Some NP-Completeness Results on Partial Steiner Triple Systems and Parallel Classes

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Abstract

The complexity of determining if a Steiner triple system on $v = 6n + 3$ points contains a parallel class, is currently unknown. In this paper, we show that the problem of determining if a partial Steiner triple system on $v = 6n + 3$ points contains a parallel class is NP-complete. We also consider the problem of determining the chromatic index of a partial Steiner triple system and show that this problem is NP-hard.

1 Introduction

In this paper, we provide several computational-complexity results related to the problem of coloring the blocks of (partial) Steiner triple systems. In 1982, Colbourn [1] presented a depth-first branch-and-bound algorithm for computing the chromatic index of Steiner triple systems. In 1983, Colbourn [2] presented two greedy algorithms for approximating the chromatic index. However, the complexity of computing the chromatic index of a Steiner triple system is still unknown. In addition, the complexity of determining the existence of a parallel class in Steiner triple system is also unknown.

Define a *set system* as a pair $(X, \mathcal{B})$ where X is a finite set of *points*, and $\mathcal{B}$ is a collection of subsets of X, called *blocks*. A (v, λ) triple system is a set system $(X, \mathcal{B})$ where $|X| = v$, such that each unordered pair from X occurs in exactly λ triples of $\mathcal{B}$. If $\lambda = 1$, $(X, \mathcal{B})$ is called a *Steiner triple system (on v points)*, which we denote by $STS(v)$. We denote $|X|$ as the *order* of the triple system. A *partial* (v, λ) triple system is a set system

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$(X, \mathcal{B})$ where $|X| = v$, such that each unordered pair from X occurs in at most λ triples of $\mathcal{B}$. A *partial Steiner triple system* is a partial $(v, 1)$ triple system. If V is a set of elements, we denote the family of k-subsets of V by $\binom{V}{k}$.

It is known that a Steiner triple system of order v exists if and only if $v \equiv 1, 3 \pmod 6$. In addition, if a $STS(v)$ exists, then it must have exactly $\frac{v(v-1)}{6}$ triples and each element occurs in exactly $\frac{v-1}{2}$ triples of the system. For an up-to-date, extensive collection of results on triple systems, see [3].

Given a $STS(v)$ where $v = 6n + 3$, we are interested in determining if the system contains a set of $\frac{v}{3}$ triples such that it partitions the $6n + 3$ points. Such a set of triples is called a *resolution (or parallel) class* of the system. It is not known if this decision problem is NP-complete or not [3]. If we relax the condition from a $STS(v)$ to a partial $STS(v)$, then we can show that this revised decision problem is NP-complete. We formally state this problem as follow:

Problem (PSTS-1-RES): Given a partial $STS(6n + 3)$, does it contain a parallel class?

In the Section 2 we prove that this problem is NP-complete. In Section 3, we show that determining the chromatic index of a partial Steiner triple system is NP-hard. In Section 4, conclusions and future work are given.

2 NP-completeness of PSTS-1-RES

We begin by proving that the PSTS-1-RES problem is NP-complete using a polynomial time transformation from the EXACT COVER BY 3-SETS (X3C) problem. The decision version of X3C can be formulated as follow: "Suppose V is a set of points where $|V| = 3q$ and $\mathcal{B}$ is a family of triples from $\binom{V}{3}$. Does $\mathcal{B}$ contain a subset $\mathcal{B}'$ such that $|\mathcal{B}'| = q$ and $\cup_{B \in \mathcal{B}'} B = V$?" This decision problem is known to be NP-complete [4]. From this point onward, if V is a set of elements, we denote the family of k-subsets of V by $\binom{V}{k}$.

A useful way of interpreting a partial Steiner triple system is to think of the points as vertices of a graph where each triple is represented by a triangle in the graph. Since each pair of points may occur at most once and each point occurs at most $(v - 1)/2$ times, each edge in the graph belongs to exactly one triangle and each vertex of the graph has a degree at most $(v - 1)/2$. Conversely, if a graph with these two properties exist, then we can extract a partial Steiner triple system from it. The graph obtained in this manner will be called the *graph representation* of the partial Steiner triple system. We now proceed to show that PSTS-1-RES is NP-complete.

