Hamiltonian decompositions of the tensor product of a complete graph and a complete bipartite graph

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Abstract

In this paper it has been proved that $K_{r,r} \times K_m$, $m \geq 3$, is hamiltonian decomposable.

1 Introduction

A k-regular graph G has a *hamiltonian decomposition* if its edge set can be partitioned into $\frac{k}{2}$ Hamilton cycles when k is even, or into $(k-1)/2$ Hamilton cycles plus a 1-factor (or a perfect matching) when k is odd. We write $G = H_1 \oplus H_2 \oplus \ldots \oplus H_k$ if $H_1, H_2, \ldots, H_k$ are edge-disjoint subgraphs of G and $E(G) = E(H_1) \cup E(H_2) \cup \ldots \cup E(H_k)$. The complete graph on m vertices is denoted by K_m and its complement is denoted by $\overline{K}_m$.

For two simple graphs G and H their *wreath product*, denoted by $G * H$, has vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1 g_2 \in E(G)$, or $g_1 = g_2$ and $h_1 h_2 \in E(H)$. Similarly, $G \times H$, the *tensor product* (also called *Kronecker product* or *direct product*) of the graphs G and H has vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$. It is well known that the tensor product is commutative and distributive over edge-disjoint union of graphs, that is, if $G = H_1 \oplus H_2 \oplus \ldots \oplus H_k$, then $G \times H =$

$(H_1 \times H) \oplus (H_2 \times H) \oplus \ldots \oplus (H_k \times H)$.

We shall use the following notation throughout the paper. Let G and H be simple graphs with $V(G) = \{x_1, x_2, \ldots, x_n\}$ and $V(H) = \{y_1, y_2, \ldots, y_m\}$. Then $V(G \times H) = V(G) \times V(H)$. For our convenience, we write $V(G) \times V(H) = \bigcup_{i=1}^{n} X_i$, where X_i stands for $\{x_i\} \times V(H)$. Further, in the sequel, we shall denote the vertices of X_i as $\{x_j^i \mid 1 \leq j \leq m\}$, where x_j^i stands for the vertex (x_i, y_j). $X_i, 1 \leq i \leq n$, is called the i^{th} *layer* of $G \times H$. If $V(G) = \{x_1, x_2, \ldots, x_n\}$, then we call $G \times H$ an n-partite graph with partite sets $X_1, X_2, \ldots, X_n$.

Let G be a bipartite graph with bipartition (X,Y), where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$. If $x_i y_j$ is an edge of G, then $x_i y_j$ is called an edge of *distance* $j - i$ if $i \leq j$, or $n - (i - j)$ if $i > j$, from X to Y. The same edge is said to be of *distance* $i - j$ if $i \geq j$, or $n - (j - i)$ if $i < j$, from Y to X. If G contains the set of edges $F_i(X,Y) = \{x_j y_{i+j} \mid 1 \leq j \leq n\}$, $0 \leq i \leq n-1$, where addition in the subscript is taken modulo n with residues $1, 2, \ldots, n$, then we say that G has the 1-factor of distance i from X to Y. Clearly, if $G = K_{n,n}$, then $E(G) = \bigcup_{i=0}^{n-1} F_i(X,Y)$. Note that $F_i(Y,X) = F_{n-i}(X,Y)$, $0 \leq i \leq n-1$. For a digraph D, by $A(D)$ we mean the arc set of D. Definitions which are not seen here can be found in [4] or [8].

Let k be a positive integer and let L be a subset of $\{1, 2, \ldots, \lfloor \frac{k}{2} \rfloor\}$. A *circulant* $X = X(k; L)$ is a graph with vertex set $V(X) = \{u_0, u_1, \ldots, u_{k-1}\}$ and edge set $E(X) = \{u_i u_{i+l} \mid i \in Z_k, l \in L\}$. The edge $u_i u_{i+l}$, where $l \in L$, is said to be of distance l, and L is called the edge distance set of the circulant X. Then it is clear that if $gcd(k, l_i) = 1$, then the circulant $X(k; \{l_i\})$ is a Hamilton cycle. We shall denote a graph isomorphic to $X(2r; \{1, r\})$ by W_{2r}.

The following result of Bermond et al [7] will be used throughout the paper.

Theorem 1.1. *Any connected circulant of degree 4 can be decomposed into Hamilton cycles.* ∎

Remark 1.2. *Examples of circulants of degree 4 that are connected include the circulants of the forms* $X(k; \{l, l+1\})$, $X(k; \{2l-1, 2l+1\})$, *and if* k *is odd,* $X(k; \{2l, 2l+2\})$, *see [14].* ∎

In this paper, we study the hamiltonian decomposition of $K_{r,r} \times K_m$.

