Upper bounds on the domination number of a graph in terms of order and minimum degree

Lutz Volkmann

Lehrstuhl II für Mathematik, RWTH Aachen University, 52056 Aachen, Germany

e-mail: volkm@math2.rwth-aachen.de

Abstract

A vertex set D of a graph G is a dominating set if every vertex not in D is adjacent to some vertex in D. The domination number γ of a graph G is the minimum cardinality of a dominating set in G.

In 1975, Payan [6] communicated without proof the inequality

$$2\gamma \leq n + 1 - \delta \qquad (*)$$

for every connected graph not isomorphic to the complement of a one-regular graph, where n is the order and δ the minimum degree of the graph. A first proof of $(*)$ was published by Flach and Volkmann [3] in 1990.

In this paper we firstly present a more transparent proof of $(*)$. Using the idea of this proof, we show that

$$2\gamma \leq n - \delta$$

for connected graphs with exception of well determined families of graphs.

Keywords: Domination number; Minimum degree

1. Terminology and introduction

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* $N(v) = N(v, G)$ of the vertex v consists of the vertices adjacent to v, and the *closed neighborhood* of v is $N[v] = N[v, G] = N(v) \cup \{v\}$. For a subset $S \subseteq V(G)$, we define $N(S) = N(S, G) = \bigcup_{v \in S} N(v)$ and $N[S] = N[S, G] = N(S) \cup S$. The vertex v is an *endvertex* if $d(v, G) = 1$, and an *isolated vertex* if $d(v, G) = 0$, where $d(v) = d(v, G) = |N(v)|$ is the degree of $v \in V(G)$. An edge incident with an endvertex is called a *pendant edge*. Let $\Omega(G)$ be the set of endvertices in a graph G. By $\delta = \delta(G)$ and $\Delta = \Delta(G)$, we denote the *minimum degree* and *maximum degree* of the graph G, respectively. We write C_n for a cycle of length n, K_n for the complete graph of order n, and $K_{p,q}$ for the complete bipartite graph with bipartition X, Y such that $|X| = p$ and $|Y| = q$.

A set $D \subseteq V(G)$ is a *dominating set* of G if $N[D, G] = V(G)$. The *domination number* $\gamma = \gamma(G)$ of G is the cardinality of any smallest dominating set.

The *corona graph* $H \circ K_1$ of the graph H is the graph constructed from a copy of H, where for each vertex $v \in V(H)$, a new vertex v' and a pendant edge vv' are added.

For detailed information on domination and related topics see the comprehensive monograph [4] by Haynes, Hedetniemi, and Slater.

1975, Payan [6] proved for each graph G without isolated vertices of order n and minimum degree δ the bound $2\gamma \leq n + 2 - \delta$. In addition, Payan [6] communicated without proof the following result.

Theorem 1.1 (Payan [6] 1975). If G is a connected graph with $\delta(G) \geq 1$, then

$$\gamma(G) \leq \frac{n(G) + 1 - \delta(G)}{2}, \tag{1}$$

with exception of the case that G is the complement of a one-regular graph.

The first proof of (1) was given by Flach and Volkmann [3]. In this paper, we will show that

$$\gamma(G) \leq \frac{n(G) - \delta(G)}{2}$$

for connected graphs G with exception of well determined families of graphs. Firstly, we present a new proof of (1), and this proof should help to understand how it can be extended to prove the inequality $\gamma \leq (n - \delta)/2$.

2. Preliminary results

The following well-known results play an important role in our investigations.

Proposition 2.1 (Ore [5] 1962). If G is a graph without isolated vertices, then

$$\gamma(G) \leq \left\lfloor \frac{n(G)}{2} \right\rfloor.$$

Theorem 2.2 (Payan, Xuong [7] 1982, Fink, Jacobson, Kinch, Roberts [2] 1985). For a graph G with even order n and no isolated vertices, $\gamma(G) = \lfloor n/2 \rfloor$ if and only if the components of G consist of the cycle C_4 or the corona graph $H \circ K_1$ for any connected graph H.

Proofs of Proposition 2.1 as well as of Theorem 2.2 can also be found in the book of Volkmann [9].

In 1998, Randerath and Volkmann [8] and independently, in 2000, Xu, Cockayne, Haynes, Hedetniemi, and Zhou [10] (cf. also [4], pp. 42-48) characterized the odd order graphs G for which $\gamma(G) = \lfloor n(G)/2 \rfloor$. In order to formulate this characterization, we define a collection of graphs in the following figures.

