Prime factors of Motzkin numbers

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Abstract. In this note, we investigate arithmetic properties of the Motzkin numbers. We prove that for large n the product of the first n Motzkin numbers is divisible by a large prime. The proofs use the Deep Subspace Theorem.

§1. Introduction

Let n be any positive integer. The *n-th Motzkin number*, denoted from here on by m_n, counts the number of lattice paths in the Cartesian plane starting at (0, 0), ending at (n, 0), and which use line steps equal to either (1, 0) (level step), or to (1, 1) (up step), or to (1, -1) (down step), and which never pass below the x-axis. Clearly, $m_1 = 1$, $m_2 = 2$, and it is known that the three-term recurrence

$$(n+2)m_n = (2n+1)m_{n-1} + 3(n-1)m_{n-2} \tag{1}$$

holds for all $n \geq 3$ (see Dulucq and Penaud [4] and Woan [15]). The sequence of Motzkin numbers begins

$$(m_n)_{n\geq 0} = (1,1,2,4,9,21,51,127,323,835,2188,5798,15511,\ldots)$$

and is listed as sequence A001006 in EIS [12]. It is convenient to set $m_0 := 1$, and the recurrence is then valid for all $n \geq 2$.

For any integer k we write $P(k)$ for the largest prime factor of k with the convention that $P(0) = P(\pm 1) = 1$.

There are several papers in the literature which address the question of finding nontrivial lower bounds for $P(u_n)$, when $(u_n)_{n\geq 0}$ is some recurrent sequence naturally arising in some number theoretical or combinatorial context. For example, there is quite a rich literature on the above question when $(u_n)_{n\geq 0}$ is a nondegenerate binary recurrent sequences $(u_n)_{n\geq 0}$ for which we refer the reader to Shorey and Tijdeman [14] and to the references therein.

The situation is less understood when $(u_n)_{n\geq 0}$ is a higher order linearly recurrent sequence, although nontrivial results here can be deduced by using techniques from transcendence theory such as in Corvaja and Zannier [3].

For every positive integers k and n, let $S(n, k)$ denote the Stirling number of the second kind which counts the number of partitions of a set with

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n elements into k nonempty disjoint subsets. For a fixed value of k, the sequence $(S(n,k))_{k\geq 0}$ is linearly recurrent or order k, which makes it possible to investigate its arithmetic properties with methods from transcendetal number theory. This has been done in Brindza and Pínter [1], Pínter [9], [10] and Klazar and Luca [6]. The behaviour of the numbers $S(n,k)$ for fixed (but large) n and $k \in \{0,\ldots,n\}$ has been investigated in Canfield and Pomerance [2].

The number

$$B_n = \sum_{k=0}^{n} S(n,k) \tag{2}$$

is called the *nth Bell* number and counts the total number of partitions of a set with n elements. The sequence of Bell number $(B_n)_{n\geq 0}$ is not linearly recurrent, but the fact that it is periodic modulo p for every prime number p makes it possible to derive nontrivial results about the arithmetic structure of its members. Such results can be found in Lunnon, Pleasants and Stephens [8], and Shparlinski [13]. For example, in [13], it is shown that if we write $B(n) = \prod_{k=1}^{n} B_k$ then the inequality

$$\omega(B(n)) > \frac{\log n}{2\log\log n} \tag{3}$$

holds for all sufficiently large positive integers n, where as usual for a positive integer m we write $\omega(m)$ for the number of distinct prime factors of m. Inequality (3) above, together with the Prime Number Theorem, implies that the inequality

$$P(B_n) > \frac{\log n}{3}$$

holds for infinitely many positive integers n. It is not known if $P(B_n)$ tends to infinity with n.