The problem of finding hamiltonian decompositions of product graphs is not new. Hamiltonian decompositions of various product graphs have been studied, see [1], [6] and [9]. For example, it has been conjectured [6] that if both G and H are Hamilton cycle decomposable, then $G \Box H$ is hamiltonian decomposable, where $\Box$ denotes the cartesian product of graphs [1]. This conjucture has been verified to be true for a large class of graphs [15]. Baranyai and Szasz [5] proved that if both G and H are even-regular hamiltonian decomposable graphs, then $G * H$ is hamiltonian decomposable. In [13] Ng has obtained a partial solution to the following conjecture of Alspach et al [1]: If D_1 and D_2 are directed Hamilton cycle decomposable digraphs, then $D_1 * D_2$ is directed Hamilton cycle decomposable. Jha [10] has raised the following conjecture: if both G and H are hamiltonian decomposable and $G \times H$ is connected, then $G \times H$ is hamiltonian decomposable. But this conjecture is disproved in [3]. Because of this, finding a hamiltonian decomposition of tensor product of hamiltonian decomposable graphs is considered to be difficult. In [2] it has been proved that $K_r \times K_s$ is hamiltonian decomposable. Here we prove that $K_{r,r} \times K_m$ is hamiltonian decomposable. In fact, we have obtained the following main

Theorem 1.3. *For $m \geq 3$, $K_{r,r} \times K_m$ has a hamiltonian decomposition.*

2 Proof of the main theorem

First we prove a few lemmas. Then using them we prove the main result of this paper.

Lemma 2.1. *Let $r \geq 3$ be odd. Then $K_{r,r}$ can be decomposed into Hamilton cycles and one copy of W_{2r} ($\cong X(2r;\{1,r\})$).*

Proof. Let $A = \{u_0, u_2, \ldots, u_{2r-2}\}$ and $B = \{u_1, u_3, \ldots, u_{2r-1}\}$ be the bipartition of $K_{r,r}$. Place these vertices in the cyclic order $u_0, u_1, u_2, \ldots, u_{2r-1}$. Thus $K_{r,r}$ is isomorphic to the circulant $X(2r; \{2i-1 \mid 1 \leq i \leq (r+1)/2\})$. We divide the proof into two cases.
Case 1. $r \equiv 1 \pmod 4$.
We decompose $K_{r,r}$ into circulants as follows:
$K_{r,r} = \left(\bigoplus_{i=1}^{(r-5)/4} X(2r; \{4i-1, 4i+1\})\right) \oplus X(2r; \{r-2\}) \oplus X(2r; \{1, r\})$. Each circulant, except the last two, in the above expression is connected and 4-regular and hence each of them can be decomposed into two Hamilton cycles, by Theorem 1.1 and Remark 1.2. The circulant $X(2r; \{r-2\})$ is a Hamilton cycle

as $gcd(2r, r-2) = 1$ and the last circulant $X(2r; \{1, r\})$ is W_{2r}, by definition.
Case 2. $r \equiv 3 \pmod 4$.
We decompose $K_{r,r}$ into circulants as follows:
$K_{r,r} = \left(\bigoplus_{i=1}^{(r-3)/4} X(2r; \{4i-1, 4i+1\})\right) \oplus X(2r; \{1, r\})$. Each circulant, except the last one, in the above expression is connected and 4-regular and hence each of them can be decomposed into two Hamilton cycles, by Theorem 1.1 and Remark 1.2. The last circulant $X(2r; \{1, r\})$ is W_{2r}, by definition. ■

Lemma 2.2. *For $m \geq 2$ and $k \geq 2$, $C_{2k} \times K_{2m}$ has a hamiltonian decomposition.*

Proof. Let the partite sets of the $2k$-partite graph $C_{2k} \times K_{2m}$ be $X_i = \{x_1^i, x_2^i, \ldots, x_{2m}^i\}$, $1 \leq i \leq 2k$. First we decompose $C_{2k} \times K_{2m}$ into $2m-2$ Hamilton cycles $H_1, H_2, \ldots, H_{2m-2}$ and a 2-factor F such that F has two cycles of equal length. Then we decompose $F \cup H_{2m-2}$ into two Hamilton cycles, say, H' and H''. Thus $\{H_1, H_2, \ldots, H_{2m-3}, H', H''\}$ is a hamiltonian decomposition of $C_{2k} \times K_{2m}$.

