STRUCTURAL PROPERTIES OF HYPER-STARS

EDDIE CHENG AND LÁSZLÓ LIPTÁK

ABSTRACT. Star graphs were introduced by [1] as a competitive model to the n-cubes. Then hyper-stars were introduced in [9] to be a competitive model to both n-cubes and star graphs. In this paper we discuss strong connectivity properties and orientability of the hyper-stars.

Keywords: Interconnection networks, hyper-stars, connectivity.

1. Introduction and Preliminaries

Interconnection networks are important in parallel computing. The first major class is the classical n-cubes. Star graphs were introduced by [1] as a competitive model to the n-cubes. The hyper-stars were introduced in [9] to be a competitive model to both n-cubes and star graphs. In particular, it was shown that the hyper-stars and their "folded" version have lower network cost (using standard degree $\times$ diameter as measure) than the n-cubes and their "folded" version and other variants. Both the n-cubes and star graphs have been studied and many of their properties are known, and star graphs have proven to be superior to the n-cubes. Since the star graphs have order $n!$, for a particular network application using the star graph topology, one may be faced with the choice of either too few or too many available vertices. Two generalizations were introduced in [5] and [7] to address this issue. Another approach is via augmentation of star graphs given in [3]. The introduction of hyper-star graphs (*hyper-stars* for short) provides another interesting class of interconnection networks addressing this issue. Since n-cubes and star graphs have the desirable tightly super-connectedness property, the hyper-stars should have this property to be competitive. We will indeed establish this in this paper. Directed interconnection networks have also gained much attention in the area of interconnection networks. In particular, [4] gave an application and an architectural model for the studies of unidirectional graph topologies. Furthermore [6] proposed the use of oriented n-cubes as the basis for high speed networking. In this paper, we propose a good orientation for the hyper-stars.

The *hyper-star* $\mathrm{HS}(n,k)$ with $1 \le k \le n-1$ is defined as follows: its vertex-set is the set of $\{0,1\}$-strings of length n with exactly k 1's, and two vertices are adjacent if and only if one can be obtained from the other by exchanging the first symbol with a different symbol (1 with 0, or 0 with 1) in another position. Hence it is a bipartite graph with bipartition sets $V_0(n,k)$ and $V_1(n,k)$, where $V_0(n,k)$ (resp. $V_1(n,k)$) is the set of vertices of $\mathrm{HS}(n,k)$ with 0 (resp. 1) in the first position. We may use V_0 and V_1 instead of $V_0(n,k)$ and $V_1(n,k)$ if it is clear from the context. Hence $\mathrm{HS}(n,k)$ has $\binom{n}{k}$ vertices, every vertex in V_0 has degree k and every vertex in V_1 has degree $n-k$. So $\mathrm{HS}(n,k)$ is regular if and only if $n = 2k$. Figure 1 gives HS(6,3). Note that it is "interconnected" by a graph isomorphic to HS(5,2) (the left subgraph with 1 in the 6th position) and a graph isomorphic to HS(5,3) (the right subgraph with 0 in the 6th position). Another simple but useful fact that can be easily checked is that $\mathrm{HS}(n,k)$ has no 4-cycles.

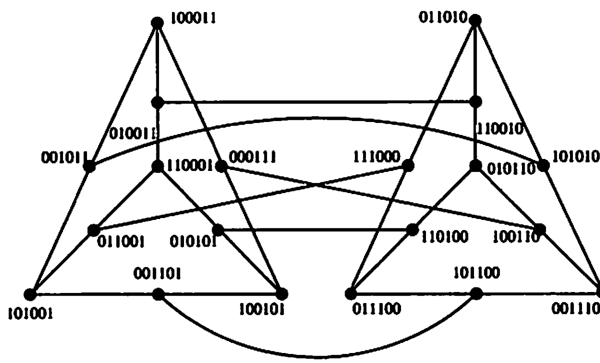

FIGURE 1. HS(6,3)

