425) \cup \{551, 552, 368, 379, 38t, 39t_1, 3t6, 3t_17, 68t, \\ \qquad\quad 79t_1\} \end{cases}$$

$$22 \in J[11] \text{ by } \begin{cases} B_R = \{78t_1, 57t, 39t, 159, 458, 967, 249, 4tt_1, 18t, 364, 13t_1, \\ \qquad\quad 238, 127, 553, 337, 774, 441, 116, 66t, t_1t_19, 998, 886\} \\ B_N = E_1(256t_1) \cup \{222, ttt\} \cup \langle 2, t\rangle \end{cases}$$

$$23 \in J[11] \text{ by } \begin{cases} B_R = \{1t3, 467, 948, 158, 265, 4t_1t, 683, 59t_1, 379, 78t_1, 16t_1, \\ \qquad\quad 57t, 28t, 129, 441, 117, 772, 33t_1, t_1t_12, 888\} \cup \langle 6, 9, t\rangle \\ B_N = E_2(3425) \end{cases}$$

Combining the above results, we have $J[11] = I[11]$.

J[13] = I[13]:

By a similar argument as in Lemma 3.3, we obtain $I[13] \setminus \{26, 27\} \subseteq J[13]$.

$$26 \in J[13] \text{ by } \begin{cases} B_R = \{1t_2t_3,\ 189,\ 167,\ 2t_1t_2,\ 29t,\ 278,\ 256,\ 3t_1t_3,\ 3tt_2,\ 379, \\ \qquad\quad 368,\ 457,\ 48t,\ 49t_1,\ 46t_3,\ 59t_3,\ 69t_2,\ 7tt_3,\ 8t_15,\ 999, \\ \qquad\quad 115,\ 22t_3,\ 335,\ 44t_2,\ 88t_2,\ t_3t_38\} \\ B_N = E_1(1234) \cup \{66t,\ 66t_1,\ tt1,\ t_1t_11,\ 1tt_1,\ 6tt_1\} \cup \langle 5, t\rangle \cup \\ \qquad\quad \langle 5, t_2\rangle \cup \langle 7, t_1\rangle \cup \langle 7, t_2\rangle \end{cases}$$

$$27 \in J[13] \text{ by } \begin{cases} B_R = \{1t_2t_3,\ 1tt_1,\ 189,\ 167,\ 2t_1t_2,\ 29t,\ 278,\ 256,\ 3t_1t_3,\ 3tt_2, \\ \qquad\quad 379, 368, 457, 48t, 49t_1, 46t_3, 59t_3, 69t_2, 7tt_3, 8t_15, 999, \\ \qquad\quad 115, 22t_3, 335, 44t_2, 88t_2, t_3t_38\} \\ B_N = E_1(1234) \cup \langle 5, t, 6, t_1, 7, t_2\rangle \cup \langle 5, t_2, 7, t_1, 6, t\rangle \end{cases}$$

J[19] = I[19]:

By a similar argument as in Lemma 3.4, we obtain $I[19] \setminus \{20, 27, 34, 41, 48, 55, 56, 57, 58, 59, 60, 61, 62\} \subseteq J[19]$.

Let (V_1, B_1) be a TETS(7) with k repeated blocks as in the above cases, where $k \in S_1 = \{0, 1, 2, \ldots, 10\}$ and $V_1 = \{a_1, a_2, \ldots, a_7\}$. We can factor the complete graph K_{12} on vertex set $V_2 = \{1, 2, \ldots, 12\}$ into a set of 4 triangles $T_0 = \{\{1, 5, 9\}, \{2, 6, 10\}, \{3, 7, 11\}, \{4, 8, 12\}\}$, a set of 4-cycles $\mathcal{C} = \{(1, 4, 7, 10), (2, 5, 8, 11), (3, 6, 9, 12)\}$ and a set of 7 1-factors $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, 7\}$, where

$$F_1 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\};$$

$$F_2 = \{\{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}, \{10, 11\}, \{12, 1\}\};$$