First we obtain the Hamilton cycles $H_1, H_2, \ldots, H_{2m-2}$ and the 2-factor F as follows: for $1 \leq i \leq 2m-2$, let $H_i = F_{2m-i}(X_1, X_2) \cup F_{2m-i}(X_2, X_3) \cup F_i(X_3, X_4) \cup F_{i+1}(X_4, X_5) \cup \left(\bigcup_{j=3}^{k} \{F_i(X_{2j-1}, X_{2j}) \cup F_{2m-i}(X_{2j}, X_{2j+1})\}\right)$ and let $F = F_1(X_1, X_2) \cup F_1(X_2, X_3) \cup F_{2m-1}(X_3, X_4) \cup F_1(X_4, X_5) \cup \left(\bigcup_{j=3}^{k} \{F_{2m-1}(X_{2j-1}, X_{2j}) \cup F_1(X_{2j}, X_{2j+1})\}\right)$, where the subscripts of X_i 's are taken modulo $2k$ with residues $1, 2, \ldots, 2k$. Clearly H_i 's are edge-disjoint Hamilton cycles of $C_{2k} \times K_{2m}$ and F is a 2-factor of it consisting of two cycles C' and C'' of equal length. In fact, the vertices $x_1^1, x_3^1, x_5^1, \ldots, x_{2m-1}^1$ are contained in a single cycle of F, say, C', and the vertices $x_2^1, x_4^1, x_6^1, \ldots, x_{2m}^1$ are contained in the other cycle of F, say, C''. Next we obtain two edge-disjoint Hamilton cycles from $H_{2m-2} \cup F$. From the construction of H_{2m-2}, it is clear that the edges $x_1^1 x_3^2$ and $x_2^2 x_4^3$ are in H_{2m-2} and from the construction of F the edges $x_1^1 x_2^2$ and $x_3^2 x_4^3$ are in F. Let $H' = (H_{2m-2} - \{x_1^1 x_3^2, x_2^2 x_4^3\}) \cup \{x_1^1 x_2^2, x_3^2 x_4^3\}$ and let $H'' = (F - \{x_1^1 x_2^2, x_3^2 x_4^3\}) \cup \{x_1^1 x_3^2, x_2^2 x_4^3\}$. That H' and H'' are indeed edge-disjoint Hamilton cycles of $C_{2k} \times K_{2m}$ as can be seen by letting $H = H_{2m-2}$ and $a = x_1^1$, $b = x_3^2$, $c = x_2^2$, $d = x_4^3$ in all the graphs of Figure 1. This completes the proof. ■

Lemma 2.3. *If $n \equiv 2 \pmod 4$, $n \geq 6$ and $m \geq 2$, then $W_n \times K_{2m}$ has a*

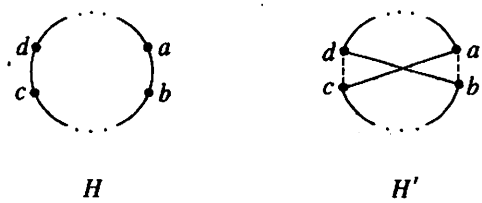

Figure 1 (a)

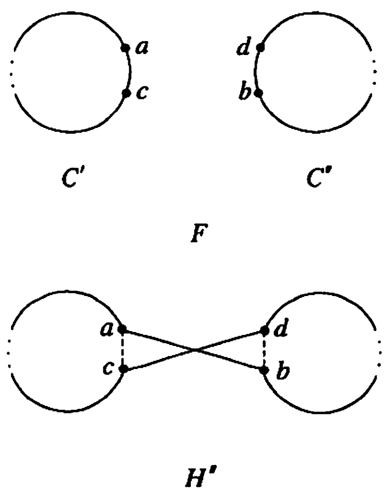

Figure 1 (b)

Broken edges of Figure 1 (a) (Figure 1 (b)) represent the edges we have deleted from H (F) for the construction of H' (H'').

Figure 1

hamiltonian decomposition, where $W_n \cong X(n;\{1,\frac{n}{2}\})$.

Proof. Throughout this lemma the subscripts of x_i's and X_j's are taken modulo n with residues $1,2,\ldots,n$. Let the vertex set of W_n be $\{x_1,x_2,\ldots,x_n\}$ taken in the cyclic order. Then its edge set can be described as $\{x_ix_{i+1} \mid 1 \le i \le n\} \cup \{x_ix_{i+\frac{n}{2}} \mid 1 \le i \le n/2\}$. Let the partite sets of the n-partite graph $W_n \times K_{2m}$ be $X_i = \{x_1^i, x_2^i, \ldots, x_{2m}^i\}$, $1 \le i \le n$. By the definition of the tensor product of graphs, the edge set of $W_n \times K_{2m}$ can be described as $\bigcup_{i=1}^{n}\left(\bigcup_{j=1}^{2m-1} F_j(X_i,X_{i+1})\right) \cup \left(\bigcup_{i=1}^{n/2}(\bigcup_{j=1}^{2m-1} F_j(X_i,X_{i+\frac{n}{2}}))\right)$.

