ON THE POINT-DISTINGUISHING CHROMATIC INDEX OF COMPLETE BIPARTITE GRAPHS

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ABSTRACT. The point-distinguishing chromatic index of a graph G, in symbols $\chi_0(G)$, is the smallest number of colours in a (not necessarily proper) edge colouring of G such that any two distinct vertices of G are distinguished by sets of colours of their adjacent edges. The exact value of $\chi_0(K_{m,n})$ is found if either $m \le 10$ or $n \ge 8m^2 - 2m + 1$.

1. INTRODUCTION

Let G be a finite simple graph with no component K_2 and at most one component K_1. Consider a general edge colouring of G (colour classes are not necessarily independent) using k colours as a mapping $\varphi : E(G) \to \{1, \cdots, k\}$. The *colour set* of a vertex $x \in V(G)$ is the set of colours of edges incident with x. The colouring φ is *point-distinguishing* if vertices of G differ by their colour sets. The *point-distinguishing chromatic index* of G is the smallest k such that there is a point-distinguishing colouring of edges of G that uses k colours and is denoted $\chi_0(G)$. This graph invariant has been introduced by Harary and Plantholt [1]. Among other things, the exact values of $\chi_0(K_n)$, $\chi_0(P_n)$, $\chi_0(C_n)$ and $\chi_0(Q_n)$ are computed in [1]. As concerns complete bipartite graphs, a natural general bound $\chi_0(G) \ge \lceil \log_2 |V(G)| \rceil$ of [1] reads as $\chi_0(K_{m,n}) \ge \lceil \log_2(m+n) \rceil$. So, with $m \le n$ we have

$$\chi_0(K_{m,n}) \ge \lceil \log_2(2m) \rceil = \lceil \log_2 m \rceil + 1. \qquad (1)$$

The mentioned paper provides also an upper bound of $\chi_0(K_{m,n})$ if $\lceil \log_2 n \rceil + 1 \le m \le n$, namely $\chi_0(K_{m,n}) \le \lceil \log_2 n \rceil + 2$. A greater effort has been concentrated on the determination of $\chi_0(K_{m,m})$. Zagaglia Salvi [5] proved that the sequence $\{\chi_0(K_{m,m})\}_{m=2}^{\infty}$ is non-decreasing with jumps of value 1. As a consequence, there is an increasing sequence $\{n_k\}_{k=2}^{\infty}$ of integers in which $n_2 = 1$ and, for any $k \ge 3$, $\chi_0(K_{m,m}) = k$ if and only if $n_{k-1} + 1 \le m \le n_k$; it is easy to see that $n_k < 2^{k-1}$. Only few exact values of n_k are known, namely $n_3 = 2, n_4 = 5, n_5 = 11, n_6 = 22$ (see Zagaglia Salvi

[4]) and $n_7 = 46$ (Horňák and Soták [2]). According to [5], $n_{k+1} \geq 2n_k$, and so the sequence $\{\frac{n_k}{2^{k-1}}\}_{k=3}^{\infty}$ (as nondecreasing and upper bounded by 1) is convergent and $0.71875 = \frac{46}{64} \leq \lim_{k\to\infty} \frac{n_k}{2^{k-1}} \leq 1$. Horňák and Soták [3] succeeded in showing that $\lim_{k\to\infty} \frac{n_k}{2^{k-1}} \geq 3 - \sqrt{5} = 0.763932\ldots$. The question of whether the mentioned limit is strictly smaller than 1 is still open. The aim of the present paper is to determine $\chi_0(K_{m,n})$ for all n sufficiently large with respect to m. Since, trivially, $\chi_0(K_{1,n}) = n$ for any $n \geq 2$, we are interested in $\chi_0(K_{m,n})$ with $m \geq 2$. Let $p, q \in \mathbb{Z}$. By $[p, q]$ we denote the interval of all integers z with $p \leq z \leq q$, and by $[p, \infty)$ the interval of all integers z with $z \geq p$. For $k \in [1, \infty)$ let $(p)_k$ be the representative of the residue class p modulo k in $[1, k]$. By $[p, q]_k$ we denote the set of all integers $(i)_k$ with $i \in [p, q]$. Consider an $m \times n$ matrix A. For $i \in [1, m]$ and $j \in [1, n]$, let $R_i(A)$ and $C_j(A)$ be the sets of elements in the ith row and in the jth column of A, respectively. Further, put $\mathcal{R}(A) := \bigcup_{i=1}^{m}\{R_i(A)\}$ and $\mathcal{C}(A) := \bigcup_{j=1}^{n}\{C_j(A)\}$; evidently, any set of $\mathcal{R}(A)$ is non-disjoint with any set of $\mathcal{C}(A)$ (the element in the ith row and the jth column is in $R_i(A) \cap C_j(A)$). Let $M_{m,n}^{(k)}$ be the set of all $m \times n$ matrices with entries from $[1, k]$ such that $|\mathcal{R}(A) \cup \mathcal{C}(A)| = m + n$ (which means that $|\mathcal{R}(A)| = m$, $|\mathcal{C}(A)| = n$ and $\mathcal{R}(A) \cap \mathcal{C}(A) = \emptyset$). If there is a point-distinguishing colouring $\varphi : E(K_{m,n}) \to [1, k]$, then, by (1), $k \geq \lceil \log_2 m \rceil + 1$. Suppose that the bipartition of $V(K_{m,n})$ is $\{\{x_i : i \in [1, m]\}, \{y_j : j \in [1, n]\}\}$, and let $F(\varphi)$ be the $m \times n$ matrix whose ith row and jth column element is $\varphi(x_i y_j)$. Clearly, the matrix $F(\varphi)$ belongs to $M_{m,n}^{(k)}$ and (as any $m \times n$ matrix) satisfies

$$\forall i \in [1, m]\ \forall j \in [1, n]\ R_i(F(\varphi)) \cap C_j(F(\varphi)) \neq \emptyset. \tag{2}$$