$F_3 = \{\{1,3\},\{5,7\},\{9,11\},\{2,4\},\{6,8\},\{10,12\}\};$

$F_4 = \{\{3,5\},\{7,9\},\{11,1\},\{4,6\},\{8,10\},\{12,2\}\};$

$F_5 = \{\{1,6\},\{11,4\},\{9,2\},\{7,12\},\{5,10\},\{3,8\}\};$

$F_6 = \{\{6,11\},\{4,9\},\{2,7\},\{12,5\},\{10,3\},\{8,1\}\};$

$F_7 = \{\{1,7\},\{2,8\},\{3,9\},\{4,10\},\{5,11\},\{6,12\}\};$

Set $T_1 = 2\{a_1xy \mid xy \in F_1\} \cup 2\{a_2xy \mid xy \in F_2\}$ or $T_1 = \{a_1xy \mid xy \in F_1 \cup F_2\} \cup \{a_2xy \mid xy \in F_1 \cup F_2\}$; $T_2 = 2\{a_3xy \mid xy \in F_3\} \cup 2\{a_4xy \mid xy \in F_4\}$ or $T_2 = \{a_3xy \mid xy \in F_3 \cup F_4\} \cup \{a_4xy \mid xy \in F_3 \cup F_4\}$; $T_3 = 2\{a_5xy \mid xy \in F_5\} \cup 2\{a_6xy \mid xy \in F_6\}$ or $T_3 = \{a_5xy \mid xy \in F_5 \cup F_6\} \cup \{a_6xy \mid xy \in F_5 \cup F_6\}$; $T_4 = 2\{a_7xy \mid xy \in F_7\}$ and $\mathcal{L} = \{2\langle x_1, x_2, \ldots, x_n, x_1\rangle$ or $\langle x_1, x_2, \ldots, x_n, x_1\rangle \cup \langle x_n, x_{n-1}, \ldots, x_1, x_n\rangle \mid c = (x_1, x_2, \ldots, x_n) \in C\}$. Then $B_1 \cup \mathcal{L} \cup 2T_0 \cup T_1 \cup T_2 \cup T_3 \cup T_4$ are indecomposable TETS(19) with the following properties:

$$J[19] \supseteq S_1 + \{10\} + \{0,12\} + \{0,12\} + \{0,12\} + \{0,4,8,12\} = \{10,11,\ldots,68\}.$$

Combining the two results, we have $J[19] = I[19]$.

$\mathbf{J[23] = I[23]}$:

By a similar argument as in Lemma 3.4, we obtain $I[23] \backslash \{83, 84, 85, 86, 87\} \subseteq J[23]$.

Let B_1 be a TETS(7) with k repeated blocks as in the above cases, where $k \in S_1 = \{3,4,5,6,7\}$. Let $B_2 = B_R \cup B_N$ be the twofold extended triple system of order 23 with a hole 7, where $B_R = \{t_7t_8d,\ t_8t_9d_1,\ t_9dd_2,\ dd_1d_3,\ d_1d_28,\ d_2d_39,\ d_38t,\ 89t_1,\ 9tt_2,\ tt_1t_3,\ t_1t_2t_4,\ t_2t_3t_4,\ t_3t_4t_6,\ t_4t_5t_7,\ t_5t_6t_8,\ t_6t_7t_9,\ 1t_7d_1,\ 19t_3,\ 1t_8d_2,\ 1tt_4,\ 1t_9d_3,\ 1t_1t_5,\ 1d8,\ 1t_2t_6,\ 2d_19,\ 2t_3t_7,\ 2d_2t,\ 2t_4t_8,\ 2d_3t_1,\ 2t_5t_9,\ 28t_2,\ 2t_6d,\ 3t_7d_2,\ 3t_1t_6,\ 3d_1t,\ 3t_5d,\ 39t_4,\ 3t_98,\ 3t_3t_8,\ 3d_3t_2,\ 4d_2t_1,\ 4t_6d_1,\ 4tt_5,\ 4d9,\ 4t_4t_9,\ 48t_3,\ 4t_8d_3,\ 4t_2t_7,\ 5t_7d_3,\ 5t_3t_9,\ 59t_5,\ 5d_1t_1,\ 5t_88,\ 79t_6,\ 5t_4d,\ 5tt_6,\ 5d_2t_2,\ 6d_3t_3,\ 6t_99,\ 6t_5d_1,\ 6t_1t_7,\ 68t_4,\ 6dt,\ 6t_6d_2,\ 6t_2t_8,\ 7t_78,\ 7t_5d_2,\ 7t_3d,\ 7t_1t_8,\ 7d_3t_4,\ 7d_1t_2,\ 7t_9t\} \cup \langle d_1, t_4, d_2, t_3\rangle \cup \langle d_3, t_6, 8, t_5\rangle$, $B_N = \langle t_7, t, t_8, 9\rangle \cup \langle t_7, 9, t_8, t\rangle \cup \langle t_9, t_1, d, t_2\rangle \cup \langle t_9, t_2, d, t_1\rangle$ and $|B_R| = 80$. Then $B_1 \cup B_2$ is an indecomposable TETS(23) with k repeated blocks, where $k \in \{83, 84, \ldots, 87\}$. Therefore we have $J[23] = I[23]$.

5 Conclusion

We now have our main result:

Main Theorem $J[v] = I[v]$, for all $v \not\equiv 0 \pmod 3$ and $v \geq 5$.

Proof. For $v \not\equiv 0 \pmod 3$, we concern v in the form of $v \equiv 1, 2, 4, 5, 7, 8, 10, 11 \pmod{12}$. If $v \equiv 4$ or $8 \pmod{12}$, then $v/2 \equiv 2$ or $4 \pmod 6$; if $v \equiv 2$ or $10 \pmod{12}$, then $v/2 \equiv 1$ or $5 \pmod 6$; if $v \equiv 1$ or $5 \pmod{12}$, then $(v-3)/2 \equiv 5$ or $1 \pmod 6$; if $v \equiv 7$ or $11 \pmod{12}$, then $(v-9)/2 \equiv 5$ or $1 \pmod 6$.

For $v = 5, 7, 8, 11, 13, 19, 23$, our statement follows from section 4. Applying those small values to Lemmas 3.1, 3.2, 3.3 and 3.4 recursively, we have $J[v] = I[v]$, for all $v \not\equiv 0 \pmod 3$ and $v \geq 5$. ∎

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