Obtain a digraph D_n from W_n as follows: replace the edge x_ix_{i+1}, $1 \le i \le n$, of W_n by two directed arcs from x_i to x_{i+1} (that is, having the same tail and head) and replace the edge $x_ix_{i+\frac{n}{2}}$, $1 \le i \le n/2$, by a symmetric pair of arcs. Thus we have a 3-regular directed graph D_n. We decompose D_n into three directed Hamilton cycles $\vec{H}_1$, $\vec{H}_2$ and $\vec{H}_3$ as follows: let

$\vec{H}_1 = \{(x_{2i-1}, x_{2i}) \mid 1 \le i \le n/2\} \cup \{(x_{\frac{n}{2}+2i-1}, x_{2i-1}) \mid 1 \le i \le n/2\}$,
$\vec{H}_2 = \{(x_{2i}, x_{2i+1}) \mid 1 \le i \le n/2\} \cup \{(x_{2i-1}, x_{\frac{n}{2}+2i-1}) \mid 1 \le i \le n/2\}$, and
$\vec{H}_3 = \{(x_i, x_{i+1}) \mid 1 \le i \le n\}$.

Clearly, $\vec{H}_1$, $\vec{H}_2$ and $\vec{H}_3$ are arc-disjoint directed Hamilton cycles of D_n. Using these three directed Hamilton cycles, we decompose $W_n \times K_{2m}$ into $3m-2$ Hamilton cycles and a 1-factor. We divide the proof into two cases.

Case 1. $m = 2$.
From W_n obtain D_n and find $\vec{H}_1, \vec{H}_2$ and $\vec{H}_3$ from D_n, as above. Corresponding to the directed Hamilton cycle $\vec{H}_i$, $i = 1,2$, of D_n we shall obtain a Hamilton cycle, H_i, $i = 1,2$, of $W_n \times K_{2m}$ as follows:

$$\text{let } H_1 = \Big(\bigcup_{(x_i,x_j)\in (A(\vec{H}_1)-\{(x_1,x_2)\})} F_1(X_i,X_j)\Big) \cup F_2(X_1,X_2) \text{ and}$$

$$H_2 = \Big(\bigcup_{(x_i,x_j)\in (A(\vec{H}_2)-\{(x_2,x_3)\})} F_1(X_i,X_j)\Big) \cup F_2(X_2,X_3).$$

Corresponding to $\vec{H}_3$, we obtain two edge-disjoint Hamilton cycles, say, H_3 and H_4 of $W_n \times K_{2m}$ as follows:
let $H_3 = F_1(X_1,X_2) \cup F_1(X_2,X_3) \cup F_2(X_3,X_4) \cup \left(\bigcup_{i=4}^{6} F_3(X_i,X_{i+1})\right) \cup \left(\bigcup_{i=7}^{n} F_2(X_i,X_{i+1})\right)$ and $H_4 = \left(\bigcup_{i=1}^{3} F_3(X_i,X_{i+1})\right) \cup \left(\bigcup_{i=4}^{6} F_2(X_i,X_{i+1})\right) \cup \left(\bigcup_{i=7}^{n} F_3(X_i,X_{i+1})\right)$. It is not difficult to check that H_1, H_2, H_3 and H_4 are edge-disjoint Hamilton cycles of $W_n \times K_{2m}$ and the edges not covered by these

Hamilton cycles form a 1-factor of it.

Case 2. $m \geq 3$.

As above, obtain $\vec{D}_n$ and $\vec{H}_1$, $\vec{H}_2$ and $\vec{H}_3$. First we decompose the graph $W_n \times K_{2m}$ into four spanning subgraphs, say, G_1, G_2, G_3 and F', where F' is a 1-factor of it. Then we find a hamiltonian decomposition of each G_i, $1 \leq i \leq 3$. First we construct our G_i's.