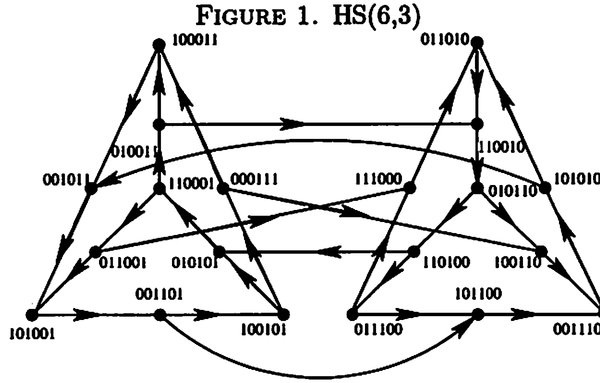

FIGURE 2. UHS(6,3)

In [8, 9] the diameter of $\mathrm{HS}(n,k)$ is obtained and more specific results are mentioned/proved for $\mathrm{HS}(2n,n)$, as regularity is often desirable. Their results concentrate on fault-diameter, w-diameter and a broadcasting scheme for HS($2n, n$). In this paper we study stronger structural properties than mere connectivity. A graph G is *maximally connected* if its connectivity is $\delta(G)$, the minimum degree in G. A maximally connected graph is *tightly super-connected* if the deletion of every minimum disconnecting set (of vertices) will always result in a graph with two components, one of which has only one vertex. This is a much stronger property than requiring the graph to be maximally connected. In the literature of interconnection networks, the deleted vertices are called *faults*. The notion of superness, first introduced in [2], is a good measure of the severity of the disconnectedness in interconnection networks when faults occur. In this paper we prove that HS($2n, n$), the regular hyper-star, is tightly super-connected except for degenerate cases. In fact, we will prove the same for HS(n, k). The second problem that we address in the paper is the orientability of HS(n, k).

We use standard graph theory terminology found in books such as [11]. The following definition will aid our presentation in subsequent sections: For $v \in V_0$ (resp. $v \in V_1$), the edge between v and the vertex obtained by exchanging the 0 (resp. 1) in the first position and the ith 1 (resp. 0) counting from the left will be called an *i-edge from v*. For example, the edge between 00010011 and 10010001 is a 2-edge from 00010011 but a 5-edge from 10010001.

2. Connectivity and Beyond

The connectivity of HS($2n, n$) is discussed in [9], and the vertex-transitivity of HS($2n, n$) is proved in [8]. In this section, we prove that HS(n, k) is edge-transitive and maximally connected. In fact, we will prove that it is tightly super-connected.

Theorem 2.1. *HS(n, k) is edge-transitive and has at most two vertex-transitive classes.*

Proof. If either k or $n - k$ is 1, then the graph is $K_{1,n-1}$ and the claim is clearly true. Hence we may assume $k, n - k \geq 2$.

Since HS(n, k) is connected, it is enough to prove that two adjacent edges are in the same edge-transitive class. Without loss of generality, we may assume that the two adjacent edges are $e_1 = (101a_4a_5 \dots a_n, 011a_4a_5 \dots a_n)$ and $e_2 = (011a_4a_5 \dots a_n, 110a_4a_5 \dots a_n)$. Let ϕ be the bijective function defined by $\phi(a_1a_2a_3a_4 \dots a_n) = a_1a_3a_2a_4 \dots a_n$ for every $a_1a_2a_3a_4 \dots a_n$ in V(HS(n, k)). Then it is easy to check that ϕ preserves adjacency and maps the end-vertices of e_1 to the end-vertices of e_2. Hence HS(n, k) is edge-transitive.

We claim that all vertices in V_1 belong to the same vertex-transitive class and all vertices in V_0 are in the same vertex-transitive class. Since HS(n, k) is connected and bipartite with bipartition V_0 and V_1, it is enough to prove that two vertices of distance 2 apart are in the same vertex-transitive class.