On the other hand, if $M_{m,n}^{(k)} \neq \emptyset$, then $\chi_0(K_{m,n}) \leq k$. Indeed, if $F = (f_{i,j}) \in M_{m,n}^{(k)}$, then the colouring $\varphi : E(K_{m,n}) \to [1, k]$, determined by $\varphi(x_i y_j) := f_{i,j}$, is point-distinguishing. Throughout the whole article we suppose (if not explicitly stated otherwise) that k, l, m, n are integers with $2 \leq m \leq n$, $l = \lceil \log_2 m \rceil$ and $k \geq \min(l + 1, 3)$ (note that with two colours we can distinguish only three vertices of a connected graph); the equality $l = \lceil \log_2 m \rceil$ is equivalent to

$$2^{l-1} < m \leq 2^l.$$

As usual, for a set X we denote by 2^X the system of all subsets of X.

2. Matrix construction

In this section we provide an upper bound for $\chi_0(K_{m,n})$. For that purpose define, for positive integers k, m,

$$d_m^{(k)} := \begin{cases} 2m, & \text{if } k \le m, \\ 2m-1, & \text{if } k = m+1, \\ m, & \text{if } k \ge m+2, \end{cases}$$

$$b_m^{(k)} := \sum_{i=0}^{m} \binom{k}{i} - d_m^{(k)},$$
$$I_m^{(k)} := [b_m^{(k-1)} + 1, b_m^{(k)}].$$

The summation term in the definition of $b_m^{(k)}$ gives the number of subsets of k items with up to m elements; this then gives the maximum possible number of distinguishing sets that can be assigned to vertices of the partite set with n elements. The adjustment made by subtracting $d_m^{(k)}$ will become clear in the proof.

Theorem 1. *If $n \in I_m^{(k)}$, then $\chi_0(K_{m,n}) \le k$.*

Proof. (a) Assume first that $n \ge k \ge 4$ and let $r := \lceil \frac{k}{2} \rceil$, $s := \min(m, r)$, $A_1 := [1,k]$, $A_i := [1,k] - \{i-1\}$, $i = 2, \ldots, s$, $B_j := [j, j+s-1]_k$, $j = 1, \ldots, k-1$, $B_k := [2,s] \cup \{k\}$ and $\mathcal{B}' := \{B_j : j \in [1,k]\}$. All 2^k subsets of $[1,k]$ can be organized into 2^{k-1} pairs of complementary sets $(A_i, [1,k] - A_i)$, $i = 1, \ldots, 2^{k-1}$, such that for any $i, j \in [1, 2^{k-1}]$ we have

$$|A_i| \ge |[1,k] - A_i|, \tag{3}$$
$$i < j \Rightarrow |A_i| \ge |A_j|, \tag{4}$$
$$i \le m \Rightarrow \{A_i, [1,k] - A_i\} \cap \mathcal{B}' = \emptyset \tag{5}$$

(note that $A_1, \ldots, A_s$ satisfy (3)–(5)). In fact, the only problem could be with (5). If $m \le k-1$, then, by (3) and (4), $|A_i| \ge k-1$ and $|[1,k] - A_i| \le 1$ for any $i \in [1,m]$, and, since $\mathcal{B}'$ consists of s-element sets with $2 \le s \le r = \lceil \frac{k}{2} \rceil \le k-2$, (5) is trivially true. On the other hand, if $m \ge k$, to fulfill (5) we only have to be sure that $m + k \le 2^{k-1}$ if k is odd and that $m + \frac{k}{2} + 1 \le 2^{k-1}$ if k is even (if $k = 2q$, $\mathcal{B}'$ contains $q-1$ complementary pairs $([j, j+q-1]_{2q}, [j+q, j+2q-1]_{2q})$, $j = 1, \ldots, q-1$). However, for an odd k the assumption $m \ge 2^{k-1} - k + 1$ yields $2^k \ge 2m + n \ge 3m \ge 3(2^{k-1} - k + 1)$ and $3(k-1) \ge 2^{k-1}$ in contradiction with $k \ge 5$. Analogously, if k is even and $m \ge 2^{k-1} - \frac{k}{2}$, then $2^k \ge 3(2^{k-1} - \frac{k}{2})$ and $3k \ge 2^k$, which contradicts $k \ge 4$. We now put $\mathcal{A} := \{A_i : i \in [1,m]\}$ and show that $\mathcal{B}'$ can be extended by subsets of $[1,k]$ of cardinality $\leq m$ to $\mathcal{B} = \{B_j : j \in [1,n]\}$ satisfying $\mathcal{A} \cap \mathcal{B} = \emptyset$ and

$$\forall i \in [1,m]\ \forall j \in [1,n]\ A_i \cap B_j \neq \emptyset \tag{6}$$