Let $G_1 = \bigcup_{(x_i,x_j)\in A(\vec{H}_1)} \left(\bigcup_{k=1}^{m-1} F_k(X_i,X_j)\right)$,

$G_2 = \bigcup_{(x_i,x_j)\in A(\vec{H}_2)} \left(\bigcup_{k=1}^{m-1} F_k(X_i,X_j)\right)$,

$G_3 = \bigcup_{i=1}^{n} \left(\bigcup_{k=m}^{2m-1} F_k(X_i,X_{i+1})\right)$ and $F' = \bigcup_{i=1}^{n/2} F_m(X_i,X_{\frac{n}{2}+i})$. Note that F' is a 1-factor of $W_n \times K_{2m}$. Clearly, G_1 and G_2 are isomorphic to G, where $G = \bigcup_{i=1}^{n} \left(\bigcup_{k=1}^{m-1} F_k(X_i,X_{i+1})\right)$. As mentioned above, to complete the proof, it is enough to decompose G and G_3 into Hamilton cycles. This is achieved by considering two subcases.

Subcase 2.1. $m \geq 4$ is even.

To obtain a hamiltonian decomposition of $G\,(\cong G_1, G_2)$, we first obtain a decomposition of G into $m-2$ Hamilton cycles, say, $H_2, H_3, \ldots, H_{m-1}$ and a 2-factor F. Again, we decompose $F \cup H_2$ into two Hamilton cycles, say, H' and H''. Then $\{H', H'', H_3, H_4, \ldots, H_{m-1}\}$ is a hamiltonian decomposition of G.

Now we define our required F and $H_2, H_3, \ldots, H_{m-1}$, as follows:

let $F = F_1(X_1,X_2) \cup F_1(X_2,X_3) \cup F_{m-1}(X_3,X_4) \cup F_{m-1}(X_4,X_5) \cup F_1(X_5,X_6) \cup F_1(X_6,X_7) \cup \left(\bigcup_{j=1}^{(n-6)/4}(F_1(X_{4j+3},X_{4j+4}) \cup F_1(X_{4j+4},X_{4j+5}) \cup F_{m-1}(X_{4j+5}, X_{4j+6}) \cup F_{m-1}(X_{4j+6},X_{4j+7}))\right)$, and let

$H_i = F_i(X_1,X_2) \cup F_i(X_2,X_3) \cup F_{m-i}(X_3,X_4) \cup F_{m-i}(X_4,X_5) \cup F_i(X_5,X_6) \cup F_{m-i+1}(X_6,X_7) \cup \left(\bigcup_{j=1}^{(n-6)/4}(F_i(X_{4j+3},X_{4j+4}) \cup F_i(X_{4j+4},X_{4j+5}) \cup F_{m-i}(X_{4j+5},X_{4j+6}) \cup F_{m-i}(X_{4j+6},X_{4j+7}))\right)$, $2 \leq i \leq m-1$.

Indeed, each H_i, $2 \leq i \leq m-1$, is a Hamilton cycle of G, H_i's are edge-disjoint and F is a 2-factor of G consisting of two cycles of equal length. In fact, the vertices $x_1^1, x_3^1, \ldots, x_{2m-1}^1$ are contained in a single cycle of F, say, C' and the vertices $x_2^1, x_4^1, \ldots, x_{2m}^1$ are contained in the other cycle of F, say, C''. Next we obtain two edge-disjoint Hamilton cycles of G from $F \cup H_2$. Let $H' = (H_2 - \{x_1^1x_3^2,\, x_2^2x_4^3\}) \cup \{x_1^1x_2^2,\, x_3^2x_4^3\}$ and $H'' =$

$(F - \{x_1^1x_2^2,\ x_3^2x_4^3\}) \cup \{x_1^1x_3^2,\ x_2^2x_4^3\}$. Clearly, H' and H'' are Hamilton cycles of G as can be seen by letting $H = H_2$ and $a = x_1^1$, $b = x_3^2$, $c = x_2^2$, $d = x_4^3$ in all the graphs of Figure 1. Thus $\{H', H'', H_3, H_4, \ldots, H_{m-1}\}$ is a required hamiltonian decomposition of G.

Next we decompose G_3 into m Hamilton cycles $H_1, H_2, \ldots, H_m$ as follows: let $H_1 = \left(\bigcup_{j=1}^{5} F_m(X_j, X_{j+1})\right) \cup F_{2m-1}(X_6, X_7) \cup \left(\bigcup_{j=7}^{n} F_m(X_j, X_{j+1})\right)$ and $H_i = F_{m+i-1}(X_1, X_2) \cup F_{m+i-1}(X_2, X_3) \cup F_{2m-i+1}(X_3, X_4) \cup F_{2m-i+1}(X_4, X_5)$ $\cup\ F_{m+i-1}(X_5, X_6) \cup F_{2m-i}(X_6, X_7) \cup \left(\bigcup_{j=1}^{(n-6)/4} (F_{m+i-1}(X_{4j+3}, X_{4j+4}) \cup F_{m+i-1}(X_{4j+4}, X_{4j+5}) \cup F_{2m-i+1}(X_{4j+5}, X_{4j+6}) \cup F_{2m-i+1}(X_{4j+6}, X_{4j+7}))\right)$, $2 \le i \le m$. Clearly, $H_1, H_2, \ldots, H_m$, are edge-disjoint Hamilton cycles of G_3.