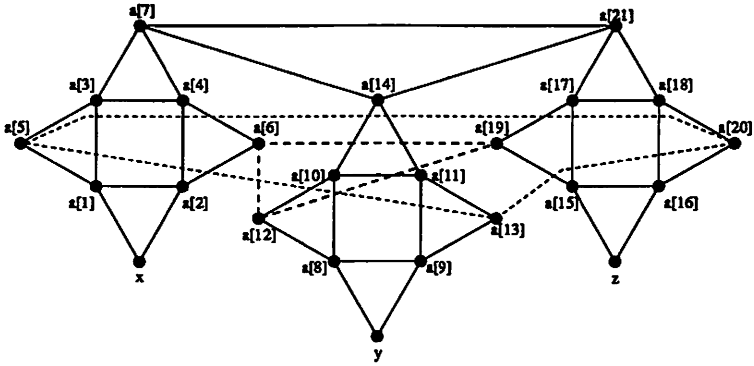

Figure 1: The component graph for a triple $c_i = \{x_i, y_i, z_i\}$. The subscript i is not shown in the diagram

Theorem 2.1 *PSTS-1-RES is NP-complete*

Proof. It is clear that PSTS-1-RES is in NP. We now introduce a polynomial transformation from X3C to PSTS-1-RES. Let (X, C) be an instance of X3C, where $|X| = 3q$, for some $q \in \mathbb{N}$. We will construct a graph representation of a PSTS using the triples of C. In this graph, each edge will belong to exactly one triangle. In addition, each vertex in the graph must have degree at most $\frac{v-1}{2}$, where v is yet to be determined.

For each $c_i = \{x_i, y_i, z_i\} \in C$, form the sub-graph given in Figure 1. This sub-graph represents the following triples in the corresponding PSTS problem instance: $\{x_i, a_i[1], a_i[2]\}$, $\{a_i[1], a_i[3], a_i[5]\}$, $\{a_i[2], a_i[4], a_i[6]\}$, $\{a_i[3], a_i[4], a_i[7]\}$, $\{y_i, a_i[8], a_i[9]\}$, $\{a_i[8], a_i[10], a_i[12]\}$, $\{a_i[9], a_i[11], a_i[13]\}$, $\{a_i[10], a_i[11], a_i[14]\}$, $\{z_i, a_i[15], a_i[16]\}$, $\{a_i[15], a_i[17], a_i[19]\}$, $\{a_i[16], a_i[18], a_i[20]\}$, $\{a_i[17], a_i[18], a_i[21]\}$ and $\{a_i[7], a_i[14], a_i[21]\}$. Notice the only vertices that could appear in more than one subgraph induced by C are those vertices which are the elements of X.

There are a total of $3q + 21 * |C|$ vertices in the constructed graph, which is divisible by 3. In addition, each edge of the graph appears in exactly one triangle. In order to make this a $PSTS(v)$ graph, we must ensure that each vertex belongs to at most $\frac{v-1}{2}$ triangles and $v = 6n + 3$ for some $n \in \mathbb{N}$. To do this, simply introduce disjoint triangles into the graph until both conditions are satisfied. It is clear that this transformation can be done in polynomial time with respect to $|C|$.

We now show that C contains an exact cover if and only if the PSTS graph contains a set of triangles that partition the vertices of the graph.

Suppose $c_1, c_2, ..., c_q$ form an exact cover of X. For $c_i = \{x_i, y_i, z_i\}$ in the exact cover, choose the triangles: $\{x_i, a_i[1], a_i[2]\}$, $\{y_i, a_i[8], a_i[9]\}$,

$\{z_i, a_i[15], a_i[16]\}$, $\{a_i[5], a_i[13], a_i[20]\}$, $\{a_i[6], a_i[12], a_i[19]\}$, $\{a_i[3], a_i[4], a_i[7]\}$, $\{a_i[10], a_i[11], a_i[14]\}$ and $\{a_i[17], a_i[18], a_i[21]\}$. For $c_i = \{x_i, y_i, z_i\}$ not in the exact cover, choose the triangles $\{a_i[1], a_i[3], a_i[5]\}$, $\{a_i[2], a_i[4], a_i[6]\}$, $\{a_i[8], a_i[10], a_i[12]\}$, $\{a_i[9], a_i[11], a_i[13]\}$, $\{a_i[15], a_i[17], a_i[19]\}$, $\{a_i[16], a_i[18], a_i[20]\}$, and $\{a_i[7], a_i[14], a_i[21]\}$. Finally, for each of the disjoint triangles added in (to satisfy the condition that each vertex belongs to at most $v-1/2$ triangles) , we trivially choose them to be in the partition. Clearly, this is a partition of the vertices of the graph into disjoint triangles.