(note that from $k \geq 4$ it follows that $\mathcal{B}'$ consists of k distinct sets). Namely, $\mathcal{B}$ can contain any of sets A_i, $[1,k] - A_i$, $i \in [m+1, 2^{k-1}]$ (if it is of cardinality $\leq m$). Indeed, if $t := |A_m|$, then, by (4), $|A_i| \geq t$ for any $i \in [1,m]$, and, by (3), $|B_j| \geq \min(t, k-t) = k-t$. Thus, we have either $|A_i| + |B_j| > t+k-t = k$ and $A_i \cap B_j \neq \emptyset$ by the Pigeonhole Principle (PP) or $|A_i| = t$, $|B_j| = k-t$ and $A_i \cap B_j \neq \emptyset$ because of the fact that $[1,k] - A_i$ does not belong to $\mathcal{B}$. The number of subsets of $[1,k]$ of cardinality $\leq m$ that cannot be in $\mathcal{B}$ is at most $2m$ (it corresponds to A_i and $[1,k] - A_i$ for $i \in [1,m]$). However, if $k = m+1$, then $|A_1| = k > m$ (so that the bound $2m$ has to be decreased to $2m-1$), and, if $k \geq m+2$, $|A_i| \geq |A_m| = k-1 > m$ for any $i \in [1,m]$ (decrease $2m$ to m). Therefore, the above extension of $\mathcal{B}'$ to $\mathcal{B}$ can be performed whenever $n \leq \sum_{i=0}^{m} \binom{k}{i} - d_m^{(k)} = b_m^{(k)}$. We prove our Theorem by constructing an $m \times n$ matrix $F = (f_{i,j})$ such that $\mathcal{R}(F) = \mathcal{A}$ and $\mathcal{C}(F) = \mathcal{B}$, which means that $F \in M_{m,n}^{(k)}$ and $\chi_0(K_{m,n}) \leq k$. We define $f_{i,j} := (i+j-1)_k$ for $i \in [1,s]$ and $j \in [1,k-1]$, $f_{1,k} := k$, and $f_{i,k} := i$ for $i \in [2,s]$. Let $F' = (f_{i,j})$ be the so far constructed $s \times k$ matrix. Then $R_1(F') = [1,k] = A_1$, $R_i(F') = \{(i+j-1)_k : j \in [1,k-1]\} \cup \{i\} = [i, i+k-1]_k \cup \{i\} = [1,k] - \{i-1\} = A_i$ for $i \in [2,s]$, $C_j(F') = \{(i+j-1)_k : i \in [1,s]\} = [j, j+s-1]_k = B_j$ for $j \in [1,k-1]$ and $C_k(F') = \{k\} \cup [2,s] = B_k$. Now suppose that $m > s$ and consider $i \in [s+1, m]$. If $j \in A_i$, define $f_{i,j} := j$, while for $j \in [1,k] - A_i$ pick $f_{i,j} \in A_i \cap B_j$ (which is possible due to (6)). For the so far defined $m \times k$ matrix $F'' = (f_{i,j})$ we have $R_i(F'') = A_i$, $i = 1, \ldots, m$, and $C_j(F'') = B_j$, $j = 1, \ldots, k$ (note that $j \in B_j$ for every $j \in [1,k]$). Finally, if $n > k$, we extend F'' to F as follows: consider $j \in [k+1, n]$, set $b_j := |B_j|$ (recall that $b_j \leq m$), and suppose that B_j consists of elements $B_j(i)$, $i \in [1, b_j]$, satisfying

$$\forall i_1, i_2 \in [1, b_j]\ (i_1 < i_2 \Rightarrow B_j(i_1) < B_j(i_2)).$$

We put $f_{i,j} := B_j(i)$ for each $i \in [1, b_j]$, and we choose $f_{i,j}$ arbitrarily from the set $A_i \cap B_j \neq \emptyset$ if $i \in [b_j+1, m]$. Then we have $B_j(1) \in A_1 = [1,k]$ and, for each $i \in [2, b_j]$, $B_j(i) \geq i > i-1$, so that $B_j(i) \in [1,k] - \{i-1\} = A_i$. Therefore it is clear that $\mathcal{R}(F) = \mathcal{A}$ and $\mathcal{C}(F) = \mathcal{B}$ as required. (b) Suppose that $n \leq k-1$. If $m \leq k-2$, then $k-1 \geq n \geq b_m^{(k-1)} + 1 \geq 1 + k - 1 + \binom{k-1}{2} - m + 1$ and $k-4 \geq m-2 \geq \binom{k-1}{2}$ which is impossible for any $k \geq 3$. Therefore, $2 \leq m = n = k-1$, $k-1 \geq 1 + k - 1 + \binom{k-1}{2} - (2k-3) + 1$ and $2k-1 \geq \binom{k-1}{2}$, hence $k \leq 4$ and $(k,m,n) \in \{(3,2,2), (4,3,3)\}$. In this case $\chi_0(K_{2,2}) = \chi_0(C_4) = 3$ (see [1]) and $\chi_0(K_{3,3}) = 4$ (see [4]). (c)

If $k=3$, then the only m, satisfying $I_m^{(3)} \cap [m,\infty) \neq \emptyset$, is $m=2$ (we have $I_3^{(3)} = [-1,2]$). .To see that $\chi_0(K_{2,n}) \leq 3$ for $n \in I_2^{(3)} \cap [3,\infty) = [3,4]$ consider matrices $\begin{pmatrix} 2 & 3 & 1 \\ 2 & 3 & 2 \end{pmatrix} \in M_{2,3}^{(3)}$ and $\begin{pmatrix} 2 & 3 & 1 & 1 \\ 2 & 3 & 2 & 3 \end{pmatrix} \in M_{2,4}^{(3)}$. □

3. Definitive results

Returning to the proof of Theorem 1 we see that the constructed matrix F in the "ordinary" case (a) has non-disjoint rows. Indeed, for any $i,j \in [1,m]$, $i \neq j$, we have $\min(|A_i|,|A_j|) \geq \lceil \frac{k}{2} \rceil$ (by (3)), and so either $|A_i| + |A_j| > k$ (and $A_i \cap A_j \neq \emptyset$ by PP) or $|A_i| = |A_j| = \frac{k}{2}$ (and then $A_i \cap A_j \neq \emptyset$ since $A_j \neq [1,k] - A_i$). We shall see in Lemma 8 that a point-distinguishing colouring $\varphi : E(K_{m,n}) \to [1,k]$ such that $F(\varphi)$ has non-disjoint rows cannot do better than it does in Theorem 1. For positive ⊔ integers k,m with $m \leq 2^k$ let $\mathcal{S}_m^{(k)}$ be the system of all systems of m distinct subsets of the set $[1,k]$, and for $\mathcal{D} \in \mathcal{S}_m^{(k)}$ put $\hat{\mathcal{D}} := \{B : (\exists D \in \mathcal{D})\, B \subseteq [1,k] - D\}$.