Subcase 2.2. $m \ge 3$ is odd.

To decompose G into Hamilton cycles, we first obtain two 2-factors, say, F_1 and F_2, and $m-3$ Hamilton cycles $H_3, H_4, \ldots, H_{m-1}$; then we decompose $F_1 \cup F_2$ into two Hamilton cycles. Now we define our F_i 's and H_j 's.

Let $F_1 = F_1(X_1, X_2) \cup F_1(X_2, X_3) \cup F_{m-1}(X_3, X_4) \cup F_{m-1}(X_4, X_5) \cup F_1(X_5, X_6)$ $\cup\ F_2(X_6, X_7) \cup \left(\bigcup_{j=1}^{(n-6)/4} (F_1(X_{4j+3}, X_{4j+4}) \cup F_1(X_{4j+4}, X_{4j+5}) \cup F_{m-1}(X_{4j+5}, X_{4j+6}) \cup F_{m-1}(X_{4j+6}, X_{4j+7}))\right)$ and let

$F_2 = F_2(X_1, X_2) \cup F_2(X_2, X_3) \cup F_{m-2}(X_3, X_4) \cup F_{m-2}(X_4, X_5) \cup F_2(X_5, X_6) \cup$ $F_1(X_6, X_7) \cup \left(\bigcup_{j=1}^{(n-6)/4} (F_2(X_{4j+3}, X_{4j+4}) \cup F_2(X_{4j+4}, X_{4j+5}) \cup F_{m-2}(X_{4j+5}, X_{4j+6}) \cup F_{m-2}(X_{4j+6}, X_{4j+7}))\right)$. For $3 \le i \le m-1$, let

$H_i = F_i(X_1, X_2) \cup F_i(X_2, X_3) \cup F_{m-i}(X_3, X_4) \cup F_{m-i}(X_4, X_5) \cup F_i(X_5, X_6) \cup$ $F_{m-i+2}(X_6, X_7) \cup \left(\bigcup_{j=1}^{(n-6)/4} (F_i(X_{4j+3}, X_{4j+4}) \cup F_i(X_{4j+4}, X_{4j+5}) \cup F_{m-i}(X_{4j+5}, X_{4j+6}) \cup F_{m-i}(X_{4j+6}, X_{4j+7}))\right)$.

It is clear that $H_3, H_4, \ldots, H_{m-1}$, are edge-disjoint Hamilton cycles of G and F_1 and F_2 are edge-disjoint 2-factors of G. If $gcd(3, 2m) = 1$, then F_1 and F_2 are Hamilton cycles, and in this case $\{F_1, F_2, H_3, H_4, \ldots, H_{m-1}\}$ is a required hamiltonian decomposition of G. If $gcd(3, 2m) = 3$, then each of the F_i, $i = 1, 2$, consists of three cycles of equal length and in fact, the vertices x_{1+3j}^1, $0 \le j \le (2m-3)/3$, are in a single cycle, say, C_i^1 of F_i, the vertices x_{2+3j}^1, $0 \le j \le (2m-3)/3$, are in another cycle, say, C_i^2, and the vertices x_{3+3j}^1, $0 \le j \le (2m-3)/3$, are in the remaining cycle, say, C_i^3 of F_i.

We obtain two edge-disjoint Hamilton cycles H' and H'' from $F_1 \cup F_2$ as follows: let $H' = (F_1 - \{x_1^1x_2^2,\ x_3^2x_4^3,\ x_3^1x_4^2,\ x_5^2x_6^3\}) \cup \{x_1^1x_3^2, x_3^1x_5^2, x_2^2x_4^3, x_4^2x_6^3\}$ and let $H'' = (F_2 - \{x_1^1x_3^2, x_3^1x_5^2, x_2^2x_4^3, x_4^2x_6^3\}) \cup \{x_1^1x_2^2, x_3^2x_4^3, x_3^1x_4^2, x_5^2x_6^3\}$. Clearly, H' and H'' are Hamilton cycles of G, see Figure 2. Hence $\{H', H'', H_3, H_4, \ldots, H_{m-1}\}$ is a hamiltonian decomposition of G.