Conversely, suppose the graph can be partition into disjoint triangles. We need to construct an exact cover of X. To do this, simple pick $c_i \in C$ where the triangle $\{a_i[7], a_i[14], a_i[21]\}$ is not in the partition. To see that this leads to a exact cover of C, we note that $\{a_i[7], a_i[14], a_i[21]\}$ is not in the partition if and only if $\{x_i, a_i[1], a_i[2]\}$, $\{y_i, a_i[8], a_i[9]\}$ and $\{z_i, a_i[15], a_i[16]\}$ are in the partition. Hence the decision problem PSTS-1-RES is NP-complete. □

It is known that the X3C problem is still NP-complete under the restriction that each element appears in at most three triples, consequently, the following result is immediate:

Corollary 2.2 *The PSTS-1-RES problem is NP-complete under the assumption that each element appears in at most three triples.*

Proof. In the proof of Theorem 2.1, start with an arbitrary instance of the X3C problem where each element occurs in at most three triples of the instance. In the construction used in Theorem 2.1, all points in the partial Steiner triple system of the form $a_i[j]$ belong to at most two triples. In addition, each occurrence of x_i (in the X3C instance) causes it's corresponding element in the partial-STS to be contained in one triple. Since each element occurs in at most three triples of C, no point in the partial-STS will be in more than three triples. □

Consider the scenario where the PSTS-1-RES problem instances are such that each element occurs in at most two triples. Let us denote this restricted problem as PSTS-1-RES-2. This problem can be solved in polynomial time. This result should not be surprising since the X3C problem can be solved in polynomial time if each element occurs in at most two triples.

Theorem 2.3 *The PSTS-1-RES-2 problem can be solved in polynomial time.*

Proof. A simple way to show this is to realize that if you have an instance of the PSTS-1-RES-2, it can be trivially reduced to an instance of the X3C problem. There is a polynomial time algorithm to solve this X3C problem instance, and hence solves the PSTS-1-RES-2 instance. □

Consider the following question: "Given a partial STS(v), does it contain (at least) two parallel classes?" We do not attempt to answer this question, but instead by answer the simpler question: "Given a partial STS(v) where each element occurs at most two times, does it contain two parallel classes?" Obviously, in order to have any hope of containing two parallel classes, each element must occur exactly twice in the partial STS(v) and hence we can assume the condition that each element occurs exactly twice.

Theorem 2.4 *Given a partial STS(v) where each element occurs exactly two times, a polynomial time algorithm exists to determine if it contains two (disjoint) parallel classes.*

Proof. By Theorem 2.3, it can be determine in polynomial time whether the partial STS(v) contains a parallel class or not. If it does, then clearly it will contain two parallel classes, as each element occurs exactly twice. Otherwise, if it does not contain a parallel class, it surely will not contain two parallel classes. □

We end this section by showing that it is not possible to approximate (in polynomial time) a solution that is at most some fixed constant away from an optimal solution for the optimization version (PSTS-1-RES-OPT) of the PSTS-1-RES problem. Given a polynomial time algorithm A and an instance I, let $A(I)$ denote the "size" of the approximate solution computed by A and let $OPT(I)$ denote the size of an optimal solution of I.

Problem (PSTS-1-RES-OPT): Given a partial $STS(6n+3)$, find the largest partial parallel class it contains.

Theorem 2.5 *If $P \neq NP$, then no polynomial time algorithm A for the PSTS-1-RES-OPT problem can guarantee*

$$OPT(I) - A(I) \leq k$$

for an instance I and for a fixed constant k.

