On the Constructions of New Families of Harmonious Graphs

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Abstract

In this paper, three methods for constructing larger harmonious graphs from one or a set of harmonious graphs are provided.

1. Introduction

All the graphs considered in this paper are finite simple graphs. Let $G = (V(G), E(G))$ be a graph with m vertices and k edges. A *vertex labeling* of G is an injection $f : V(G) \to \{0, 1, 2, \ldots, k-1\}$. And a vertex labeling f of G is called *harmonious* if the induced edge labeling $f^* : E(G) \to \{0, 1, 2, \ldots, k-1\}$ given by

$$f^*(\{u,v\}) = f(u) + f(v) \pmod{k} \text{ for every edge } \{u,v\}$$

is 1-1 when G is not a tree. If G is a tree, exactly one label can be used on two vertices and the resulting edge labels are distinct. A graph G with a harmonious labeling is called a *harmonious graph.*

In what follows, we let $f^{\#}$ denote the induced edge labeling $f^{\#} : E(G) \to \{1, 2, \ldots, 2k-3\}$ defined as

$$f^{\#}(\{u,v\}) = f(u) + f(v) \quad \text{for every edge } \{u,v\}.$$

Most of the results on constructing larger harmonious graphs focus on particular classes of graphs and methods [2, 3, 4, 7, 8, 9]. In the present paper, we discuss three kinds of methods for constructing larger harmonious graphs from one or a set of harmonious graphs.

2. The constructions

Construction I : Suppose that $G_1, G_2, \ldots, G_n$ are disjoint graphs and let v_i be any vertex of G_i, $i = 1, 2, \ldots, n$. The vertex-amalgamated graph on $G_1, G_2, \ldots, G_n$, denoted by $\odot(G_1, G_2, \ldots, G_n)$, is the graph obtained from $G_1, G_2, \ldots, G_n$ by identifying the vertices $v_1, v_2, \ldots, v_n$ on the graph $\bigcup_{i=1}^{n} G_i$. See Fig.1. We shorten $\odot(G_1, G_2, \ldots, G_n)$ to $\odot G^n$ if $G_i \cong G$ for $i = 1, 2, \ldots, n$.

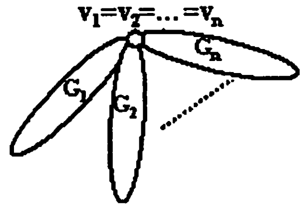


Fig. 1

Throughout the paper, let G be a graph with m vertices $\{w_1, w_2, \ldots, w_m\}$ and k edges $\{e_1, e_2, \ldots, e_k\}$.

Theorem 2.1. *If G is harmonious, then $\odot G^{2n+1}$ is harmonious.*

Proof. For the convenience of notation, let G_i be the i-th copy of G in $\odot G^{2n+1}$ and let $\{u_{i,1}, u_{i,2}, \ldots, u_{i,m}\}$ be the vertex set of G_i with the property that the vertex $u_{i,j}$ is the isomorphic image of w_j, for $j = 1, 2, \ldots, m$, and $i = 1, 2, \ldots, 2n+1$.

Choose any vertex of G, say w_1, to be the amalgamated vertex in $\odot G^{2n+1}$ and let $f(w_1) = c$, where f is a harmonious labeling of G. Therefore the vertices $u_{1,1}, u_{2,1}, \ldots, u_{2n+1,1}$ will be identified in $\odot G^{2n+1}$.

Let's introduce a vertex labeling g of $\odot G^{2n+1}$ defined by $(1 \le i \le 2n+1)$

$$g(u_{i,j}) = \begin{cases} c & , \text{ if } \quad j = 1, \\ f(w_j) + (i-1)k & , \text{ if } \quad 2 \le j \le m. \end{cases}$$

Clearly, g is one-to-one and $\left|E(\odot G^{2n+1})\right| = (2n+1)k$. To prove that g is harmonious, we shall show that the labels of edges in $\odot G^{2n+1}$ are all distinct.