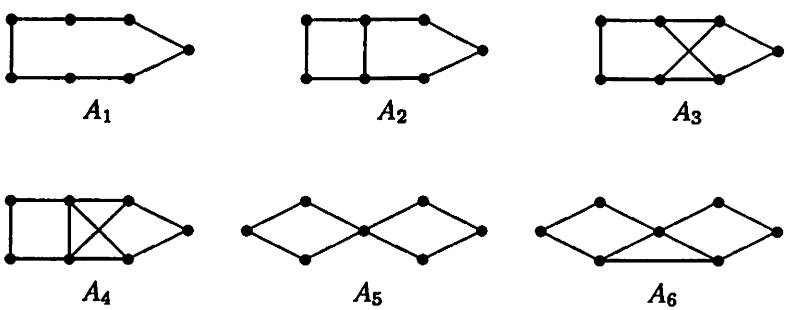

Figure 1

Let $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ be the family of six graphs in Figure 1.

In the next figure, we define a further family $\mathcal{B} = \{B_1, B_2, B_3, B_4, B_5\}$, consisting of five graphs.

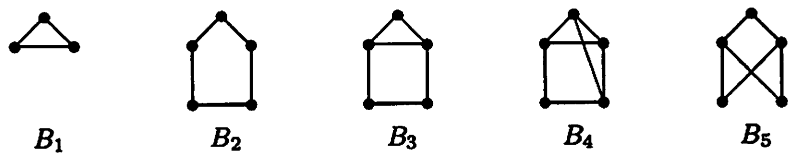

Figure 2

Theorem 2.3 (Randerath, Volkmann [8] 1998, Xu, Cockayne, Haynes, Hedetniemi, Zhou [10] 2000). If G is a connected graph of odd order n with $\delta(G) \geq 1$ and $\gamma(G) = \lfloor n/2 \rfloor$, then $\delta(G) \leq 2$.

If $\delta(G) = 2$, then G belongs to the families $\mathcal{A}$ or $\mathcal{B}$.

If $\delta(G) = 1$, then the following eight cases are possible:
(1) $|N(\Omega(G), G)| = |\Omega(G)| - 1$ and $G - N[\Omega(G), G] = \emptyset$.
(2) $|N(\Omega(G), G)| = |\Omega(G)|$ and $G - N[\Omega(G), G]$ is an isolated vertex.
(3) $|N(\Omega(G), G)| = |\Omega(G)|$ and $G - N[\Omega(G), G]$ is a $K_{1,2}$.
(4) $|N(\Omega(G), G)| = |\Omega(G)|$ and $G - N[\Omega(G), G]$ is a $K_{2,3}$ such that exactly one vertex of degree 2 of the $K_{2,3}$ is adjacent to vertices of $N(\Omega(G), G)$.
(5) $|N(\Omega(G), G)| = |\Omega(G)|$ and $G - N[\Omega(G), G]$ is a bipartite graph H_1 with one endvertex u, which is also a cut vertex of G, and $H_1 - u = C_4$.
(6) G consists of a cycle C_3 and a graph $H \circ K_1$ and arbitrary additional edges between H and one or two vertices of the cycle C_3 such that G is connected.
(7) G consists of a cycle $C_5 = v_1v_2v_3v_4v_5v_1$ and a graph $H \circ K_1$ and arbitrary additional edges between H and v_1 such that G is connected. Furthermore, one or two chords of the form v_1v_4 and v_2v_5 are also admissible.
(8) G consists of a cycle $C_5 = v_1v_2v_3v_4v_5v_1$ and a graph $H \circ K_1$ and arbitrary additional edges between H and v_1 and v_3 such that G is connected. Furthermore, the chord v_1v_3 is also admissible.

Theorem 2.4 (Clark, Dunning [1] 1997). Let G be a graph of order n and minimum degree δ. If $n = 10$ and $\delta \geq 3$ or $n = 11$ and $\delta \geq 4$ or $n = 12$ and $\delta \geq 5$, then $\gamma(G) \leq 3$. If $n = 10$ and $\delta \geq 5$ or $n = 12$ and $\delta \geq 7$ or $n = 13$ and $\delta \geq 8$, then $\gamma(G) \leq 2$.

3. A new proof of Theorem 1.1

Proof of Theorem 1.1. If $\delta = \delta(G) = 1$, then Theorem 1.1 follows from Proposition 2.1. Now let $\delta \geq 2$, $\gamma = \gamma(G)$, and let x be a vertex of minimum degree. Furthermore, let I be a the set of isolated vertices in the subgraph $G - N[x]$ and $R = G - (N[x] \cup I)$.

If $I = \emptyset$, then Proposition 2.1 yields the desired bound

$$\gamma \leq 1 + \frac{|V(R)|}{2} = \frac{2+n-\delta-1}{2} = \frac{n-\delta+1}{2}.$$

In the case that $|I| \geq 1$, the set $\{x, y\}$ dominates $N[x] \cup I$ for each vertex $y \in N(x)$.

If $|I| \geq 2$, then Proposition 2.1 leads to the desired bound

$$\gamma \leq 2 + \frac{|V(R)|}{2} = \frac{4+n-\delta-1-|I|}{2} \leq \frac{n-\delta+1}{2}.$$

Finally, we discuss the case that $|I| = 1$. If $R = \emptyset$ and $\Delta(G) = n-1$, then $\gamma = 1$ and (1) is valid. If $R = \emptyset$ and $\Delta(G) = \delta = n-2$, then G is the complement of a one-regular graph.