In this paper, we address the above question for the Motzkin numbers. Some results about the arithmetic properties of these numbers have already appeared in Klazar and Luca [7]. For example, in [7], it is shown that if $(M_n)_{n\geq 0} := (M_n(\lambda,\mu))_{n\geq 0}$ is a sequence of rational numbers satisfying recurrence (1) with the initial values $M_0 := \lambda,\ M_1 := \mu$ then $(M_n)_{n\geq 0}$ consists of rational numbers whose denominators are divisible by arbitrarily large primes provided that $\lambda \neq \mu$. In particular, the only sequences of integers satisfying the Motzkin recurrence (1) are the integer multiples of the Motzkin numbers $(m_n)_{n\geq 0}$. In the same paper, it is shown that $(m_n)_{n\geq 0}$ is not eventually periodic (i.e., periodic from some point on) modulo any positive integer $T > 1$. Here, we look at the prime factors of the numbers $(m_n)_{n\geq 0}$. It would be interesting to know whether $P(m_n)$ tends to infinity with n, but we have not been able to answer this question. In fact, we haven't even been able to decide whether $(m_n)_{n\geq 0}$ contains finitely or infinitely many powers of 2. However, we prove that an inequality of the same type as (3) holds for the Motzkin numbers $(m_n)_{n\geq 0}$.

We have the following results.

Proposition.

Let $\mathcal{P}$ be any fixed finite set of prime numbers. There exists a positive integer $n_{\mathcal{P}}$ depending on $\mathcal{P}$ such that for $n > n_{\mathcal{P}}$ the number $m_n \cdot m_{n+1} \cdot m_{n+2} \cdot m_{n+3}$ is divisible by a prime number p not in $\mathcal{P}$.

Theorem.

Let $M(n) := \prod_{k=1}^{n} m_k$. Then the inequalities

$$\omega(M(n)) > 10^{-4}\log n \quad \textit{and} \quad P(M(n)) > 10^{-5}\log n \log\log n \tag{4}$$

hold for all $n \geq 3$.

The two results from the present note extend easily to other sequences which naturally arise in enumerative combinatorics. One of such sequences is the Schröder sequence $(s_n)_{n\geq 1} = (1, 1, 3, 11, 45, 197, \ldots)$ (A001003 of EIS [12]). For $n \geq 0$, the number s_{n+1} counts the number of lattice paths in the Cartesian plane starting at (0, 0), ending at $(n,\ n)$, and which use line steps equal to either $(k,\ 0)$ (level step), where k is any fixed positive integer, or to (0, 1) (up step), and which never pass above the line $y = x$. Using the same method of proof as in the present note, one can show that inequalities (4) asserted by our Theorem hold with $M(n)$ replaced by $S(n) := \prod_{k=1}^{n} s_k$. We do not give further details and proceed to the proofs of our results.

§2. The Proofs

The Proof of the Proposition. Let $n \geq 2$ and rewrite relation (1) as

$$n(m_n - 2m_{n-1} - 3m_{n-2}) = -2m_n + m_{n-1} - 3m_{n-2}. \tag{5}$$

Note that the two sides of the above equation are nonzero. Indeed, if the two sides of the above equation were zero, we then get that

$$0 = m_n - 2m_{n-1} - 3m_{n-2} = -2m_n + m_{n-1} - 3m_{n-2},$$

therefore

$$m_n = 2m_{n-1} + 3m_{n-2} \qquad \text{and} \qquad m_n = \frac{1}{2}\cdot m_{n-1} - \frac{3}{2}\cdot m_{n-2} \tag{6}$$

and it is clear that equations (3) cannot hold simultaneously because since m_{n-1} and m_{n-2} are both positive, the expression from the left of (6) representing m_n is larger than the expression on the right of (6) representing m_n.