Proposition 7. *If k,m are positive integers with $m \leq 2^k$ and $\mathcal{D} \in \mathcal{S}_m^{(k)}$, then $|\hat{\mathcal{D}}| \geq m$.*

Proof. For any $\mathcal{D} \in \mathcal{S}_m^{(k)}$ set $\bar{\mathcal{D}} := \{[1,k] - D : D \in \mathcal{D}\}$. Clearly, $\hat{\mathcal{D}} \supseteq \bar{\mathcal{D}}$, and we have $|\hat{\mathcal{D}}| \geq |\bar{\mathcal{D}}| = |\mathcal{D}| = m$. □

Lemma 8. *If a point-distinguishing colouring $\varphi : E(K_{m,n}) \to [1,k]$ is such that the sets in $\mathcal{R}(F(\varphi))$ are pairwise non-disjoint, then $n \leq b_m^{(k)}$.*

Proof. Denote for short $\mathcal{R} = \mathcal{R}(F(\varphi))$ and $\mathcal{C} = \mathcal{C}(F(\varphi))$. (This convention will be applied in the sequel whenever working with a point-distinguishing colouring φ.) Then, by the assumption of our Lemma, $\hat{\mathcal{R}} \cap \mathcal{R} = \emptyset$, and, with respect to (2), $\hat{\mathcal{R}} \cap \mathcal{C} = \emptyset$. The colouring φ is point-distinguishing, and so $\mathcal{R} \cap \mathcal{C} = \emptyset$. By Proposition 7, $|\hat{\mathcal{R}}| \geq m$ (note that $m \leq 2^l \leq 2^{k-1}$). (a) Assume that $k \leq m+1$. Then $n + 2m = |\mathcal{C}| + |\mathcal{R}| + m \leq |\mathcal{C}| + |\mathcal{R}| + |\hat{\mathcal{R}}| = |\mathcal{C} \cup \mathcal{R} \cup \hat{\mathcal{R}}| \leq |2^{[1,k]}| = 2^k$, hence $n + 2m \leq \sum_{i=0}^{m} \binom{k}{i}$ if $k \leq m$ and $n + 2m \leq \sum_{i=0}^{m} \binom{k}{i} + 1$ if $k = m+1$, which results in $n \leq \sum_{i=0}^{m} \binom{k}{i} - d_m^{(k)} = b_m^{(k)}$ in both cases. (b) Let $k \geq m+2$. Consider a set $R \in \mathcal{R}$ of minimum cardinality r. Any subset of $[1,k] - R$ belongs to $\hat{\mathcal{R}}$, hence it is out of $\mathcal{C}$. The system $\mathcal{C}$ contains exclusively subsets of $[1,k]$ of cardinality $\leq m$, therefore $n = |\mathcal{C}| \leq \sum_{i=0}^{m} \binom{k}{i} - \sum_{i=0}^{m} \binom{k-r}{i}$. If $r \leq k-m$, then $k-r \geq m$ and $n \leq \sum_{i=0}^{m} \binom{k}{i} - \sum_{i=0}^{m} \binom{m}{i} \leq \sum_{i=0}^{m} \binom{k}{i} - m - 2 = b_m^{(k)} - 2$. On the other hand, $r \geq k-m+1$ means that, for any $\hat{R} \in \hat{\mathcal{R}}$, $|\hat{R}| \leq |[1,k] - R| = k - r \leq m-1$. Thus $\mathcal{C}$ and $\hat{\mathcal{R}}$ are disjoint subsystems of the system of all subsets of $[1,k]$ with cardinality $\leq m$, so that $n + d_m^{(k)} = |\mathcal{C}| + m \leq |\mathcal{C}| + |\hat{\mathcal{R}}| = |\mathcal{C} \cup \hat{\mathcal{R}}| \leq \sum_{i=0}^{m} \binom{k}{i}$ and $n \leq b_m^{(k)}$. □

Theorem 9. *The inequality $\chi_0(K_{m,n}) \geq k+1$ follows from either of the assumptions*

1. $k \leq m+1$ *and* $n \geq 2^k - m - l$;
2. $k \geq m+2$ *and* $n \geq b_m^{(k)} + 1$.

Proof. Suppose that one of the mentioned conditions is fulfilled and there is a point-distinguishing colouring $\varphi : E(K_{m,n}) \to [1,k]$. 1. Let $\mathcal{S}$ be the system of 1-element sets in $\mathcal{R} \cup \mathcal{C}$, $S := \{c \in [1,k] : \{c\} \in \mathcal{S}\}$, and $s := |S|(= |\mathcal{S}|)$. Then $2^k - l \leq m + n = |\mathcal{R}| + |\mathcal{C}| \leq s + \sum_{i=2}^{m} \binom{k}{i}$, hence $s \geq \sum_{i=0}^{k} \binom{k}{i} - \sum_{i=2}^{m} \binom{k}{i} - l \geq 1 + k - l \geq 2$. Clearly, since φ is point-distinguishing, (2) implies that all sets of $\mathcal{S}$ are either in $\mathcal{R}$ or in $\mathcal{C}$. If they are in $\mathcal{R}$, then, by (2), sets in $\mathcal{C}$ are distinguished only by $k-s$ colours of the set $[1,k]-S$, and so $2^k - m - l \leq n = |\mathcal{C}| \leq 2^{k-s} \leq 2^{k-2}$. Since $m \leq n$, we have $2 \cdot 2^{k-2} \geq m+n \geq 2^k - l$ and $l \geq 2^{k-1} \geq 2^l$, a contradiction Now suppose that all sets of $\mathcal{S}$ are in $\mathcal{C}$. Sets in $\mathcal{R}$ are distinguished only by $k-s$ colours of the set $[1,k]-S$, hence $2^{k-s} \geq m > 2^{l-1}$, $k-s > l-1$, $k-s \geq l$ and $s \leq k-l$ in contradiction with the above inequality $s \geq 1+k-l$. 2. The result comes from Lemma 8 because of the circumstance that its proof did not use the fact that sets in $\mathcal{R}(F(\varphi))$ are pairwise disjoint. □