Note that $g^{\#}(E(G_1)) = f^{\#}(E(G))$. And for any edge e_j of G, let's consider the isomorphic images $\dot{e}_{i,j} \in E(G_i)$, $i = 1, 2, \ldots, 2n+1$, of e_j. If $f^{\#}(e_j) = s_j$, the labels of these edges are given as follows

(1) If w_1 is one of the endpoints of e_j, then $\{ g^{\#}(\dot{e}_{i,j}) \mid i = 1, 2, \ldots, 2n+1\} = \{ s_j, s_j+k, s_j+2k, \ldots, s_j+2nk\}$.

(2) If w_1 is not an endpoint of e_j, then $\{ g^{\#}(\dot{e}_{i,j}) \mid i = 1, 2, \ldots, 2n+1\} = \{ s_j, s_j+2k, s_j+4k, \ldots, s_j+4nk\}$.

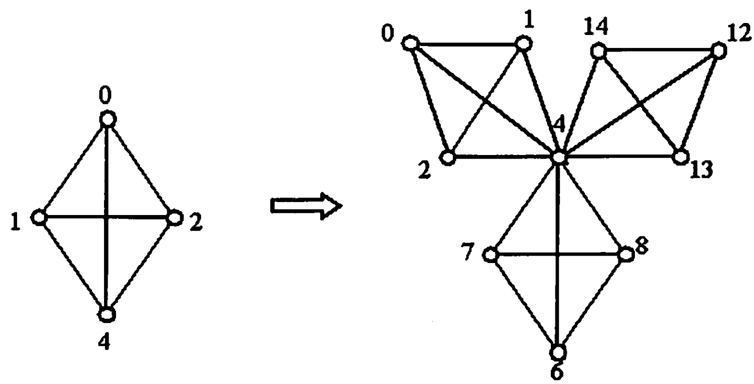
In both cases, $\{ g^{\#}(\dot{e}_{i,j}) \mid i = 1, 2, \ldots, 2n+1\} \equiv \{ s_j, s_j+k, s_j+2k, \ldots, s_j+2nk\}$ (mod $(2n+1)k$).

So $\{ g^{*}(\dot{e}_{i,j}) \mid i = 1, 2, \ldots, 2n+1\} =$

$$\begin{cases} \{s_j, s_j+k,\ s_j+2k,\ \ldots,\ s_j+2nk\} & , \quad \text{if } s_j < k \\ \{s_j-k, s_j,\ s_j+k,\ \ldots,\ s_j+(2n-1)k\} & , \quad \text{if } s_j \ge k \end{cases}$$

One can easily observe that g is a harmonious labeling of $\odot G^{2n+1}$. ∎

The graph $\odot(K_4)^{2n+1}$ is actually a windmill $K_4^{(2n+1)}$ [6], as shown in Fig.2.



(1) K_4 (2) $\odot(K_4)^3 = K_4^{(3)}$ (choose $c = 4$)

Fig. 2

Remarks:

(1) According to the dynamic survey by Gallian [3], Figueroa-Centeno, Ichishima and Muntaner-Batle [2] have shown a similar result using the vertex labeled 0 as the amalgamated vertex.

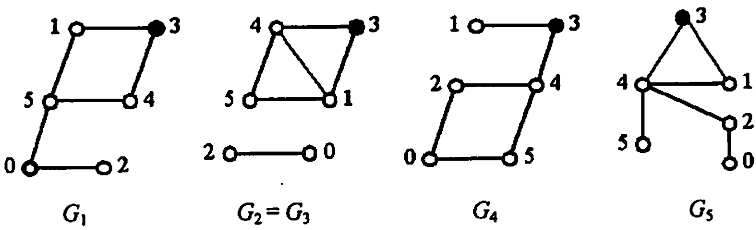
(2) In [1] Deb and Limaye use the notation $C(m,k)$ to denote the set of cycles C_m with k cords sharing a common endpoint. They call the graph $C(m,m-3)$ a shell and define a multiple shell to be a collection of edge disjoint shells that have their apex in common. They showed that a variety of multiple shells are harmonious and conjecture that all multiple shells are harmonious. Since $C(m,m-3)$ is the same as the fan f_{m-1} which has been proven to be harmonious [5], by theorem 2.1, we conclude that any multiple shell consisting of an odd number of shells is harmonious.