To decompose G_3 into Hamilton cycles, we first decompose G_3 into $m-1$ Hamilton cycles $H_2, H_3, \ldots, H_m$ and a 2-factor F. Then we decompose $F \cup H_2$ into two Hamilton cycles. For $2 \leq i \leq m$, let $H_i = F_{m+i-1}(X_1, X_2) \cup F_{m+i-1}(X_2, X_3) \cup F_{2m-i}(X_3, X_4) \cup F_{2m-i}(X_4, X_5) \cup F_{m+i-1}(X_5, X_6) \cup F_{2m-i+1}(X_6, X_7) \cup \left(\bigcup_{j=1}^{(n-6)/4}(F_{m+i-1}(X_{4j+3}, X_{4j+4}) \cup F_{m+i-1}(X_{4j+4}, X_{4j+5}) \cup F_{2m-i+1}(X_{4j+5}, X_{4j+6}) \cup F_{2m-i+1}(X_{4j+6}, X_{4j+7}))\right)$ and $F = F_m(X_1, X_2) \cup F_m(X_2, X_3) \cup F_{2m-1}(X_3, X_4) \cup F_{2m-1}(X_4, X_5) \cup F_m(X_5, X_6) \cup F_m(X_6, X_7) \cup \left(\bigcup_{j=7}^{n} F_m(X_j, X_{j+1})\right)$. Clearly $H_2, H_3, \ldots, H_m$ are edge-disjoint Hamilton cycles of G_3 and F is a 2-factor consisting of two cycles of equal length. In fact, the vertices $x_1^1, x_3^1, \ldots, x_{2m-1}^1$ are contained in a single cycle of F, say, C', and the vertices $x_2^1, x_4^1, \ldots, x_{2m}^1$ are contained in the other cycle, say, C'' of F. Next we obtain two edge-disjoint Hamilton cycles of G_3 from $F \cup H_2$ as follows: let $H' = (H_2 - \{x_1^1x_{m+2}^2, x_{m+1}^2x_2^3\}) \cup \{x_1^1x_{m+1}^2, x_{m+2}^2x_2^3\}$ and let $H'' = (F - \{x_1^1x_{m+1}^2, x_{m+2}^2x_2^3\}) \cup \{x_1^1x_{m+2}^2, x_{m+1}^2x_2^3\}$. Clearly, H' and H'' are Hamilton cycles of G_3 as can be seen by letting $H = H_2$ and $a = x_1^1$, $b = x_{m+2}^2$, $c = x_{m+1}^2$, $d = x_2^3$ in all the graphs of Figure 1. Thus $\{H', H'', H_3, H_4, \ldots, H_m\}$ is a hamiltonian decomposion of G_3. This completes the proof. ∎

Lemma 2.4. *For* $k \geq 1$, $C_{2k+1} \times K_{r,r} \cong C_{2(2k+1)} * \overline{K}_r$.

Proof. Let X and Y be the bipartition of $K_{r,r}$. Let $V(C_{2k+1}) = \{v_1, v_2, \ldots, v_{2k+1}\}$. Clearly, $V(C_{2k+1} \times K_{r,r}) = \bigcup_{i=1}^{2k+1}\{(v_i \times X) \cup (v_i \times Y)\}$. From the definition of the tensor product of graphs, the subgraphs induced by $(v_i \times X) \cup (v_{i+1} \times Y)$ and $(v_i \times Y) \cup (v_{i+1} \times X)$ are complete bipartite subgraphs of $C_{2k+1} \times K_{r,r}$. Then $C_{2k+1} \times K_{r,r}$ is isomorphic to $C_{2(2k+1)} * \overline{K}_r$; this can be seen by arranging the vertex subsets $(v_1 \times X)$, $(v_2 \times Y)$, $(v_3 \times X), \ldots, (v_{2k+1} \times X)$, $(v_1 \times Y)$, $(v_2 \times X)$, $(v_3 \times Y), \ldots, (v_{2k+1} \times Y)$, in order, wherein any two consecutive subsets, taken in the cyclic order, induce a complete bipartite graph. ∎

Proof of Theorem 1.3. We prove this theorem in two cases.