Lemma 10. *If $n \geq b_m^{(k)} + 1$ and $\chi_0(K_{m,n}) \leq k$, then $k \leq 2l$.*

Proof. Because of the inequality $n \geq b_m^{(k)} + 1$, from Lemma 8 it follows that for any point-distinguishing colouring $\varphi : E(K_{m,n}) \to [1,k]$ there are $R', R'' \in \mathcal{R}$ with $R' \cap R'' = \emptyset$. We can choose those sets in such a way that $r' := |R'| \leq |R''| =: r''$. By (2), $C \cap R' \neq \emptyset$ for any $C \in \mathcal{C}$, hence $\mathcal{C} \cap 2^{[1,k]-R'} = \emptyset$ and

$$n = |\mathcal{C}| \leq 2^k - 2^{k-r'}. \tag{7}$$

Since $n \geq \sum_{i=0}^{m} \binom{k}{i} - d_m^{(k)} + 1 = 2^k - 2m + 1$ for any $k \leq m+1$, (7) yields

$$2m \geq 2^k + 1 - (2^k - 2^{k-r'}) > 2^{k-r'}. \tag{8}$$

With respect to $m \leq 2^l$ then (8) leads to $2^{l+1} > 2^{k-r'}$, $l+1 > k-r'$ and $k \leq l + r' \leq l + k/2$ (which follows from $r' \leq r''$ and $r' + r'' \leq k$). As a consequence, $k \leq 2l$. □

Theorem 11. *The equivalence $\chi_0(K_{m,n}) = k \Leftrightarrow n \in I_m^{(k)}$ follows from any of the assumptions*

1. $k \geq 2l+2$;
2. $m \in \{2,3\}$.

Proof. 1. Suppose first that $n \in I_m^{(k)}$. By Theorem 1, $\chi_0(K_{m,n}) \leq k$. If $k \geq m+3$, then $k-1 \geq m+2$, hence the inequality $n \geq b_m^{(k-1)} + 1$ together with Theorem 9.2 imply $\chi_0(K_{m,n}) \geq k$. If $k \leq m+2$, then $2l+1 \leq k-1$, and so $n \geq b_m^{(k-1)} + 1$ with respect to Lemma 10 means that $\chi_0(K_{m,n}) \leq k-1$ is impossible. Thus, for $n \in I_m^{(k)}$, $\chi_0(K_{m,n}) = k$. ¿From $m > 2^{l-1}$ we have $m > l$ and $b_m^{(l)} = \sum_{i=0}^{m} \binom{l}{i} - 2m + 1 = 2^l - 2m + 1 < 1$, hence the integer interval $[m, \infty)$ is covered by the system $\{I_m^{(k)} : k \in [l+1, \infty)\}$ of pairwise disjoint integer intervals. Therefore, if $n \in [m, \infty) - I_m^{(k)}$, there is a unique $k' \in [l+1, \infty) - \{k\}$ such that $n \in I_m^{(k')}$. If $k' \geq 2l+2$, by the first part of the proof, $\chi_0(K_{m,n}) = k' \neq k$. On the other hand, if $k' \leq 2l+1$, then, by Theorem 1, $\chi_0(K_{m,n}) \leq k' \leq 2l+1 < k$. 2. If $m \in \{2, 3\}$, we have $I_m^{(l+1)} \cap [m, \infty) = \emptyset$. Thus, by Theorem 11.1 it is sufficient to treat the cases $k \in [l+2, 2l+1]$ so that (k, m) can only be one of the pairs $(3, 2), (4, 3)$ and $(5, 3)$. Theorem 9.1 then yields $\chi_0(K_{m,n}) \geq k$ for any $n \in I_m^{(k)}$ (satisfying $n \geq m$), hence $\chi_0(K_{m,n}) = k$ for $n \in I_m^{(k)}$ and we are done. □

For some values of m and n Theorem 11 yields an explicit formula for $\chi_0(K_{m,n})$.

Corollary 12. *If $m \in [7, \infty)$ and $n \in [2 \cdot 4^l - 2m + 1, 2^{m+1} - 2m]$, then $\chi_0(K_{m,n}) = \lceil \log_2(n + 2m) \rceil$. In particular, $\chi_0(K_{m,n}) = \lceil \log_2(n + 2m) \rceil$ if $n \in [8m^2 - 2m + 1, 2^{m+1} - 2m]$.*