(3) The method used in theorem 2.1 only works when an odd number of copies of a harmonious graph are amalgamated. In the case of even copies, there will be some vertex labels repeated. So the method fails to be true. In this paper, we deal only with the cases of odd copies.

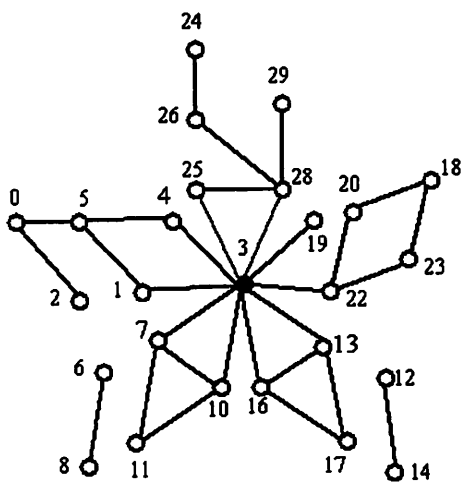
If G_i's are not all isomorphic, throughout this paper, we assume that G_i has m_i vertices and k edges, and f_i is a harmonious labeling of G_i. Suppose that $d(u,v)$ denotes the length of the shortest path joining the vertices u and v. In particular, $d(u,u) = 0$. Let $adj_i(u) = \{ v \mid d(u,v) = 1\}$ be the set of vertices of G_i which are adjacent to the vertex u. By the argument similar to theorem 2.1, we have the following result.

Theorem 2.2. *Let $G_1, G_2, \ldots, G_{2n+1}$ be harmonious with k edges. If $f_1^{*}(E(G_1)) = f_2^{*}(E(G_2)) = \cdots = f_{2n+1}^{*}(E(G_{2n+1}))$ and there exists a vertex v_i of G_i with $f(v_i) = c$ for every $i = 1, 2, \ldots, 2n+1$, such that $adj_1(v_1) = adj_2(v_2) = \cdots = adj_{2n+1}(v_{2n+1})$, then $\odot(G_1, G_2, \ldots, G_{2n+1})$ is harmonious.*

Fig.3 gives an example for illustration. The solid dots in the graphs indicate the amalgamated vertex.



(1) $f_i^{*}(E(G_i))=\{2, 4, 5, 6, 7, 9\}$ and $adj_i(3) = \{1, 4\}$, $i = 1, 2, 3, 4, 5$



(2) $\odot(G_1, G_2, \ldots, G_5)$ $(c = 3)$

Fig. 3

By setting $c = 0$ in theorem 2.1 and 2.2 , we have similar results when $G_1, G_2, \ldots, G_{2n+1}$ are trees.

Corollary 2.3.

(1) *If G is a harmonious tree with k edges and $f(V(G)) = \{0, 1, \ldots, k\}$, then $\odot G^{2n+1}$ is harmonious.*

(2) *Let* $G_1, G_2, \ldots, G_{2n+1}$ *be harmonious trees with k edges. If* $f_i(V(G_i)) = \{0, 1, \ldots, k\}$, $i = 1, 2, \ldots, 2n+1$, *and* $f_1^{\#}(E(G_1)) = f_2^{\#}(E(G_2)) = \cdots = f_{2n+1}^{\#}(E(G_{2n+1}))$ *with* $adj_1(0) = adj_2(0) = \cdots = adj_{2n+1}(0)$, *then* $\odot(G_1, G_2, \ldots, G_{2n+1})$ *is harmonious.*

Remark. It's now easy to see that $\odot C_{2m+1}^{2n+1}$ and $\odot P_{m+1}^{2n+1}$ are harmonious , for all $m, n \geq 1$.