To see the first claim, we observe that ϕ constructed above is the desired automorphism. The second claim can be proved similarly. □

Corollary 2.2. *If $k \neq \frac{n}{2}$, then HS(n,k) has exactly two vertex-transitive classes. Moreover, HS(2n,n) is vertex-transitive.*

Proof. If $k \neq \frac{n}{2}$, then the claim follows easily from Theorem 2.1 as vertices in V_0 and V_1 have different degrees. To prove the second part, it follows from Theorem 2.1 that it is enough to prove that 1^n0^n and 0^n1^n belong to the same vertex-transitive class. Let ϕ be the function that switch a 0 to a 1 and vice versa. Then ϕ is the desired automorphism. □

A theorem of Watkins [10] says that a connected edge-transitive graph with minimum degree r is r-connected. Using this and Theorem 2.1, we have the following result:

Theorem 2.3. *HS(n,k) is maximally connected, that is, the connectivity of HS(n,k) is* $\min\{k, n-k\}$.

As we mentioned in Section 1, it is important to look at the severity of the disconnection when faults are present. It will be ideal for HS(n,k) to be tightly super-connected. We note that HS(2,1) is K_2 which is a complete graph and HS(3,1) (which is isomorphic to HS(3,2)) is $K_{1,2}$ which is tightly super-connected. So we assume $n \geq 4$. If k or $n-k$ is 1, then HS(n,k) is isomorphic to $K_{1,n-1}$; so clearly it is not tightly super-connected. It is also easy to see that HS(4,2) is a 6-cycle which is not tightly super-connected. The next result shows that these are the only exceptional cases.

Theorem 2.4. *Let $k, n-k \geq 2$. Then HS(n,k) is tightly super-connected if $n \geq 5$.*

Proof. We observe that HS(n,2) can be obtained by subdividing every edge in K_{n-1} once (that is, every edge is replaced by a path of length 2). It is easy to check that it is tightly super-connected for $n \geq 5$. Hence we may assume $k, n-k \geq 3$. In fact we may assume $n-k \geq k \geq 3$ by symmetry.

Let G^0 be the subgraph of HS(n,k) with 0 in the nth position and G^1 be the subgraph of HS(n,k) with 1 in the nth position. Then G^0 is isomorphic to HS($n-1,k$) and G^1 is isomorphic to HS($n-1,k-1$). (See, for example, Figure 1.) We note that only some vertices in G^0 are adjacent to vertices in G^1 and vice versa. In fact, only the vertices in G^0 with 1 in the first position have a neighbour in G^1, and only the vertices in G^1 with 0 in the first position have a neighbour in G^0; hence there are $\binom{n-2}{k-1}$ such edges and they are independent. Let T be a set of faults with $|T| = k$. Our first goal is to prove that either HS(n,k)$-T$ is connected or T is the neighbour-set of a vertex; moreover, there is only one non-singleton component if the graph is disconnected. Let $T_0 = T \cap V(G^0)$ and $T_1 = T \cap V(G^1)$. We consider three cases.

Case 1: $|T_0| < \min\{n-1-k, k\}$ and $|T_1| < \min\{n-k, k-1\}$. Then $G^0 - T_0$ and $G^1 - T_1$ are connected by Theorem 2.3. There are $\binom{n-2}{k-1}$ independent edges between G^0 and G^1 and it is easy to see that $\binom{n-2}{k-1} > k$ as $n-k \geq k \geq 3$. Hence $\mathrm{HS}(n,k) - T$ is connected.