Proof. We have $2l + 1 \leq m$: for $m \in [7, 9]$ the inequality can be directly checked, and for $m \in [10, \infty)$ it suffices to show that $m - 2\log_2 m - 3 \geq 0$. Therefore, $2 \cdot 4^l - 2m + 1 = 2^{2l+1} - 2m + 1 = \sum_{i=0}^{m} \binom{2l+1}{i} - d_m^{(2l+1)} + 1 = b_m^{(2l+1)} + 1$, and, as $2^{m+1} - 2m = \sum_{i=0}^{m} \binom{m+1}{i} - (2m - 1) = b_m^{(m+1)}$, n belongs to exactly one of the intervals $I_m^{(k)}$ with $k \in [2l+2, m+1]$. Since $k - 1 \leq m$, we have $n \geq \sum_{i=0}^{m} \binom{k-1}{i} - d(k-1, m) + 1 = 2^{k-1} - 2m + 1$. On the other hand, $\sum_{i=0}^{k} \binom{k}{i} - d_m^{(k)} = 2^k - 2m$ for any $k \in [2l+2, m+1]$, and so $n \leq 2^k - 2m$. The pair of inequalities $2^{k-1} - 2m + 1 \leq n \leq 2^k - 2m$ is equivalent to $2^{k-1} + 1 \leq n + 2m \leq 2^k$, and also, by Theorem 11, with $\lceil \log_2(n + 2m) \rceil = k = \chi_0(K_{m,n})$. The second claim of the Corollary follows from the inequality $8m^2 > 2 \cdot 4^l$ (coming from $m > 2^{l-1}$). □

From Theorem 11 and Corollary 12 it is clear that the value of $\chi_0(K_{m,n})$ is exactly determined for any $m \geq 7$ and $n \geq 2 \cdot 4^l - 2m + 1$ (and so for $n \geq 8m^2 - 2m + 1$), though for $n \geq 2^{m+1} - 2m + 1$ we have no explicit formula.

Proposition 13. *If $m \geq 4$ and $k \in [l+2, 2l+1]$, then $b_m^{(k-1)} + 1 \leq 2^{k-1} - m - l - 1 < b_m^{(k)}$.*

Proof. From $k \leq 2l + 1$ we obtain $k - 1 \leq 2l \leq m + 1$: the last inequality is easy to check for $m \in [4, 6]$ and for $m \geq 7$ we can use the inequality

$2l+1 \leq m$ from the proof of Corollary 12. As $k-1 \leq m+1$, we have $b_m^{(k-1)} = \sum_{i=0}^{m} \binom{k-1}{i} - d_m^{(k-1)} = 2^{k-1} - 2m$, and so $b_m^{(k-1)} + 1 = 2^{k-1} - 2m + 1 \leq 2^{k-1} - m - l - 1$, since the last inequality is equivalent to $l+2 \leq m$. On the other hand, if $k \leq m+1$, then $b_m^{(k)} = \sum_{i=0}^{m} \binom{k}{i} - d_m^{(k)} = 2^k - 2m > 2^{k-1} - m - l$, which follows from $2^{k-1} + l > 2^{k-1} \geq 2^l \geq m$. If $k = m+2$, then $b_m^{(k)} = \sum_{i=0}^{m} \binom{m+2}{i} - d_m^{(m+2)} = 2^{m+2} - (m+2) - 1 - m = 2^{m+2} - 2m - 3 > 2^{m+1} - m - l$, as a consequence of $2^{m+1} + l > m+3$. □

By Proposition 13, with $m \geq 4$ and $k \leq 2l+1$, the interval $I_m^{(k)}$ decomposes into two subintervals, the left one, $L_m^{(k)} := [b_m^{(k-1)}+1, 2^{k-1} - m - l - 1]$, and the right one, $R_m^{(k)} := [2^{k-1} - m - l, b_m^{(k)}]$.

Theorem 14. *The equality $\chi_0(K_{m,n}) = k$ follows from either of the assumptions*

1. $k \in [l+2, 2l+1]$ *and* $n \in R_m^{(k)}$;
2. $k = l+1$ *and* $n \in I_m^{(k)}$.

Proof. 1. Since $R_m^{(k)} \subseteq I_m^{(k)}$, from Theorem 1 it follows that $\chi_0(K_{m,n}) \leq k$. As in the proof of Proposition 13, $k-1 \leq m+1$. Therefore, the inequality $n \geq 2^{k-1} - m - l$ with respect to Theorem 9.1 implies $\chi_0(K_{m,n}) \geq k$. 2. Use (1) and Theorem 1. □

Proposition 15. *If $k \in [l+2, 2l+1]$ and $n \in L_m^{(k)}$, then $\chi_0(K_{m,n}) \in \{k-1, k\}$.*

Proof. Again, from $L_m^{(k)} \subseteq I_m^{(k)}$ we obtain $\chi_0(K_{m,n}) \leq k$. If $k \geq l+3$, then $n \geq b_m^{(k-1)} + 1 = \sum_{i=0}^{m} \binom{k-1}{i} - d_m^{(k-1)} + 1 = 2^{k-1} - 2m + 1 \geq 2^{k-2} - m - l$, because the last inequality is equivalent to $2^{k-2}+l+1 \geq m$ and $2^{k-2}+l+1 \geq 2^{l+1}+l+1 \geq 2m+l+1 > m$. Since $k-2 \leq 2l-1$, we have also $k-2 \leq m+1$. Thus, by Theorem 9.1, $\chi_0(K_{m,n}) \geq k-1$. If $k = l+2$, then, by (1), $\chi_0(K_{m,n}) \geq l+1 = k-1$. □

From Theorems 11 and 14 and from Proposition 15 we see that if $n \in I_m^{(k)}$, then $\chi_0(K_{m,n})$ is either $k-1$ or k. All definitive results we have proved so far are such that $\chi_0(K_{m,n}) = k$. There are, however, also pairs (m,n) for which the former possibility applies.