Construction II: Suppose that $G_1, G_2, \ldots, G_n$ are disjoint graphs and let v_i be any vertex of G_i, $i = 1, 2, \ldots, n$. The vertex-edge-attached graph on $G_1, G_2, \ldots, G_n$, denoted by $\oplus(G_1, G_2, \ldots, G_n | v_1, v_2, \ldots, v_n)$, is the graph obtained from $G_1, G_2, \ldots, G_n$ by adjoining to graph $\bigcup_{i=1}^{n} G_i$ with a new vertex w accompanied by the edges $\{w, v_1\}$, $\{w, v_2\}$, …, $\{w, v_n\}$, as shown in Fig.4.

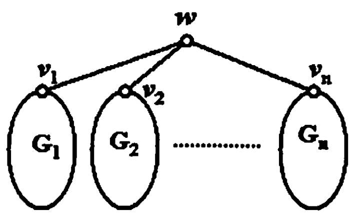


Fig. 4

In the case when $G_i \cong G$ for $i = 1, 2, \ldots, n$, we simply write $\oplus(G_1, G_2, \ldots, G_n | v_1, v_2, \ldots, v_n)$ as $\oplus(G^n | v_1, v_2, \ldots, v_n)$. Recall that $\{ u_{i,1}, u_{i,2}, \ldots, u_{i,m} \}$ is the vertex set of G_i such that the vertex $u_{i,j}$ is the isomorphic image of w_j for $j = 1, 2, \ldots, m$, and $i = 1, 2, \ldots, 2n+1$.

Theorem 2.4. *If G is harmonious with* $f^{\#}(E(G)) = \{ r, r+1, \ldots, r+k-1 \}$ *and* $f(w_1) = r$. *Let* $u_{i,1} \in V(G_i)$ *be the isomorphic image of* w_1 *for* $i = 1, 2, \ldots, 2n+1$, *then* $\oplus(G^{2n+1} | u_{1,1}, u_{2,1}, \ldots, u_{2n+1,1})$ *is harmonious.*

Proof. In $\oplus(G^{2n+1} | u_{1,1}, u_{2,1}, \ldots, u_{2n+1,1})$, the new vertex w is adjoined to the vertices $u_{i,1}$'s, $i = 1, 2, \ldots, 2n+1$. Let's construct a vertex labeling g of $\oplus(G^{2n+1} | u_{1,1}, u_{2,1}, \ldots, u_{2n+1,1})$ defined as $(1 \leq i \leq 2n+1)$

$g(w) = 0$, and

$$g(u_{i,j}) = f(w_j) + 1 + (i-1)(k+1), \quad 1 \leq j \leq m.$$

One can easily observe that all the vertex labels are distinct and $|E(\oplus G^{2n+1})| = (2n+1)(k+1)$. To prove that g is harmonious, it suffices to show that all the edge labels are distinct.

Note that $g^{*}(E(G_i)) = \{ r+2+2(i-1)(k+1), r+3+2(i-1)(k+1), \ldots, r+k+1+2(i-1)(k+1) \}$, $i = 1, 2, \ldots, 2n+1$. For any edge e_j of G, let's consider the isomorphic images $\dot{e}_{i,j} \in E(G_i)$, $i = 1, 2, \ldots, 2n+1$, of e_j. If $f^{*}(e_j) = s_j$, the labels of these edges are given as

$\{ g^{\#}(\dot{e}_{i,j}) \mid i = 1, 2, \ldots, 2n+1 \}$

$= \{ s_j+2, s_j+2+2(k+1), s_j+2+4(k+1), \ldots, s_j+2+4n(k+1)\}$

$\equiv \{ s_j+2, s_j+2+(k+1), s_j+2+2(k+1), \ldots, s_j+2+2n(k+1)\} \pmod{(2n+1)(k+1)}$.