Now let $R \neq \emptyset$. In the case that $\gamma(R) \leq (|V(R)| - 1)/2$, we obtain

$$\gamma \leq 2 + \frac{|V(R)|-1}{2} = \frac{4+n-\delta-3}{2} = \frac{n-\delta+1}{2}.$$

If $\gamma(R) = |V(R)|/2$, then, according to Theorem 2.2, the components of R consist of the cycle C_4 or the corona graph $H \circ K_1$, where H is connected.

Firstly, assume that the subgraph R has a component $H \circ K_1$ with $V(H) = \{u_1, u_2, \ldots, u_k\}$ and $\Omega(H \circ K_1) = \{v_1, v_2, \ldots, v_k\}$ such that $k \geq 2$ and $u_i v_i \in E(R)$ for $i = 1, 2, \ldots, k$. Since $\delta \geq 2$, there exists an edge yv_1 with $y \in N(x)$. Therefore, $\{x, y, u_2, u_3, \ldots, u_k\}$ dominates $N[x] \cup I \cup V(H \circ K_1)$. If $T = G - (N[x] \cup I \cup V(H \circ K_1))$, then Proposition 2.1 leads to

$$\gamma \leq k+1+\frac{|T|}{2} = \frac{2k+2+n-\delta-2-2k}{2} = \frac{n-\delta}{2}.$$

Secondly, assume that R has a component $K_1 \circ K_1 = K_2$ with the vertex set $\{u, v\}$. Then u is adjacent to $\delta - 1$ vertices $y_1, y_2, \ldots, y_{\delta-1} \in N(x)$. If y_δ is the remaining vertex in $N(x)$, then $\{u, y_\delta\}$ dominates $N[x] \cup I \cup \{u, v\}$, and we receive at the desired inequality (1) as above.

Thirdly, assume that $C_4 = v_1 v_2 v_3 v_4 v_1$ is a component of R. Since G is connected, there exists an edge, say $yv_1 \in E(G)$, with $y \in N(x)$. Since $\{x, y, v_3\}$ dominates $N[x] \cup I \cup V(C_4)$, we obtain easily the desired inequality. □

4. Main results

In order to present the main results, we define a collection of graphs in the following figures.

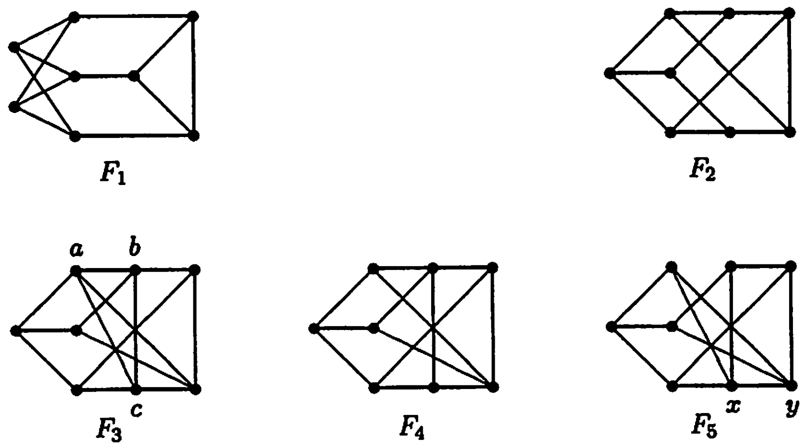

Figure 3

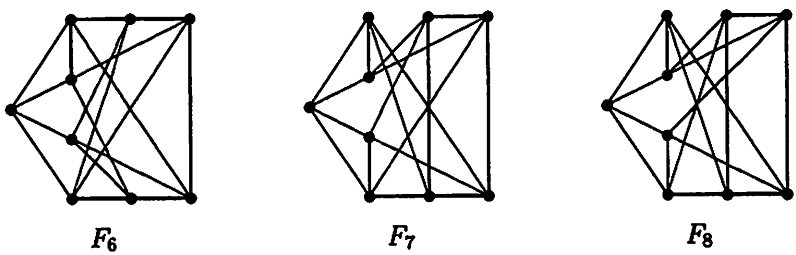

Figure 4

F_9

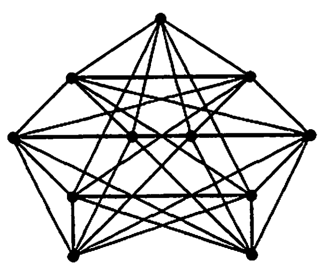

Figure 5

Let $\mathcal{F} = \{F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9\}$ be the family of graphs in Figures 3, 4, and 5. Note that $F_4 = F_3 - \{ac\}$, $F_5 = F_3 - \{ab\}$, $F_2 = F_5 - \{xy\}$, and

$$\gamma(F_i) = 3 > \frac{5}{2} = \frac{n(F_i) - \delta(F_i)}{2}$$

for $F_i \in \mathcal{F}$.