Theorem 16. *If $m \in [4, 10]$ and $n \in L_m^{(k)}$, then $\chi_0(K_{m,n}) = k-1$ if and only if $(k,m,n) = (6, 10, 13)$.*

Proof. Consider systems $\mathcal{A} = \{A_i : i \in [1,10]\}$ and $\mathcal{B} = \{B_j : j \in [1,13]\}$ of subsets of the set $[1,5]$, in which $A_1 := [1,5]$, $A_i := [1,5] - \{i-1\}$ for $i \in [2,6]$, $A_7, \cdots, A_{10}$ are successively $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{2,5\}$, $B_j := [j, j+2]_5$ for $j \in [1,4]$, and $B_5, \cdots, B_{13}$ are successively $\{2,3,5\}$,

$\{1,3,5\}$, $\{2,4,5\}$, $\{1,2\}$, $\{1,5\}$, $\{2,3\}$, $\{2,4\}$, $\{3,5\}$, $\{4,5\}$. We can apply to $\mathcal{A}$ and $\mathcal{B}$ the construction of Theorem 1 with $k=5$ (though $13 \notin I_{10}^{(5)}$), since the sets A_i, $i \in [1,3]$, and B_j, $j \in [1,5]$, are defined in the same way as there and $A_i \cap B_j \neq \emptyset$ holds for any $i \in [1,10]$ and $j \in [1,13]$. The constructed matrix F belongs to $M_{10,13}^{(5)}$, hence $\chi_0(K_{10,13}) \leq 5$, and, by Proposition 15 (as $13 \in L_{10}^{(6)}$ and $\chi_0(K_{10,13}) \in \{5,6\}$), $\chi_0(K_{10,13}) = 5$. For every $m \in [4,10]$ there are only few values of n for which $\chi_0(K_{m,n})$ could be equal (by Proposition 15) to $k-1$. More precisely, $\chi_0(K_{m,n})$ is undetermined (by the statements preceding Theorem 16) only if $n \in U_m := [m,\infty) \cap \bigcup_{k=l+1}^{2l+1} L_m^{(k)}$. For example, $U_{10} = \bigcup_{k=6}^{9} L_{10}^{(k)}$, where $L_{10}^{(6)} = [13,17]$, $L_{10}^{(7)} = [45,49]$, $L_{10}^{(8)} = [109,113]$ and $L_{10}^{(9)} = [237,241]$. Suppose that there is $n \in U_{10} - \{13\}$ such that $n \in L_{10}^{(k)}$ and $\chi(K_{10,n}) = k-1$. Consider a point-distinguishing colouring $\varphi : E(K_{10,n}) \to [1,k-1]$. For $p \in [1,k-1]$ let r_p (c_p, respectively) be the number of p-element sets in the system $\mathcal{R}$ (in $\mathcal{C}$). Since $n \geq b_{10}^{(k-1)} + 1$, by Lemma 8 there are two disjoint sets in $\mathcal{R}$, so that, using (2), $c_1 = 0$. Let first $k=6$. Clearly, $r_1 \geq 2$ is impossible, for then $2^{5-r_1} \leq 8 < n$ (sets in $\mathcal{C}$ differ only in colours out of singleton colour sets). Assume that $r_1 = 1$, and that, say, $\{1\} \in \mathcal{R}$. As $\sum_{i=1}^{5}(r_i + c_i) = 10 + n$, we have $r_2 + c_2 = 10 + n - 1 - \sum_{i=3}^{5}(r_i + c_i) \geq 9 + n - \sum_{i=3}^{5}\binom{5}{i} = n - 7 \geq 7$, hence at least three out of six 2-element subsets of $[2,5]$ (which cannot be in $\mathcal{C}$) are in $\mathcal{R}$. If two of those 2-element subsets in $\mathcal{R}$ are disjoint, then, by (2), $|\mathcal{C}| \leq (2^2-1)^2 = 9 < n$, a contradiction. Assume that three of those 2-element subsets in $\mathcal{R}$ have a common colour, say $\{2,3\},\{2,4\},\{2,5\} \in \mathcal{R}$. Then, evidently, each $C \in \mathcal{C}$ is a superset of either $\{1,2\}$ or $\{1,3,4,5\}$, and $|\mathcal{C}| \leq 2^3 + 1 = 9 < n$, a contradiction. Finally, we may suppose without loss of generality that $\{2,3\},\{2,4\},\{3,4\} \in \mathcal{R}$. Then any set $C \in \mathcal{C}$ contains 1 and at least two colours of $[2,4]$, hence $|\mathcal{C}| \leq (\binom{3}{2} + \binom{3}{3}) \cdot 2 = 8 < n$, a contradiction again. Thus $r_1 = 0$ and $r_2 + c_2 \geq n - 6 \geq 8$. If there are two disjoint 2-element sets in $\mathcal{R}$, then $c_2 \leq 2 \cdot 2 = 4$ and $r_2 \geq 8 - c_2 \geq 4$. Consider graphs G, $G(\mathcal{R})$ and $G(\mathcal{C})$ with $V(G) = V(G(\mathcal{R})) = V(G(\mathcal{C})) = [1,5]$, whose edge sets consist of 2-element sets in $\mathcal{R} \cup \mathcal{C}$, $\mathcal{R}$ and $\mathcal{C}$, respectively. Let $p \in [1,5]$ be such that $d := \deg_{G(\mathcal{R})}(p) = \Delta(G(\mathcal{R}))$. From $|E(G(\mathcal{R}))| = r_2 \geq 4$ we obtain $d \geq \lceil (2\cdot 4)/5 \rceil = 2$, and so, by (2) (which means that any edge of $G(\mathcal{C})$ is adjacent to any edge of $G(\mathcal{R})$ in G), it is easy to see that $c_2 \leq 2$. Therefore $r_2 \geq 6$ and $d \geq \lceil (2\cdot 6)/5 \rceil = 3$. Then, again by (2), $c_2 \leq 1$ and $r_2 \geq 7$. Complements (in the set $[1,5]$) of 2-element sets of $\mathcal{R}$ can only be in $\mathcal{R}$, and, as $r_3 \leq 10 - r_2$, the number of 3-element subsets of $[1,5]$ that are out of $\mathcal{R} \cup \mathcal{C}$ is at least $r_2 - (10 - r_2) \geq 4$. So, $|\mathcal{R} \cup \mathcal{C}| \leq \sum_{i=2}^{5}\binom{5}{i} - 4 = 22 < 10 + n$, a contradiction. The last possibility is that there are disjoint sets in $\mathcal{R}$ of cardinalities 2 and 3, say $\{1,2\},[3,5] \in \mathcal{R}$. Since $r_2 + c_2 \geq 8$ and 2-element subsets of $[3,5]$ are not in $\mathcal{C}$, at least one of them is in $\mathcal{R}$ and we have the situation from above. Now assume that $k \in [7,9]$ and let $r \in [1, \lfloor (k-1)/2 \rfloor]$ be such that there are disjoint sets $R, \bar{R} \in \mathcal{R}$ with $r = |R| \le |\bar{R}|$. For any $C \in \mathcal{C}$ both sets $C \cap R$ and $C \cap ([1,k-1] - R) \supseteq C \cap \bar{R}$ are nonempty, and so $n = |\mathcal{C}| \le (2^r - 1)(2^{k-1-r} - 1) = 2^{k-1} - 2^{k-1-r} - 2^r + 1$. We have $n \ge b_{10}^{(k-1)} + 1 = \sum_{i=0}^{10} \binom{k-1}{i} - 20 + 1 = 2^{k-1} - 19$, hence $2^r + 2^{k-1-r} \le 20$, and it is easy to see that this is possible only if $k = 7$ and $r = 3$. Evidently, $r_1 > 0$ implies $|\mathcal{C}| \le 2^{6-r_1} \le 2^5 < n$, a contradiction. Thus $r_1 = 0$ and $\binom{6}{2} \ge r_2 + c_2 \ge 10 + n - \sum_{i=3}^{6} \binom{6}{i} = n - 32 \ge 13$. Therefore, if (say) $[1,3]$ and $[4,6]$ are in $\mathcal{R}$, at least one of 2-element subsets of $[4,6]$ (which are all out of $\mathcal{C}$) must be in $\mathcal{R}$, but then $|\mathcal{C}| \le (2^3 - 1)(2^2 - 1) \cdot 2 = 42 < n$, a contradiction. In a similar (and simpler) way it is possible to treat the cases $m \in [4,9]$. The analysis is left to the reader. □