So $\{ g^{*}(\dot{e}_{i,j}) \mid i = 1, 2, \ldots, 2n+1 \} =$

$$\begin{cases} \{ s_j+2, s_j+2+(k+1), s_j+2+2(k+1), \ldots, s_j+2+2n(k+1)\}, & \text{if } s_j+2 < k+1, \\ \{ s_j+2-(k+1), s_j+2, s_j+2+(k+1), \ldots, s_j+2+(2n-1)(k+1)\}, & \text{if } s_j+2 \geq k+1, \end{cases}$$

where $s_j \in \{ r, r+1, \ldots, r+k-1 \}$.

Next, $g^{*}(\{w, u_{i,1}\}) = g(u_{i,1}) = r+1+(i-1)(k+1)$, $i = 1, 2, \ldots, 2n+1$. One can now easily verify that g is harmonious. ∎

We provide an example when G is the complete graph K_4 in Fig.5. (The solid dots indicate the vertices to which the new vertex is adjoined.)

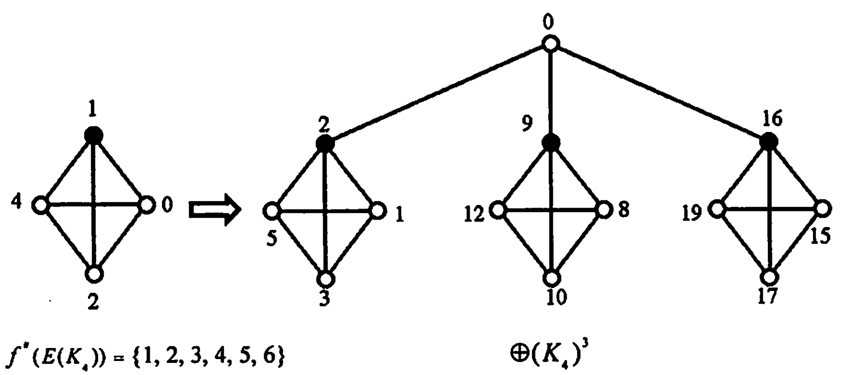
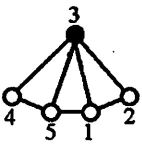


Fig. 5

When $G_1, G_2, \ldots, G_{2n+1}$ are not all isomorphic, we assume that $\{ u_{i,1}, u_{i,2}, \ldots, u_{i,m_i} \}$ is the vertex set of G_i for $i = 1, 2, \ldots, 2n+1$. A similar result is given as follows.

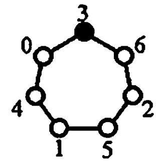
Theorem 2.5. *If $G_1, G_2, \ldots, G_{2n+1}$ are harmonious with $f_i^{*}(E(G_i)) = \{ r, r+1, \ldots, r+k-1 \}$ and $f_i(u_{i,1}) = r$ for $i = 1, 2, \ldots, 2n+1$, then $\oplus(G_1, G_2, \ldots, G_{2n+1} | u_{1,1}, u_{2,1}, \ldots, u_{2n+1,1})$ is harmonious.*

Fig.6 gives an example for this case.