Theorem 4.1 Let G be a connected graph of order n, minimum degree $\delta \geq 2$, and domination number γ. Then

$$\gamma \leq \frac{n - \delta}{2}, \qquad (2)$$

with exception of the cases that G is a member of the families $\mathcal{A}, \mathcal{B}, \mathcal{F}$ or G is the complete graph or $n - 3 = \delta \leq \Delta(G) \leq n - 2$.

Proof. Let x be a vertex of minimum degree with $N(x) = \{y_1, y_2, \ldots, y_\delta\}$. In addition, let I be the set of isolated vertices in $G - N[x]$ and $R = G - (N[x] \cup I)$. If $|I| \geq 1$, then the set $\{x, y_i\}$ dominates $N[x] \cup I$ for each vertex $y_i \in N(x)$.

Case 1. Let $|I| \geq 3$. In this case, Proposition 2.1 yields the desired bound

$$\gamma \leq 2 + \frac{|V(R)|}{2} = \frac{4 + n - \delta - 1 - |I|}{2} \leq \frac{n - \delta}{2}.$$

Case 2. Let $|I| = 2$. If $R = \emptyset$, then $\delta = n - 3$. If $\Delta(G) = n - 1$, then $\gamma = 1$ and (2) is valid. In the remaining case we arrive at the family of exceptional graphs with $n - 3 = \delta \leq \Delta(G) \leq n - 2$. Let now $R \neq \emptyset$. If $\gamma(R) < |V(R)|/2$, then we obtain the desired bound

$$\gamma \leq 2 + \frac{|V(R)| - 1}{2} = \frac{4 + n - \delta - 2 - 2}{2} = \frac{n - \delta}{2}.$$

If $\gamma(R) = |V(R)|/2$, then, in view of Theorem 2.2, the components of R consist of the cycle C_4 or the corona graph $H \circ K_1$, where H is connected.

Firstly, assume that the subgraph R has a component $H \circ K_1$ with $V(H) = \{u_1, u_2, \ldots, u_k\}$ and $\Omega(H \circ K_1) = \{v_1, v_2, \ldots, v_k\}$ such that $k \geq 2$ and $u_i v_i \in E(R)$ for $i = 1, 2, \ldots, k$. Since $\delta \geq 2$, there exists an edge between $\Omega(H \circ K_1)$ and $N(x)$, say $y_1 v_1$. Therefore, $\{x, y_1, u_2, u_3, \ldots, u_k\}$ dominates $N[x] \cup I \cup V(H \circ K_1)$. If $T = G - (N[x] \cup I \cup V(H \circ K_1))$, then Proposition 2.1 leads to

$$\gamma \leq k + 1 + \frac{|T|}{2} = \frac{2k + 2 + n - \delta - 3 - 2k}{2} = \frac{n - \delta - 1}{2}.$$

Secondly, assume that R has a component $K_1 \circ K_1 = K_2$ with the vertex set $\{u, v\}$. Then u is adjacent to $\delta - 1$ vertices in $N(x)$, say $y_1, y_2, \ldots, y_{\delta-1}$. Now $\{u, y_\delta\}$ dominates $N[x] \cup I \cup \{u, v\}$, and we receive at the desired inequality (2) as above.

Thirdly, assume that $C_4 = v_1v_2v_3v_4v_1$ is a component of R. Since G is connected, there exists an edge between $N(x)$ and C_4, say $y_1v_1 \in E(G)$. Since $\{x, y_1, v_3\}$ dominates $N[x] \cup I \cup V(C_4)$, we obtain easily the desired inequality.

Case 3. Let $|I| = 1$. If $R = \emptyset$, then $\delta = n - 2$. If $\Delta(G) = n - 1$, then $\gamma = 1$ and (2) is valid. In the remaining case we arrive at the family of exceptional graphs with $n - 2 = \delta = \Delta(G)$. Let now $R \neq \emptyset$. If $\gamma(R) \leq (|V(R)| - 2)/2$, then

$$\gamma \leq 2 + \frac{|V(R)| - 2}{2} = \frac{4 + n - \delta - 2 - 2}{2} = \frac{n - \delta}{2}.$$

If $\gamma(R) = |V(R)|/2$, then, in view of Theorem 2.2, the components of R consist of the cycle C_4 or the corona graph $H \circ K_1$, where H is connected. In this case, the bound (2) follows analogously to Case 2.

Finally, let $\gamma(R) = (|V(R)| - 1)/2$. If one of the components of R is a C_4 or a corona graph, then we obtain (2) as in the Case 2. It remains the case that all components of R are odd. However, if there are at least two odd components in R, then we conclude that $\gamma(R) \leq (|V(R)| - 2)/2$, a contradiction.

Thus it remains the case that R is a connected odd order graph such that $\gamma(R) = (|V(R)| - 1)/2$. Now we will apply Theorem 2.3.