Thus, we may use (5) to write

$$n = \frac{-2m_n + m_{n-1} - 3m_{n-2}}{m_n - 2m_{n-1} - 3m_{n-2}} \qquad \text{for all } n \geq 2. \tag{7}$$

Replacing n by $n+1$ in (7) and taking the difference of the two equations we obtain

$$\frac{-2m_{n+1} + m_n - 3m_{n-1}}{m_{n+1} - 2m_n - 3m_{n-1}} - \frac{-2m_n + m_{n-1} - 3m_{n-2}}{m_n - 2m_{n-1} - 3m_{n-2}} = 1, \tag{8}$$

which can be rewritten as

$$m_{n+1}m_n - 5m_{n+1}m_{n-1} - 12m_{n+1}m_{n-2} + m_n^2 + 10m_nm_{n-1}$$
$$+15m_nm_{n-2} - 3m_{n-1}^2 + 9m_{n-1}m_{n-2} = 0. \tag{9}$$

For $n \geq 1$ we put $X_n := m_n/m_{n-1}$ and then relation (9) becomes

$$X_{n+1}X_n^2X_{n-1}^2 - 5X_{n+1}X_nX_{n-1}^2 - 12X_{n+1}X_nX_{n-1} + X_n^2X_{n-1}^2$$
$$+10X_nX_{n-1}^2 + 15X_nX_{n-1} - 3X_{n-1}^2 + 9X_{n-1} = 0. \tag{10}$$

Assume now that $\mathcal{P}$ is a fixed finite set of prime numbers containing the primes 2, 3 and 5. Let $\mathcal{S}$ be the set of all nonzero rational numbers having the property that both their numerator and denominator are divisible only by primes in $\mathcal{P}$. Assume further that $n \geq 2$ is such that $m_{n-2} \cdot m_{n-1} \cdot m_n \cdot m_{n+1}$ is a member of $\mathcal{S}$. Write

$$Y_1 := 3^{-2}X_{n+1}X_n^2X_{n-1}, \qquad Y_2 := -3^{-2} \cdot 5X_{n+1}X_nX_{n-1},$$
$$Y_3 := -2^2 \cdot 3^{-1}X_{n+1}X_n, \qquad Y_4 := 3^{-2}X_n^2X_{n-1}$$
$$Y_5 := 2 \cdot 3^{-2} \cdot 5X_nX_{n-1}, \qquad Y_6 := 3^{-1} \cdot 5X_n,$$
$$Y_7 := -3^{-1}X_{n-1}, \qquad Y_8 := 1. \tag{11}$$

Formally, the rational numbers Y_i for $i = 1, \ldots 8$ appearing in (11) above depend on n, but we shall omit the dependence on n in order not to complicate the notation. Then $Y_i \in \mathcal{S}$ for $i = 1, \ldots 8$, and equation (10) becomes

$$\sum_{i=1}^{8} Y_i = 0. \tag{12}$$

We now recall the following Theorem from Evertse [5] on the solutions of *nondegenerate S-unit equations.*

Let $\mathcal{P}$ be any fixed set of primes, and let $\mathcal{S}$ be the set of rational numbers defined previously. Let $k \geq 2$ be a fixed positive integer. An equation of the form

$$\sum_{i=1}^{k} Y_i = 0 \tag{13}$$

with $Y_i \in \mathcal{S}$ for $i = 1, \ldots k$ is called a *nondegenerate $\mathcal{S}$-unit equation* if

$$\sum_{i \in I} Y_i \neq 0$$

holds for every nonempty proper subset I of $\{1, \ldots, k\}$.

Theorem [5].

If the nondegenerate $\mathcal{S}$-unit equation (13) admits a solution, then there exist a positive integer L and L solutions $\mathbf{Y}^{(j)} := (Y_1^{(j)}, \ldots, Y_k^{(j)})$ with $j = 1, \ldots L$ of equation (13) whose components are in $\mathcal{S}$ such that if $\mathbf{Y} := (Y_1, \ldots, Y_k)$ is any other solution of the nondegenerate $\mathcal{S}$-unit equation (13), then there exists $\rho \in \mathcal{S}$ and $j \in \{1, \ldots, L\}$ such that $\mathbf{Y} = \rho \cdot \mathbf{Y}^{(j)}$. Moreover, if we write $t := \#\mathcal{P}$, then the number L can be bounded above by