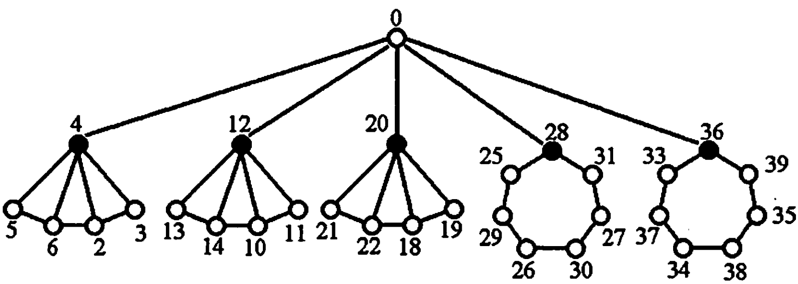
$f_i^*(E(G_i)) = \{3, 4, 5, 6, 7, 8, 9\}$, $i = 1, 2, 3$

(1) $G_1 = G_2 = G_3 = f_4$ (the fan)



$f_i^*(E(G_i)) = \{3, 4, 5, 6, 7, 8, 9\}$, $i = 4, 5$

(2) $G_4 = G_5 = C_7$ (the cycle)



(3) $\oplus(G_1, G_2, \ldots, G_5)$

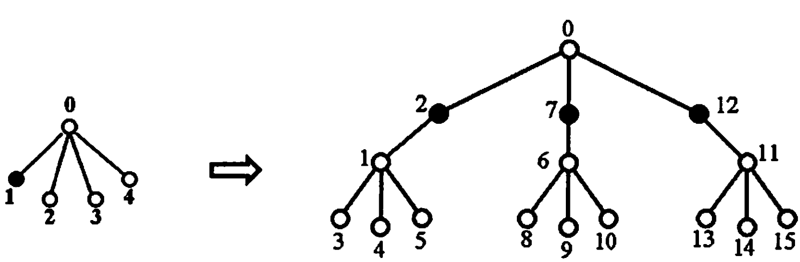
Fig. 6

The analogous result follows immediately when it comes to trees.

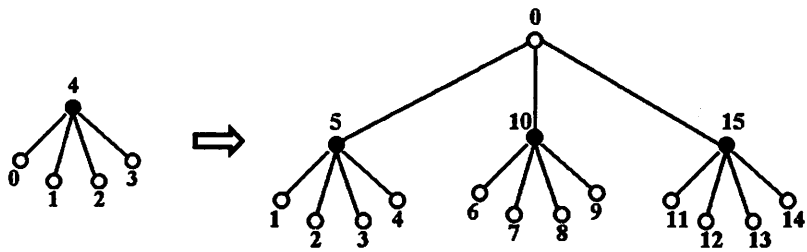
Corollary 2.6. *Let* $G_1, G_2, \ldots, G_{2n+1}$ *be harmonious trees with k edges . If* $f_i(V(G_i)) = \{0,\ 1,\ \ldots,\ k\}$ *with* $f_i(u_{i,1}) = r$ *and* $f_i^*(E(G_i)) = \{\ r,\ r+1,\ \ldots,\ r+k-1\ \}$, $i = 1, 2, \ldots, 2n+1$, *then* $\oplus(G_1, G_2, \ldots, G_{2n+1} | u_{1,1}, u_{2,1}, \ldots, u_{2n+1,1})$ *is harmonious.*

Remark. $\oplus C_{2m+1}^{2n+1}$, $\oplus S_{m+1}^{2n+1}$ and $\oplus P_{m+1}^{2n+1}$ are harmonious, for all $n, m \geq 1$.

Two possible cases of the graph $\oplus(S_5)^3$ are shown in Fig.7.



(1) $f^*(E(S_5)) = \{\ 1, 2, 3, 4\}$ $\oplus(S_5)^3$



(2) $f^*(E(S_5)) = \{4, 5, 6, 7\}$ $\qquad\qquad$ $\oplus(S_5)^3$

Fig. 7

Construction III : Suppose that $G_1, G_2, \ldots, G_n$ are disjoint graphs and let v_i be any vertex of G_i , i= 1, 2, ..., n. The edge-attached graph on $G_1, G_2, \ldots, G_n$, denoted by $\Theta(G_1, G_2, \ldots, G_n | v_1, v_2, \ldots, v_n)$, is the graph obtained from $G_1, G_2, \ldots, G_n$ by adjoining to graph $\bigcup_{i=1}^{n} G_i$ the edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, ..., $\{v_{n-1}, v_n\}$. See Fig.8.