Case 3.1. Let $\delta(R) = 1$. In view of Theorem 2.3, we have to distinguish eight cases.

Case 3.1.1. Let $|N(\Omega(R), R)| = |\Omega(R)| - 1$ and $R - N[\Omega(R), R] = \emptyset$. If $|V(R)| \geq 5$, then choose a vertex $u \in N(\Omega(R))$ with the unique neighbor $v \in \Omega(R)$. Let, without loss of generality, $y_1, y_2, \ldots, y_{\delta-1}$ be the neighbors of v in $N(x)$. Then we observe that $(N(\Omega(R) - \{u\}) \cup \{v, y_\delta\}$ is a dominating set of G, and we deduce that

$$\gamma \leq 1 + \frac{|V(R)| - 1}{2} = \frac{2 + n - \delta - 3}{2} = \frac{n - \delta - 1}{2}.$$

Let now $|V(R)| = 3$ with $R = v_1uv_2$. Assume first that $\delta \geq 3$. It follows that, without loss of generality, v_1 has the neighbors $y_1, y_2, \ldots, y_{\delta-1}$ in $N(x)$. If $v_2y_\delta \in E(G)$, then $\{v_1, y_\delta\}$ is a dominating set in G and we are done. If not, then v_2 has also the neighbors $y_1, y_2, \ldots, y_{\delta-1}$ and y_δ has a further neighbor in $N(x)$, say y_1. Now $\{y_1, v_1\}$ is a dominating set of G. In the remaining case that $\delta = 2$, we arrive at the inequality (2) or at the exceptional graph A_5.

Case 3.1.2. Let $|N(\Omega(R), R)| = |\Omega(R)|$ and let $R - N[\Omega(R), R]$ be an isolated vertex. Analogously to Case 3.1.1, we obtain (2).

Case 3.1.3. Let $|N(\Omega(R), R)| = |\Omega(R)|$ and let $R - N[\Omega(R), R]$ be a $K_{1,2}$. Let $u \in N(\Omega(R))$ with the unique neighbor $v \in \Omega(R)$, and let $K_{1,2} = a_1a_2a_3$. If, without loss of generality, $y_1, y_2, \ldots, y_{\delta-1}$ are the neighbors of v in $N(x)$, then $(N(\Omega(R) - \{u\}) \cup \{y_\delta, v, a_2\}$ is a dominating set of G, and this leads to (2).

Case 3.1.4. Let $|N(\Omega(R), R)| = |\Omega(R)|$ and let $R - N[\Omega(R), R]$ be a $K_{2,3}$ such that exactly one vertex of degree 2 of the $K_{2,3}$ is adjacent to vertices of $N(\Omega(R), R)$. Let $u \in N(\Omega(R))$ with the unique neighbor $v \in \Omega(R)$, and let b_1, b_2 be the vertices of the $K_{2,3}$ of degree 3. If, without loss of generality, $y_1, y_2, \ldots, y_{\delta-1}$ are the neighbors of v in $N(x)$, then $(N(\Omega(R) - \{u\}) \cup \{y_\delta, v, b_1, b_2\}$ is a dominating set of G, and this leads to (2).

Case 3.1.5. Let $|N(\Omega(R), R)| = |\Omega(R)|$ and let $R - N[\Omega(R), R]$ be a bipartite graph H_1 with one endvertex u, which is also a cut vertex of R, and $H_1 - u = C_4$. We obtain (2) analogously to Case 3.1.4.

Case 3.1.6 Let R consist of a cycle C_3 and a graph $H \circ K_1$ and arbitrary additional edges between H and one or two vertices of the cycle C_3 such that R is connected. We obtain (2) analogously to Case 3.1.3.

Case 3.1.7. Let R consist of a cycle $C_5 = v_1v_2v_3v_4v_5v_1$ and a graph $H \circ K_1$ and arbitrary additional edges between H and v_1 such that R is connected. Furthermore, one or two chords of the form v_1v_4 and v_2v_3 are also admissible. We obtain (2) analogously to Case 3.1.4.

Case 3.1.8. Let R consist of a cycle $C_5 = v_1v_2v_3v_4v_5v_1$ and a graph $H \circ K_1$ and arbitrary additional edges between H and v_1 and v_3 such that R is connected. Furthermore, the chord v_1v_3 is also admissible. We obtain (2) analogously to Case 3.1.4.

Case 3.2. Let $\delta(R) = 2$. According to Theorem 2.3, the graph R belongs to the families $\mathcal{A}$ or $\mathcal{B}$.

If $R \in \mathcal{A}$, then the connectivity of G leads easily to $\gamma \leq 4 < 9/2 = (n - \delta)/2$ and we are done.

Let now $R \in \mathcal{B}$. If $\delta = 2$, then the connectivity leads easily to $\gamma \leq 2 < 5/2 = (n - \delta)/2$ when $R = B_1$ and $\gamma \leq 3 < 7/2 = (n - \delta)/2$ when $R \in \{B_2, B_3, B_4, B_5\}$ and we are done.