Proof. Suppose not. That is, suppose a polynomial time algorithm A exists that satisfies the conditions stated above. We will use A to derive a polynomial time algorithm for solving the PSTS-1-RES decision problem and hence contradicting $P \neq NP$. Given an instance $I = (X, \mathcal{B})$, we construct $k+1$ disjoint isomorphic copies of I. We will denote this new instance by $J = (Y, \mathcal{C})$. We note that $OPT(J) = (k+1)OPT(I)$. Now, we construct a partial parallel class for I of size at least $\lceil A(J)/(k+1) \rceil$, by taking the largest partial parallel class among the disjoint copies of I. As

$(k+1)OPT(I) - A(J) = OPT(J) - A(J) \leq k$, dividing by $k+1$, we obtain $OPT(I) - 1 < \frac{A(J)}{k+1} \leq OPT(I)$, and consequently $OPT(I) = \lceil \frac{A(J)}{k+1} \rceil$. This means the constructed partial parallel class for I must be optimal. Hence we have designed a polynomial time algorithm (based on the algorithm A) for computing the optimal size of a partial parallel class in I, which implies $P = NP$. This is a contradiction. □

3 Chromatic index of partial Steiner triple systems

We say that a partial STS(v) can be *(properly) k-colored* if the blocks of the systems can be colored such that no two intersecting triples have the same color. Given a partial STS(v), its *chromatic index* is the smallest value of k such that it can be properly k-colored. In this section we provide some results about the chromatic index of partial Steiner triple systems under certain restrictions. We begin by showing that determining if the triples of a partial Steiner triple system can be properly 3-colored is NP-complete. In order to do this, we will use the result by Holyer [5] that deciding if a (3-regular) graph is (edge) 3-colorable is NP-complete.

Lemma 3.1 *Deciding whether the blocks of a partial Steiner triple system can be properly 3-colored is NP-complete.*

Proof. The problem clearly belongs in NP. We will construct a polynomial-time reduction from the 3-colorability of 3-regular graphs problem to our problem. Suppose $G = (V, E)$ is a 3-regular graph. For each edge $e_i = \{u, v\} \in E$, introduce a new point ∞_i and construct the triple $\{\infty_i, u, v\}$. It is easy to see that the set system $(X, \mathcal{B})$ where

$$X = V \bigcup \left(\bigcup_{e_i \in E} \{\infty_i\} \right)$$

and

$$\mathcal{B} = \bigcup_{e_i = \{u,v\} \in E} \{\{\infty_i, u, v\}\}$$

is a partial Steiner triple system. If the value of $|X|$ is not admissible, simply add points to X until it becomes admissible. This transformation can be done in polynomial time.

If G can be (properly) 3-colored, then color the triple $\{\infty_i, u, v\}$ the same color as the edge $e_i = \{u, v\} \in E$. Clearly this is a 3-coloring of the blocks of $\mathcal{B}$. Conversely, if the blocks of $\mathcal{B}$ can be 3-colored, then for each block $\{\infty_i, u, v\} \in \mathcal{B}$, color the corresponding edge $e_i = \{u, v\} \in E$ the same color. This properly 3-colors the graph G. □

This immediately leads to the following result, assuming $P \neq NP$.

Corollary 3.2 *Determining the chromatic index of a partial Steiner system is NP-Hard.*

4 Conclusions

In this paper, we have derived several complexity results on partial Steiner triple systems and parallel classes. We would like to be able to extend these results to (full) Steiner triple systems. The difficulty arises from the fact that it is difficult to identify a known NP-complete problem from which a polynomial time reduction to instances of the Steiner triple system can be derived. Another possible research direction is to consider developing better approximation algorithms for approximating the chromatic index of partial Steiner triple systems than those currently available.

References

[1] C.J. COLBOURN *Computing the chromatic index of Steiner triple systems*, Computer J., 25(1982), 338-339.

[2] C.J. COLBOURN AND M.J. COLBOURN *Greedy colourings of Steiner triple systems*, Ann. Discr. Math. 18(1983), 201-207.

[3] C.J. COLBOURN AND A. ROSA. *Triple systems*, Oxford University Press, 1999.

[4] M.R. GAREY AND D.S. JOHNSON *Computers and intractability A guide to the theory of NP-completeness*, W.H. Freeman and Company, 1979.

[5] I. HOLYER *The NP-completeness of edge-colouring*, SIAM J. Computing, 10(1981), 718-720.