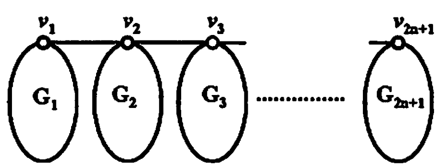


Fig. 8

In the following theorem, we leave the statement of the choices of the v_i's in the proof because it is complicated.

Theorem 2.7. *If* G *is harmonious with* $f^*(E(G)) = \{ r, r+1, \ldots, r+k-1 \}$ *and* $\{ 0, r, k-1 \} \subset f(V(G))$, *then there is a vertex* v_i *of* G_i *for every* $i = 1, 2, \ldots,$ $2n+1$, *such that* $\Theta(G_1, G_2, \ldots, G_{2n+1} | v_1, v_2, \ldots, v_{2n+1})$ *is harmonious.*

Proof. We may assume $f(w_1) = 0$, $f(w_2) = r$, $f(w_3) = k-1$. Recall that $u_{i,j}$, i = 1, 2, ..., $2n$+1, are the isomorphic images of w_j. Let $v_{n+1\pm 2j} = u_{n+1\pm 2j,2}$, $0 \le j \le [n/2]$, $v_{n-2j} = u_{n-2j,1}$, $0 \le j \le [(n-1)/2]$, *and* $v_{n+2+2j} = u_{n+2+2j,3}$, $0 \le j \le [(n-1)/2]$. We join v_i to v_{i+1} , for every i = 1, 2, ..., $2n$. The vertex labeling g of $\Theta(G_1, G_2, \ldots, G_{2n+1} | v_1, v_2, \ldots, v_{2n+1})$ can be described by the following formula. ($1 \le i \le 2n+1$)

$$g(u_{i,j}) = f(w_j) + (i-1)(k+1), \quad 1 \le j \le m .$$

Clearly, all vertex labels are distinct and $| \ \Theta G^{2n+1} \ | = k\ (2n+1) + 2n$. Therefore, to prove that g is harmonious, we shall show that all edge labels are also distinct. Since $g^*(E(G_i)) = \{ r+2(i-1)(k+1),\ r+1+2(i-1)(k+1), ..., r+k-1+2(i-1)(k+1) \}$, by reducing them modulo $k(2n+1)+2n$, we have

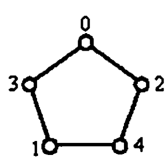
$$g^*(E(G_1)) \equiv \{ r,\ r+1, ...,\ r+k-1 \},$$
$$g^*(E(G_2)) \equiv \{ r+2k+2,\ r+2k+3, ...,\ r+3k+1 \},$$
$$\vdots$$
$$g^*(E(G_{n+1})) \equiv \{ r+2n(k+1),\ r+1+2n(k+1), ...,\ k(2n+1)+2n-1,\ 0,\ 1, ...,\ r-1 \},$$
$$g^*(E(G_{n+2})) \equiv \{ r+k+2,\ r+k+3, ...,\ r+2k+1 \},$$
$$\vdots$$
$$g^*(E(G_{2n+1})) \equiv \{ r+(2n-1)(k+1)+1,\ r+(2n-1)(k+1)+2, ...,\ r+(2n-1)(k+1)+k \}.$$

In addition, a routine computation shows

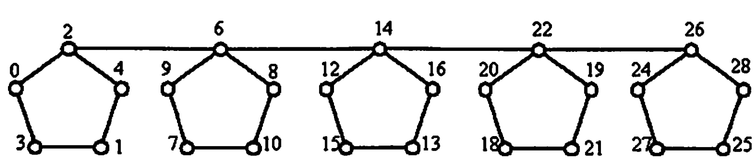
$$g^*(\{v_j, v_{j+1}\}) \equiv$$
$$\begin{cases} \{ r+k+1,\ r+3k+3,\ r+5k+5, ...,\ r+(2n-1)(k+1) \} & , \quad 1 \le j \le n, \\ \{ r+k,\ r+3k+2,\ r+5k+4, ...,\ r+(2n-1)(k+1)-1 \} & , \quad n+1 \le j \le 2n. \end{cases}$$

It can now be easily seen that g is harmonious. ▮

We illustrate for $\Theta(C_5)^5$ in Fig.9.