$$L < F(k,t) := (2^{35}(k-1)^2)^{(k-1)^3(t+1)}. \tag{14}$$

From the above Theorem, it follows that if the equation shown at (12) is nondegenerate, then since $Y_8 = 1$, equation (12) can have at most $F(8,t)$ solutions $\mathbf{Y}$. Since

$$(X_{n+1},\ X_n,\ X_{n-1}) = (-2^{-2} \cdot 5Y_3Y_6^{-1},\ 3 \cdot 5^{-1}Y_6,\ -3Y_7),$$

it follows that $(X_{n+1},\ X_n,\ X_{n-1})$ can take at most $F(8,t)$ values.

It remains to investigate the instances in which equation (12) is degenerate.

From now on, we assume that (12) is degenerate. We recall that the inequality

$$3 - \frac{6}{n+2} < X_n < 3 - \frac{4}{n+2} \tag{15}$$

holds for all $n \geq 2$ (see [15]). Thus, if we write $Z_i := \lim_{n\to\infty} Y_i$ for $i = 1, \ldots, 8$, then

$$Y_i = Z_i + O\Big(\frac{1}{n}\Big) \qquad \text{holds for } i = 1, \ldots, 8 \tag{16}$$

and

$$\mathbf{Z} := (Z_1,\ Z_2,\ Z_3,\ Z_4,\ Z_5,\ Z_6\ Z_7,\ Z_8) = (9,\ -15,\ -12,\ 3,\ 10,\ 5,\ -1,\ 1). \tag{17}$$

Assume now that $\ell \geq 2$ and that $I_1, I_2, \ldots, I_\ell$ is a partition of $\{1,\ldots,8\}$ is such that equation (12) implies that

$$\sum_{i \in I_j} Y_i = 0 \qquad \text{holds for } j = 1, \ldots, \ell, \tag{18}$$

and such that everyone of the $\mathcal{S}$-unit equations shown at (18) is nondegenerate. Using (16), we get that

$$\sum_{i \in I_j} Z_i = O\Big(\frac{1}{n}\Big) \qquad \text{holds for } j = 1, \ldots, \ell. \tag{19}$$

In fact, using (15) and (17) one can check that the constant understood in the O-symbol appearing in (18) above can be taken to be 10^3. We deduce that if $n > 10^3$ then relations (19) imply that

$$\sum_{i \in I_j} Z_i = 0 \qquad \text{holds for } j = 1, \ldots, \ell. \tag{20}$$

Assume that $2 \in I_1$. If $3 \in I_1$, then it is easy to see that $\ell = 2$, $I_1 = \{1, 2, 3, 4, 5, 6\}$ and $I_2 = \{7, 8\}$, and so the only possibility is

$$\sum_{i=1}^{6} Y_i = 0 \quad \text{and} \quad Y_7 + Y_8 = 0. \tag{21}$$

If $3 \notin I_1$, then it is clear that I_1 must contain either 1 or 5 but not both. If I_1 contains 1 then it is easy to see that $\ell = 2$, $I_1 = \{1, 2, 6, 8\}$ and $I_2 = \{3, 4, 5, 7\}$ and therefore the only possibility is

$$Y_1 + Y_2 + Y_6 + Y_8 = 0 \quad \text{and} \quad Y_3 + Y_4 + Y_5 + Y_7 = 0. \tag{22}$$