Case 3.2.1. Let $R \in \{B_2, B_3, B_4, B_5\}$ and $\delta \geq 3$. If $3 \leq \delta \leq 5$, then, by Theorem 2.4, $\gamma \leq 3 < 7/2 = (n-\delta)/2$. If $\delta \geq 6$, then we observe that there exist at least $5\delta - 14 > 2\delta$ edges from R to $N(x)$. Therefore, there exists a vertex, say y_1, in $N(x)$ with at least three neighbors in R. From these three neighbors of y_1 in R, we can choose two vertices, say u and v, such that $R - \{u, v\}$ contains a path of length two, say $a_1a_2a_3$. Now $\{x, y_1, a_2\}$ is a dominating set of G, and we are done.

Case 3.2.2. Let $R = B_1 = v_1v_2v_3v_1$.

Case 3.2.2.1. Let $\delta \geq 7$. This implies that there exist at least $3(\delta-2) > 2\delta$ edges from R to $N(x)$. Therefore there exists a vertex, say y_1, in $N(x)$ with three neighbors in B_1. So $\{x, y_1\}$ is a dominating set in G, and (2) is proved.

Case 3.2.2.2. Let $\delta = 5$. According to Theorem 2.4, we have $\gamma \leq 2$, and hence (2) is valid.

Case 3.2.2.3. Let $\delta = 3$. Firstly, assume that there exists an edge, say y_1y_2, in $G[N(x)]$. If there is a second edge in $G[N(x)]$, for example y_2y_3, then $\{y_2, v_1\}$ is a dominating set of G and we are done. If not, then y_3 is adjacent to a vertex of R, say v_1, and $\{v_1, y_1\}$ is a dominating set of G. Secondly, assume that $N(x)$ is an independent set. If one vertex v_i has two neighbors in $N(x)$, say y_1, y_2, then $\{v_i, y_3\}$ is a dominating set of G. If not, then we arrive at the exceptional graph F_1.

Case 3.2.2.4. Let $\delta = 4$. If $\{v_1, v_2, v_3\} \subset N(y_i)$ for any $i \in \{1, 2, 3, 4\}$, then $\{x, y_i\}$ is a dominating set of G and we are done. If any v_j has three neighbors in $N(x)$, say $\{y_1, y_2, y_3\} \subseteq N(v_j)$, then $\{v_j, y_4\}$ is a dominating set of G. It remains the case that each vertex v_j has exactly two neighbors in $N(x)$ for $j \in \{1, 2, 3\}$, and each vertex y_i has at most two neighbors in R for $i \in \{1, 2, 3, 4\}$. Consequently, there exists at least one edge, say y_2y_3, in $G[N(x)]$.

Assume that there is a further edge in $G[(N(x)]$. Firstly, let $y_1y_2 \in E(G)$. If there is an edge y_4v_i, then $\{v_i, y_2\}$ is a dominating set of G. If not, then, $y_2y_4 \in E(G)$ or $y_1y_4, y_3y_4 \in E(G)$. If $y_2y_4 \in E(G)$, then $\{y_2, v_1\}$ is a dominating set. If $y_1y_4, y_3y_4 \in E(G)$, then there exists, without loss of generality, an edge v_1y_1 and $\{y_3, v_1\}$ is a dominating set of G. Secondly, let $y_1y_4 \in E(G)$. If there is a third edge in $G[N(x)]$, then we arrive at $\gamma = 2$ as in the first case. If not, then assume, without loss of generality, that $y_1v_1, y_1v_2 \in E(G)$. If there is an edge y_iv_3 for $i = 2, 3$, then $\{y_1, y_i\}$ is a dominating set of G. If not, then $y_1v_3 \in E(G)$ and $\{y_1, y_2\}$ is a dominating set of G.

Assume that there is no further edge in $G[N(x)]$. This implies that G is 4-regular. Let, without loss of generality, $y_1v_1, y_1v_2 \in E(G)$ and thus, without loss of generality, $y_4v_1 \in E(G)$. This implies that $\{v_1, y_2\}$ is a dominating set of G.

Case 3.2.2.5. Let $\delta = 6$. Analogously to Case 3.2.2.1, it remains the case that each vertex v_i has exactly 4 neighbors in $N(x)$ for $i = 1, 2, 3$ and each vertex y_j has exactly two neighbors in R for $j = 1, 2, 3, 4, 5, 6$. If there is a vertex y_i in $N(x)$ with at least three neighbors in $N(x)$, say y_1 such that $y_1y_2, y_1y_3, y_1y_4 \in E(G)$, then there exists a vertex v_i such that $v_iy_5, v_iy_6 \in E(G)$ for any $i \in \{1, 2, 3\}$. This implies that $\{y_1, v_i\}$ is a dominating set of G and we are done. Therefore we only consider the case that each vertex y_j has exactly degree two in $G[N(x)]$ for $j = 1, 2, 3, 4, 5, 6$.