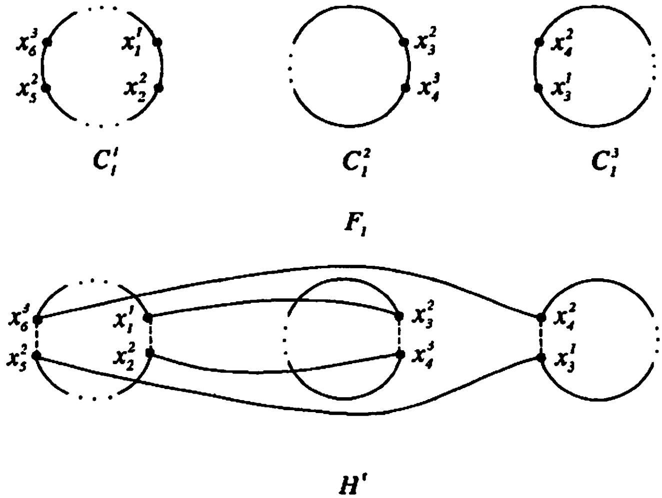

Figure 2 (a)

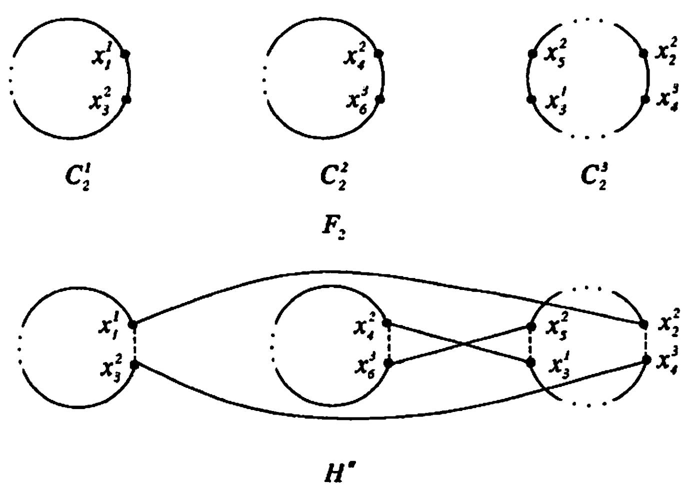

Figure 2 (b)

Broken edges of Figure 2 (a) (Figure 2 (b)) represent the edges we have deleted from F_1 (F_2) for the construction of $H'(H'')$.

Figure 2

Case 1. r is even.
As $K_{r,r}$ is hamiltonian decomposable, (a hamiltonian decomposition of $K_{r,r}$ is $F_{2i-2} \cup F_{2i-1}$, $i = 1, 2, \ldots, r/2$, where F_k denotes the 1-factor of distance k from one part to the other), $K_{r,r} = H_1 \oplus H_2 \oplus \ldots \oplus H_{r/2}$, where H_i's are Hamilton cycles of $K_{r,r}$. As the tensor product is distributive over edge-disjoint subgraphs, $K_{r,r} \times K_m = \bigoplus_{i=1}^{r/2}(H_i \times K_m)$. If m is odd, then $H_i \times K_m$ can be decomposed into Hamilton cycles by Lemma 3.2 of [12]. If m is even, then $H_i \times K_m$ can be decomposed into Hamilton cycles by Lemma 2.2.
Case 2. r is odd.
If $r = 1$, then the result is obvious as $K_{1,1} \times K_m \cong K_{m,m} - F$, where F is a 1-factor of $K_{m,m}$. So we may assume that $r \geq 3$. We complete the proof of Case 2 in two subcases.
Subcase 2.1. $m \geq 3$ is odd.
As K_m is hamiltonian decomposable, $K_m = H_1 \oplus H_2 \oplus \ldots \oplus H_{(m-1)/2}$, where H_i's are Hamilton cycles of K_m. Now $K_{r,r} \times K_m \cong K_m \times K_{r,r} = \bigoplus_{i=1}^{(m-1)/2}(H_i \times K_{r,r})$. $H_i \times K_{r,r} \cong C_{2m} * \overline{K}_r$ by Lemma 2.4, and $C_{2m} * \overline{K}_r$ has a hamiltonian decomposition [11], the result follows.
Subcase 2.2. $m \geq 4$ is even.
Now $K_{r,r} = H_1 \oplus H_2 \oplus \ldots \oplus H_{(r-3)/2} \oplus W_{2r}$, where H_i's are Hamilton cycles of $K_{r,r}$ and $W_{2r} \cong X(2r; \{1, r\})$ by Lemma 2.1. $K_{r,r} \times K_m = \bigoplus_{i=1}^{(r-3)/2}(H_i \times K_m) \oplus (W_{2r} \times K_m)$. The graph $H_i \times K_m$ has a hamiltonian decomposition by Lemma 2.2. The graph $W_{2r} \times K_m$ is hamiltonian decomposable by Lemma 2.3. This completes the proof. ∎

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