Finally, if I_1 contains 5, then it also contains 6. Moreover, up to relabelling the I_j's, we get that I_2 must contain 1, 3 and 4, and either $\ell = 2$ in which case both 7 and 8 belong to either I_1 or I_2, or $\ell = 3$ in which case I_3 consists of 7 and 8. Thus, we get the possibilities

$$Y_2 + Y_5 + Y_6 + Y_7 + Y_8 = 0 \quad \text{and} \quad Y_1 + Y_3 + Y_4 = 0, \tag{23}$$

or

$$Y_2 + Y_5 + Y_6 = 0 \quad \text{and} \quad Y_1 + Y_3 + Y_4 + Y_7 + Y_8 = 0, \tag{24}$$

or

$$Y_2 + Y_5 + Y_6 = 0, \quad Y_1 + Y_3 + Y_4 = 0 \quad \text{and} \quad Y_7 + Y_8 = 0. \tag{25}$$

Assume that we are in the situation given by (21). Then $X_{n-1} = -3Y_7 = 3$ is uniquely determined. Writing $Y_i' := Y_i/X_n$ for $i = 1, \ldots 6$, we get the nondegenerate equation

$$\sum_{i=1}^{6} Y_i' = 0, \tag{26}$$

with $Y_6' = 3^{-1} \cdot 5$, and $Y_5' = 2 \cdot 3^{-2} \cdot 5 \cdot X_{n-1} = 2 \cdot 3^{-1} \cdot 5$. Thus, $Y_5' + Y_6' = 5$, and we can therefore regard equation (26) as a nondegenerate $\mathcal{S}$-unit equation in 5 terms, and as such it has at most $F(5,t)$ solutions $\mathbf{Y}' = (Y_1{}', \ldots, Y_6')$. Since $X_n = 9Y_4'X_{n-1}^{-1} = 3Y_4'$ and $X_{n+1} = -2^{-2} \cdot 3 \cdot Y_3'$, it follows that (X_{n+1}, X_n, X_{n-1}) can take at most $F(5,t) < F(8,t)$ values. Similar arguments can be employed to show that the pair of nondegenerate equations (22) leads to at most $F(4,t)^2 < F(8,t)$ values for the triple (X_{n+1}, X_n, X_{n-1}), that each one of the two systems of nondegenerate equations (23) and (24) leads to at most $F(3,t)F(5,t) < F(8,t)$ values for the triple (X_{n+1}, X_n, X_{n-1}), and finally that the system of equations (22) leads to at most $F(3,t)^2 < F(8,t)$ values for the triple (X_{n+1}, X_n, X_{n-1}). In conclusion, equation (12) implies that the triple (X_{n+1}, X_n, X_{n-1}) can take a totality of at most $6F(8,t)$ values. Since formula (7) implies that

$$n = \frac{-2X_nX_{n-1} + X_{n-1} - 3}{X_nX_{n-1} - 2X_{n-1} - 3},$$

it follows that $n > 10^3$ can take at most $6F(8,t)$ values. This shows that there exists $n_{\mathcal{P}} > 10^3+2$ such that if $n > n_{\mathcal{P}}-2$ then $m_{n-2} \cdot m_{n-1} \cdot m_n \cdot m_{n+1}$ is not a member of $\mathcal{S}$, which implies the conclusion of our Proposition.

The Proof of the Theorem. We first prove the inequality appearing on the left of (4). It clearly suffices to assume that $\log n \geq 10^4$ for otherwise the lower bound on $\omega(M(n))$ appearing in (4) is smaller than 1. Write $\omega = \omega(M(n))$. The argument from the above proof of our Proposition implies that

$$n - 1001 < 6F(8, \omega),$$

therefore

$$n < 1001 + 6F(8, \omega) < 1001 + 6(2^{35} \cdot 7^2)^{7^3(\omega+1)} < 12(2^{35} \cdot 7^2)^{7^3(\omega+1)}$$

$$= \exp(7^3(\omega + 1)(35 \log 2 + 2 \log 7) + \log 12) < \exp(9700(\omega + 1)). \tag{27}$$

One checks computationally that the first 100 members of the Motzkin sequence are divisible by a totality of more than 40 primes. Since $n > 100$, it follows that the inequality

$$(\omega + 1) \leq \frac{10000}{9700} \cdot \omega$$

holds, as it is equivalent to $300\omega > 9700$, which is implied by $\omega \geq 40$. Together with (27), the above inequality gives

$$n < \exp(10^4\omega),$$

which is precisely the first inequality asserted at (4).