Case 2: $|T_1| \geq \min\{n-k, k-1\}$. Since $k \leq n-k$, $\min\{n-k, k-1\} = k-1$; thus $|T_1|$ is either $k-1$ or k. Suppose $|T_1| = k$. Then $T_0 = \emptyset$, hence $G^0 - T_0 = G^0$ is connected. Let X be the component of $\mathrm{HS}(n,k) - T$ containing G^0, and let Y be a component of $G^1 - T_1$. If Y is not a singleton, then it has at least one edge, hence it has a vertex of the form $0a_2 \ldots a_{n-1}1$. Now this vertex has a unique neighbour in G^0, namely $1a_2 \ldots a_{n-1}0$, which is not in T. Hence Y is part of X. Suppose Y is a singleton with vertex y. If y has 0 in the first position, then y is adjacent to a vertex in $G^0 = G^0 - T$, hence Y is part of X. If y has 1 in the first position, then the degree of y in G^1 is $(n-1)-(k-1) = n-k \geq k$. But Y is a component, hence we have found all the vertices in T_1 and they are the neighbours of y. (Note that since the degree is $n-k \geq k$, this actually occurs only when $n = 2k$; in the other cases, Y is part of X.) Therefore the only possible components, other than X, in $\mathrm{HS}(n,k) - T$ are singleton components, and T is the set of neighbours of a vertex.

Now suppose $|T_1| = k-1$. We use a similar argument, but the analysis is slightly tighter. In this case, $|T_0| = 1$. Since G^0 is isomorphic to $\mathrm{HS}(n-1,k)$ and $n-k, k \geq 3$, G^0 has connectivity $\min\{n-1-k, k\} \geq 2$ by Theorem 2.3, hence $G^0 - T_0$ is connected. Let X be the component of $\mathrm{HS}(n,k) - T$ containing $G^0 - T_0$. If Y is a component other than X, then Y is a component of $G^1 - T_1$. We claim that Y has a vertex with 0 in its first position. If not, then Y must be a singleton with a vertex with 1 in its first position. Let this vertex be v. Since v is a singleton in $G^1 - T_1$, all its neighbours are in T_1. But v has $n-k \geq k$ neighbours and $|T_1| = k-1$, a contradiction. Hence we may assume $z = 0a_2 \ldots a_{n-1}1$ is a vertex in Y. Now this vertex has a neighbour in G^0, namely $w = 1a_2 \ldots a_{n-1}0$. If $w \notin T$, then Y is part of X, so suppose $w \in T$. Now z has $k-1$ neighbours in G^1, and each of them has a 1 in the first position. So each of its $n-k-1$ neighbours different from z has a neighbour in G^0 different from w as the graph has no 4-cycles. Since the graph has no 4-cycles, the tree of depth 2 rooted at z with $k-1$ vertices at level 1 and $(k-1)(n-k-1)$ vertices at level 2 are all distinct. Since $n-k \geq k \geq 3$, either z and hence X is part of Y, or all paths from z to X contain a vertex in T. This can only happen if all $k-1$ neighbours of z are in T_1, since $|T_1| = k-1$ and $n-k-1 \geq 2$. This completely identifies T_1, and together with w, we have T equal to the set of neighbours of z. So Y must be singleton component.

Case 3: $|T_0| \geq \min\{n-1-k, k\}$. By assumption $n-k \geq k$. We consider two subcases: $n-k > k$ and $n-k = k$.

Suppose $n-k>k$. Then $\min\{n-1-k,k\}=k$, hence $T_1=\emptyset$. Thus $G^1-T_1=G^1$ is connected. Let X be the component of $\mathrm{HS}(n,k)-T$ containing G^1, and let Y be a component of G^0-T_0. If Y is not a singleton, then it has at least one edge, hence it has a vertex of the form $1a_2\dots a_{n-1}0$. Now this vertex has a neighbour in G^1, namely $0a_2\dots a_{n-1}1$, which is not in T, hence Y is part of X. Suppose Y is a singleton with vertex y. If y has 1 in the first position, then y is adjacent to a vertex in $G^1=G^1-T$, hence Y is part of X. If y has 0 in the first position, then the degree of y in G^0 is k. But Y is a component, hence we have found all the vertices in T_0 and they are the neighbours of y. Therefore the only possible components, other than X, in $\mathrm{HS}(n,k)-T$ are singleton components, and T is the set of neighbours of a vertex.