Firstly, assume that the subgraph $G[N(x)]$ consists of two cycles of length 3, say $y_1y_2y_3y_1$ and $y_4y_5y_6y_4$. Assume, without loss of generality, that $y_1v_1, y_1v_2 \in E(G)$. It follows that there exists an edge y_jv_3 for any $j \in \{4,5,6\}$. Hence $\{y_1, y_j\}$ is a dominating set of G.

Secondly, assume that the subgraph $G[N(X)]$ has a Hamiltonian cycle, say $y_1y_2y_3y_4y_5y_6y_1$. If $y_iv_s, y_iv_t \in E(G)$ for $s \neq t$ and $y_{i+3}v_k \in E(G)$ for $k \neq s,t$, then $\{y_i, y_{i+3}\}$ is a dominating set of G. If not, then we arrive at the exceptional graph F_9.

Case 4. Let $|I| = 0$ and assume that $R = \emptyset$ or that R consists of exactly one component. If $R = \emptyset$, then $G = K_\delta$, and G belongs to the family of exceptional graphs. If $R \neq \emptyset$ and $\gamma(R) \leq (|V(R)| - 1)/2$, then Proposition 2.1 yields

$$\gamma \leq 1 + \frac{|V(R)| - 1}{2} = \frac{2 + n - \delta - 2}{2} = \frac{n - \delta}{2}.$$

In view of Theorem 2.2, it remains the case that $R \neq \emptyset$ and R is the cycle C_4 or the corona graph $H \circ K_1$, where H is connected.

Case 4.1. Assume that $R = H \circ K_1$ with $|V(R)| \geq 6$. Let $V(H) = \{u_1, u_2, \ldots, u_k\}$ and $\Omega(R) = \{v_1, v_2, \ldots, v_k\}$ with $k \geq 3$ such that $u_iv_i \in E(R)$ for $i = 1, 2, \ldots, k$. Assume, without loss of generality, that $H - u_1$ is connected and $y_1, y_2, \ldots, y_{\delta-1} \in N(v_1)$. If y_δ is adjacent to a vertex $v_i \neq v_1$, say to v_2, then $\{v_1, y_\delta\} \cup \{u_3, u_4, \ldots, u_k\}$ is a dominating set of G and we deduce that

$$\gamma \leq k = \frac{n - \delta - 1}{2}.$$

If not, then $N(v_i) \cap N(x) = \{y_1, y_2, \ldots, y_{\delta-1}\}$ for all $1 \leq i \leq k$. Now $\{x, y_1\}$ is a dominating set of $N[x] \cup \Omega(R)$, and because of $k \geq 3$, Proposition 2.1 implies

$$\gamma \leq 2 + \frac{|V(H)|}{2} = 2 + \frac{n - \delta - 1}{4} \leq \frac{n - \delta}{2}.$$

Case 4.2. Let $R = K_1 \circ K_1$. This leads to $\delta = n - 3$. If $\Delta(G) = n - 1$, then $\gamma = 1$ and (2) is valid. In the remaining case we arrive at the family of exceptional graphs with $n - 3 = \delta \leq \Delta(G) \leq n - 2$.

Case 4.3. Let $R = K_2 \circ K_1$. Let $K_2 = u_1u_2$ and $\Omega(R) = \{v_1, v_2\}$ such that $v_1u_1, v_2u_2 \in E(R)$.

Case 4.3.1. Let $\delta = 2$. Then it is easy to verify that (2) is valid or G is an element of the family $\mathcal{A}$.

Case 4.3.2. Let $\delta = 3$. Assume, without loss of generality, that $v_1y_1, v_1y_2 \in E(G)$ and $v_2y_2 \in E(G)$.

Case 4.3.2.1. Assume that $y_1y_2 \in E(G)$. If $y_2y_3 \in E(G)$, then $\{y_2, u_1\}$ is a dominating set of G, and we are done. If $y_1y_3 \in E(G)$, then $\{y_1, u_2\}$ is a dominating set of G. If $y_3u_1 \in E(G)$, then $\{y_2, u_1\}$ is a dominating set of G. If $y_3u_2 \in E(G)$, then $\{y_2, u_2\}$ is a dominating set of G. It remains the case that $y_3v_1, y_3v_2 \in E(G)$. If $u_1y_1 \in E(G)$, then $\{y_1, v_2\}$ is a dominating set of G. Thus, we only have to investigate the case that $u_1y_2 \in E(G)$, and this leads to the dominating set $\{y_2, v_2\}$ of G.