4. Concluding remarks

We have seen that triples (k,m,n) such that $n \in L_m^{(k)}$ and $\chi_0(K_{m,n}) = k-1$ appear quite rarely. In spite of this fact the number of those triples is infinite. Recall the number n_k related to $\chi_0(K_{m,m})$ and mentioned in the introductory section. From results of [5] it follows that $\chi_0(K_{m,m}) = k$ for all $m \in [\lfloor 2^k/3 \rfloor, n_k]$, hence, using $n_7 = 46$ and $n_{k+1} \ge 2n_k$, we obtain $n_k \ge 23 \cdot 2^{k-6}$ for every $k \in [7,\infty)$. Suppose that $k \in [8,\infty)$ and $m \in [\lceil 2^{k-1}/3 \rceil, 23 \cdot 2^{k-7}]$. Then $\chi_0(K_{m,m}) = k-1$, for $\lfloor 2^{k-1}/3 \rfloor \le m \le 23 \cdot 2^{k-7} \le n_{k-1}$. We have also $k-1 \le m$, as a consequence of $k \le 2^{k-1}/3$. Therefore, $m \ge b_m^{(k-1)} + 1 = \sum_{i=0}^{m} \binom{k-1}{i} - 2m + 1 = 2^{k-1} - 2m + 1$, since this inequality is equivalent to either of $3m \ge 2^{k-1} + 1$, $3m > 2^{k-1}$ and $m \ge \lceil 2^{k-1}/3 \rceil$. On the other hand, $l = \lceil \log_2 m \rceil \le \lceil \log_2 23 + k - 7 \rceil = k-2$, $2m + 1 + l \le 23 \cdot 2^{k-6} + k - 1 \le 2^{k-1} = 32 \cdot 2^{k-6}$, and so $m \le 2^{k-1} - m - l - 1$. Thus $m \in L_m^{(k)}$, and, as mentioned above, $\chi_0(K_{m,m}) = k-1$. With fixed $k \ge 8$ the number of integers in the interval $[\lceil 2^{k-1}/3 \rceil, 23 \cdot 2^{k-7}]$ is $\ge 23 \cdot 2^{k-7} - 2^{k-1}/3 = 5 \cdot 2^{k-7}/3 \ge 3$, and so the number of triples (k,m,m) such that $m \in L_m^{(k)}$ and $\chi_0(K_{m,m}) = k-1$ is infinite. If $n \in I_m^{(k)}$ and $\chi_0(K_{m,n}) = k-1$, then, by Theorems 11 and 14, $k \in [l+2, 2l+1]$ and $n \in L_m^{(k)}$. In the above infinite sequence of examples of this type we have $m = n$, and from $m \ge \lceil 2^{k-1}/3 \rceil \ge 2^{k-1}/3 > 2^{k-3}$ it follows that $k - 3 < \log_2 m$, $k - 2 \le l$, and so $k = l+2$. In the only exceptional case of Theorem 16 the involved parameters are $m = 10$, $n = 13$ and $k = 6 = l+2$. Thus, the following could be of interest:

Problem. *Decide whether there is a triple of integers (k,m,n) such that $m \ge 11$, $k \in [l+3, 2l+1]$, $n \in L_m^{(k)}$ and $\chi_0(K_{m,n}) = k-1$.*

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