We now consider the subcase when $n-k=k$. Then $\min\{n-1-k,k\}=n-k-1=k-1$, hence $|T_1|=1$. Since G^1 is isomorphic to $\mathrm{HS}(n-1,k-1)$ and $n-k,k\geq 3$, G^1 has connectivity $\min\{n-k,k-1\}\geq 2$ by Theorem 2.3, thus G^1-T_1 is connected. Let X be the component of $\mathrm{HS}(n,k)-T$ containing G^1-T_1. If Y be another component, then Y is a component of G^0-T_0. Since any vertex in G^0-T_0 with 0 in its first position has k neighbours in G^0, and $|T_0|=n-k-1=k-1$, Y is not a singleton with a vertex with 0 in its first position. Thus Y has a vertex with 1 in its first position. Hence we may assume $z=1a_2\dots a_{n-1}0$ is a vertex in Y. Now this vertex has a neighbour in G^1, namely $w=0a_2\dots a_{n-1}1$. If $w\notin T$, then Y is part of X, so suppose $w\in T$. Now z has $n-1-k$ neighbours in G^0 and each of them has a 0 in the first position, so each of its $k-1$ neighbours different from z has a neighbour in G^1 different from w as the graph has no 4-cycles. Since the graph has no 4-cycles, the tree of depth 2 rooted at z with $n-1-k$ vertices at level 1 and $(n-1-k)(k-1)$ vertices at level 2 are all distinct. Since $n-k\geq k\geq 3$, either z and hence X is part of Y, or all paths from z to X contain a vertex in T. This can only happen if all $n-k-1$ neighbours of z are in T_0, since $|T_0|=n-k-1=k-1\geq 2$. This completely identifies T_0, and together with w, we have T equal to the set of neighbours of z; hence Y is a singleton component.

Now if there are at least two singleton components, then the deleted vertices form the set of neighbours of two vertices. This is impossible, since $k,n-k\geq 3$, and $\mathrm{HS}(n,k)$ has no 4-cycles. □

3. Unidirectional hyper-stars

In this section, we orient $\mathrm{HS}(n,k)$ such that the difference of in-degree and out-degree of every vertex is at most 1 in the resulting directed graph, that is, the orientation is "balanced." Moreover, this orientation is determined by a *local orientation rule*, that is, the orientation of an edge is given by the labels of its two ends. Let v be a vertex in $\mathrm{HS}(n,k)$. Define

$\psi(v)$ to be the number of 0's, except the first 0 from the left, in odd positions. In other words, $\psi(v)$ is the number of 0's of v in odd positions, excluding the first 0 from the left if it is in an odd position. For example, $\psi(01100) = 1, \psi(11000) = 1$, and $\psi(10000) = 2$. Since HS(n,k) is a bipartite graph, it is enough to give the orientation rule for edges incident to vertices in V_0. Let $v \in V_0$, and let u be its neighbour through an i-edge from v. If $\psi(v)$ is even, then $v \longrightarrow u$[1] if i is even and $v \longleftarrow u$ if i is odd. If $\psi(v)$ is odd, then $v \longrightarrow u$ if i is odd and $v \longleftarrow u$ if i is even. In other words, $v \longrightarrow u$ if and only if $\psi(v)$ and i have the same parity. Denote the resulting directed graph by UHS(n,k) and call it the *unidirectional hyper-star*. Figure 2 shows UHS(6,3). We note that UHS(n,k) is "interconnected" by two directed graphs isomorphic to two unidirectional hyper-stars of smaller order or one with the orientation reversed. In fact, the subgraph G^0 (defined in the proof of Theorem 2.4) is directed to be isomorphic to UHS$(n-1,k)$ and the subgraph G^1 is directed to be isomorphic to UHS$(n-1,k-1)$ or one with the orientation reversed. Our next result shows that this orientation is "balanced" in terms of the in-degree and out-degree of a vertex. To aid the presentation of the paper, we let $f(v)$ be the position of the first 0 in v.