For the second inequality (4), we may first assume that $n > 10$ for otherwise the right hand side of it is smaller than 1. Since $n > 10$, it follows that $P(M(n)) \geq 547 > 500$, so we may assume that

$$10^{-5} \log n \log\log n > 500,$$

and the above inequality forces $\log n > 3 \cdot 10^6$. Now let $p_1 < p_2 < \dots$ be the increasing sequence of all prime numbers. By the first inequality (4), we certainly have that $P(M(n)) \geq p_\omega$, where ω is bounded from below as shown in (4). Note that when $\log n > 3 \cdot 10^6$ the lower bound on ω shown at (4) is larger than 1. It is known that if $\ell \geq 1$ is any positive integer, then $p_\ell > \ell \log \ell$ (see Rosser and Schoenfeld [11]). Thus,

$$P(M(n)) > p_\omega > \omega \log \omega \geq 10^{-4} \log n \log\big(10^{-4} \log n\big). \tag{28}$$

It suffices to show that in our range we have

$$\log\big(10^{-4} \log n\big) > 10^{-1} \log n \tag{29}$$

and the desired inequality will follow. However, inequality (29) is equivalent to

$$10^{-4} \log n > (\log n)^{1/10},$$

which is equivalent to

$$\log n > (10^4)^{10/9} = 10^{40/9},$$

which certainly holds because $\log n > 3{\cdot}10^6 > 10^5 > 10^{40/9}$. This completes the proof of our Theorem.

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References

[1] B. Brindza and Á. Pintér, *On the power values of Stirling numbers*, Acta Arith. 60 (1991), 169–175.

[2] E. R. Canfield and C. Pomerance, *On the problem of uniqueness for the maximum Stirling number(s) of the second kind*, Integers **2** (2002), A1.

[3] P. Corvaja and U. Zannier, *Diophantine equations with power sums and universal Hilbert sets*, Indag. Mathem., N. S. **9** (1998), 317–332.

[4] S. Dulucq and J-G. Penaud, *Interprétation bijective d'une récurrence des nombres de Motzkin*, Discrete Math. **256** (2002), 671–676.

[5] J.-H. Evertse, *On equations in S-units and the Thue-Mahler equation*, Invent. Math. **75** (1984), 561–584.

[6] M. Klazar and F. Luca, *Diophantine equations with polynomial expressions involving Stirling numbers*, Acta Arith. **107** (4) (2003), 357–372.

[7] M. Klazar and F. Luca, *On the integrality of the Motzkin number*, Aequationes Math., to appear.

[8] V.F. Lunnon, P.A.B. Pleasants and N.M. Stephens, *Arithmetic properties of Bell numbers to a composite modulus*, Acta Arith. **35** (1979), 1–16.

[9] Á. Pintér, *On some arithmetical properties of Stirling numbers*, Publ. Math. (Debrecen) 40 (1992), 91–95.

[10] Á. Pintér, *On a diophantine problem concerning Stirling numbers*, Acta Math. Hung. **65** (1994), 361–364.

[11] J.B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. of Math. **6** (1962), 64–94.

[12] N.J.A. Sloane, editor (2002), *The On-Line Encyclopedia of Integer Sequences*, published electronically at http://www.research.att.com/~njas/sequences/.

[13] I. Shparlinski, *The number of different prime divisors of recurrent sequences*, (Russian) Mat. Zametki **42** (4) (1987), 494–507, 622.

[14] T.N. Shorey and R. Tijdeman, *Exponential Diophantine Equations*, Cambridge U. Press, 1986.

[15] W.-J. Woan, *A combinatorial proof of a recurrence relation of the Motzkin sequence by lattice paths*, Fibonacci Quart. 40 (2002), 3–8.

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