(1) $f^*(E(C_5)) = \{ 2, 3, 4, 5, 6\}$



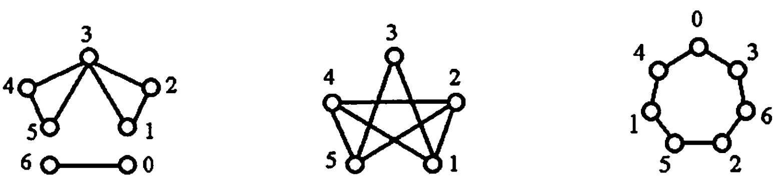
(2) $\Theta(C_5)^5$

Fig. 9

The next result considers the edge-attached graph when G_i's are not all isomorphic. It looks complicated because we have to make sure that each vertex label that should be assigned to the vertex v_i does appear in $f_i(V(G_i))$ for $i = 1, 2, \ldots, 2n+1$.

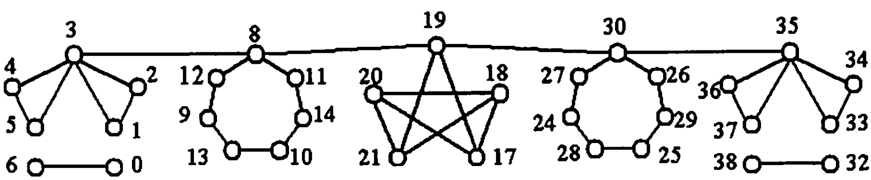
Theorem 2.8. *Let* $G_1, G_2, \ldots, G_{2n+1}$ *be harmonious. If* $f_i^{\#}(E(G_i)) = \{ r, r+1, \ldots, r+k-1 \}$ *and* $f_{n-2j}(u_{n-2j,1}) = 0$, *for* $0 \le j \le [(n-1)/2]$; $f_{n+1\pm 2j}(u_{n+1\pm 2j,1}) = r$, *for* $0 \le j \le [n/2]$ *and* $f_{n+2+2j}(u_{n+2+2j,1}) = k-1$, *for* $0 \le j \le [(n-1)/2]$, *then* $\Theta(G_1, G_2, \ldots, G_{2n+1} | u_{1,1}, u_{2,1}, \ldots, _{2n+1,1})$ *is harmonious.*

Fig.10 illustrates for this case.



$G_1 = G_5$ G_3 $G_2 = G_4$

(1) $f_i^{\#}(E(G_i))=\{3, 4, 5, 6, 7, 8, 9\}$, $i = 1, 2, \ldots, 5$.



(2) $\Theta(G_1, G_2, \ldots, G_5)$

Fig. 10

With the same choices of the $u_{i,1}$'s stated in theorem 2.8. We have a similar result about trees given below.

Corollary 2.9. *Let* $G_1, G_2, \ldots, G_{2n+1}$ *be harmonious trees with k edge. If* $f_i(V(G_i)) = \{ 0, 1, \ldots, k\}$, *and* $f_i^{\#}(E(G_i)) = \{ r, r+1, \ldots, r+k-1 \}$, $i = 1, 2, \ldots, 2n+1$, *then the graph* $\Theta(G_1, G_2, \ldots, G_{2n+1} | u_{1,1}, u_{2,1}, \ldots, _{2n+1,1})$ *is harmonious.*

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