Theorem 3.1. *Let v be a vertex in UHS(n,k). Suppose $v \in V_0$ (V_1). If $\psi(v) + f(v)$ is even (odd), then the in-degree and out-degree of v in UHS(n,k) are $\lceil \frac{k}{2} \rceil$ ($\lceil \frac{n-k}{2} \rceil$) and $\lfloor \frac{k}{2} \rfloor$ ($\lfloor \frac{n-k}{2} \rfloor$), respectively, and vice versa if $\psi(v) + f(v)$ is odd (even).*

Proof. The statement concerning the in-degree and out-degree of a vertex in V_0 follows directly from the definition of the orientation of edges as $f(v) = 1$.

Let $v \in V_1$. The cases when $\psi(v)$ is even and $\psi(v)$ is odd are similar, so we will only give the argument for one of them. Assume $\psi(v)$ is even. It is enough to show that the i-edges from v are oriented to and away from v alternately, and the "first" edge is oriented toward v if and only if $f(v)$ is odd. Suppose $f(v)$ is odd. Then v is of the form $11^{2i+1}0\alpha$, where 1^{2i+1} denotes $2i+1$ 1's and α is some $\{0,1\}$-string. Then the first edge is between v and $u = 01^{2i+1}1\alpha$. This is a 1-edge from v, but a $(2i+2)$-edge from u. Since $\psi(v) = \psi(u)$, $\psi(u)$ is also even. So the orientation is $u \longrightarrow v$. Similarly, the orientation is $u \longleftarrow v$ if $f(v)$ is even.

The next step is to show that the orientation of the i-edges from v are oriented alternately to and away from v. Let v be of the form $1\alpha01^j0\beta$, where the 0 before 1^j is the ith 0 from the left, and α, β are some $\{0,1\}$-strings of appropriate sizes. Therefore, the neighbours from v through the i-edge from v and the $(i+1)$-edge from v are $u_1 = 0\alpha11^j0\beta$ and $u_2 = 0\alpha01^j1\beta$,

[1]This signifies that the edge is oriented from v to u.

respectively. Observe that u_1 and u_2 are almost the same, and that if v is the neighbour of u_1 via a p-edge from u_1 for some p, then v is the neighbour of u_2 via a $(p+j)$-edge from u_2. Now if j is even, then $\psi(u_1)$ and $\psi(u_2)$ are of different parity, so the i-edge and $(i+1)$-edge from b are oriented in opposite directions. If j is odd, then $\psi(u_1)$ and $\psi(u_2)$ are of the same parity, and again these two edges are oriented in opposite directions. □

The definition that we gave for the orientation rule is asymmetric but illuminating. An equivalent symmetric (though more cryptic) definition exists, and it can be easily extracted from the statement and proof of Theorem 3.1. This is given in the next result without proof.

Corollary 3.2. *Let v be a vertex in HS(n,k). Then UHS(n,k) is obtained by applying the following rule on the edge between v and u via an i-edge from v: $v \longrightarrow u$ if and only if $\psi(v)$ and $f(v)+i$ have opposite parity.*

The orientation will be useless if UHS(n,k) is not strongly connected. Since HS(n,k) is connected, the next result implies that UHS(n,k) is strongly connected.

Proposition 3.3. *Suppose $n-k, k \geq 2$. Then every arc in UHS(n,k) lies on either a directed 6-cycle or a directed 8-cycle.*

Proof. Consider the arc between two vertices u and v, where u is of the form $1\alpha 0\beta$, and v is of the form $0\alpha 1\beta$. Since $n-k \geq 2$, there is another 0 in u. We consider two cases.

Case 1: In u, the next 0 to the right of the given 0 is separated by a sequence of 1's or the next 0 to the left of the given 0 is separated by a sequence of 1's. We will assume the first scenario as the other is symmetrical. We consider two subcases.

The first subcase is when u is of the form $1\alpha 01^{2i}10\beta$. Then it follows from the orientation rule that the following vertices form a directed 6-cycle: $1\alpha 01^{2i}10\beta$, $0\alpha 11^{2i}10\beta$, $1\alpha 11^{2i}00\beta$, $0\alpha 11^{2i}01\beta$,
$1\alpha 01^{2i}01\beta$, $0\alpha 01^{2i}11\beta$.

The second subcase is when u is of the form $1\alpha 01^{2i}110\beta$. Then it follows from the orientation rule that the following vertices form a directed 8-cycle: $1\alpha 01^{2i}110\beta$, $0\alpha 11^{2i}110\beta$, $1\alpha 11^{2i}010\beta$,
$0\alpha 11^{2i}011\beta$, $1\alpha 11^{2i}001\beta$, $0\alpha 11^{2i}101\beta$, $1\alpha 01^{2i}101\beta$, $0\alpha 01^{2i}111\beta$.

Case 2: In u, there is a block of 0's to the left and to the right of the given 0. This is also straightforward and we omit the proof. □

The most relevant unidirectional hyper-star is perhaps the regular one, UHS$(4n,2n)$. In this case, one can apply the following easy result (see, for example, [3]): Let $H=(V,E)$ be a $(2k)$-regular $(2k)$-edge-connected graph, and let G be an orientation of H. If G is k-regular, then G is k-arc-connected. Summarizing, we have the following result:

Theorem 3.4. *Suppose $n-k, k \geq 2$. Then UHS(n, k) is strongly connected. Moreover, UHS(n, k) is maximally arc-connected if $n = 2k$ and k is even.*

4. Concluding Remarks

In this paper, we proved that HS(n, k) is tightly super-connected and showed that it can be oriented in a nice way. Although the regular uni-directional hyper-stars are the most relevant, and they are maximally arc-connected for our orientation, perhaps an interesting question is whether every UHS(n, k) is maximally arc-connected or even maximally (vertex)-connected.

References

[1] S.B. Akers, D. Harel, and B. Krishnamurthy. The star graph: An attractive alternative to the n-cube. *Proc. Int'l Conf. Parallel Processing*, pages 393–400, 1987.

[2] D. Bauer, F. Boesch, C. Suffel, and R. Tindell. Connectivity extremal problems and the design of reliable probabilistic networks. In *The theory and application of graphs*, pages 89–98. Wiley, New York, 1981.

[3] E. Cheng and M.J. Lipman. Increasing the connectivity of the star graphs. *Networks*, 40(3):165–169, 2002.

[4] S.C. Chern, J.S. Jwo, and T.C. Tuan. Uni-directional alternating group graphs. In *Lecture Notes in Comput. Sci. 959*, pages 490–495. Springer, Berlin, 1995.

[5] W.K. Chiang and R.J. Chen. The (n, k)-star graph: a generalized star graph. *Inform. Proc. Lett.*, 56:259–264, 1995.

[6] C.H. Chou and D.H.C. Du. Unidirectional hypercubes. *Proc. Supercomputing'90*, pages 254–263, 1990.

[7] K. Day and A. Tripathi. Arrangement graphs: a class of generalized star graphs. *Inform. Proc. Lett.*, 42:235–241, 1992.

[8] Jong-Seok Kim, Eunseuk Oh, Hyeong-Seok Lim, and Teong-Nam Heo. Topological and communication aspects of hyper-star graphs. In *Lecture Notes in Comput. Sci. 2869*, pages 51–58. Springer, Berlin, 2003.

[9] Hyeong-Ok Lee, Jong-Seok Kim, Eunseuk Oh, and Hyeong-Seok Lim. Hyper-star graph: a new interconnection network improving the network cost of the hypercube. In *Lecture Notes in Comput. Sci. 2510*, pages 858–865. Springer, Berlin, 2002.

[10] M.E. Watkins. Connectivity of transitive graphs. *J. Comb. Theory*, 8:23–29, 1970.

[11] D.B. West. *Introduction to Graph Theory*. Prentice Hall, 1996.

Eddie Cheng, Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309. Email: echeng@oakland.edu.

László Lipták, Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309. Email: liptak@oakland.edu.

Research partially supported by an Oakland University Faculty Research Fellowship.