Case 4.3.2.2. Assume that $y_2y_3 \in E(G)$ and $y_1y_2 \notin E(G)$. If $y_1u_1 \in E(G)$, then $\{y_2, u_1\}$ is a dominating set of G and we are done. If $y_1u_2 \in E(G)$, then $\{y_2, u_2\}$ is a dominating set of G. If $y_2u_2 \in E(G)$, then $\{y_2, v_1\}$ is a dominating set of G. It remains the case that $u_2y_3 \in E(G)$. If $y_3v_2 \in E(G)$, then $\{y_3, v_1\}$ is a dominating set of G. If $y_3v_2 \notin E(G)$, then it follows that $y_1v_2 \in E(G)$. If $u_1y_2 \in E(G)$, then $\{y_2, v_2\}$ is a dominating set of G. If not, then $u_1y_3 \in E(G)$ and $\{y_1, y_3\}$ is a dominating set of G.

Case 4.3.2.3. Assume that $y_1y_3 \in E(G)$, $y_1y_2 \notin E(G)$, and $y_2y_3 \notin E(G)$.

Case 4.3.2.3.1. Let $y_3v_2 \in E(G)$. If $y_1u_1 \in E(G)$, then $\{y_1, v_2\}$ is a dominating set of G. If $y_3u_2 \in E(G)$, then $\{y_3, v_1\}$ is a dominating set of G. Hence assume in the following that $y_1u_1, y_3u_2 \notin E(G)$.

Let now $u_1y_2 \in E(G)$ or $u_2y_2 \in E(G)$, say $u_1y_2 \in E(G)$. If $y_2u_2 \in E(G)$, then $\{y_2, x\}$ is a dominating set of G. If $u_2y_1 \in E(G)$, then $\{y_1, y_2\}$ is a dominating set of G.

It remains the case that $u_1y_2, u_2y_2 \notin E(G)$. It follows that $u_1y_3, u_2y_1 \in E(G)$. However, this is the exceptional graph F_5. If there is a further edge, for example y_1v_2, then $\{y_1, v_1\}$ is a dominating set of G.

Case 4.3.2.3.2. Let $y_3v_2 \notin E(G)$. This implies that $y_1v_2 \in E(G)$. If $y_2u_1 \in E(G)$, then $\{y_1, u_1\}$ is a dominating set of G. If $y_2u_2 \in E(G)$, then $\{y_1, u_2\}$ is a dominating set of G. If $y_1u_1 \in E(G)$, then $\{y_1, v_2\}$ is a dominating set of G. Thus assume in the following that $y_2u_1, y_2u_2, y_1u_1 \notin E(G)$. This yields $y_3u_1 \in E(G)$. If $y_3u_2 \in E(G)$, then $\{y_2, y_3\}$ is a dominating set of G. If $y_3u_2 \notin E(G)$, then $y_1u_2 \in E(G)$ and $\{y_1, v_1\}$ is a dominating set of G.

Case 4.3.2.4. Assume that $N(x)$ is an independent set.

Case 4.3.2.4.1. Let $y_3v_2 \in E(G)$. If $y_3u_2 \in E(G)$, then $\{y_3, v_1\}$ is a dominating set of G. If $y_1u_1 \in E(G)$, then $\{y_1, v_2\}$ is a dominating set of G. Hence assume in the following that $y_3u_2, y_1u_1 \notin E(G)$.

Let now $u_1y_2 \in E(G)$. If $y_2u_2 \in E(G)$, then $\{y_2, x\}$ is a dominating set of G. If $y_2u_2 \notin E(G)$, then it follows that $y_1u_2 \in E(G)$.

If $y_3u_1 \in E(G)$, then we have the exceptional graph F_5. If we add on the one hand the edge y_1v_2, then $\{y_1, u_1\}$ is a dominating set of G. If we add on the other hand the edge y_3v_1, then we arrive at the exceptional graph F_3.

If $y_3u_1 \notin E(G)$, then we conclude that $y_3v_1 \in E(G)$ and we have again the exceptional F_5. If in addition $y_1v_2 \in E(G)$, then $\{y_1, v_1\}$ is a dominating set of G.

It remains the case that $u_1y_2 \notin E(G)$. It follows that $u_1y_3 \in E(G)$. In the case that $u_2y_2 \in E(G)$, we have a symmetric situation to above. Hence we can assume that $u_2y_2 \notin E(G)$ and this implies that $y_1u_2 \in E(G)$. This yields the exceptional graph F_2. If in addition there exists the edge v_1y_3 or v_2y_1 in G, then we arrive at F_5. If there exist both of these edges, then $\{y_1, v_1\}$ is a dominating set of G.

Case 4.3.2.4.2. Let $y_3v_2 \notin E(G)$. It follows that $y_1v_2 \in E(G)$.

Case 4.3.2.4.2.1. Assume that $y_3u_1, y_3u_2 \in E(G)$. This is the exceptional graph F_1. If in addition $y_3v_1 \in E(G)$, then $\{y_3, v_2\}$ is a dominating set of G. If in addition $y_1u_2 \in E(G)$, then $\{y_2, u_2\}$ is a dominating set of G. If in addition $y_2u_2 \in E(G)$, then $\{y_1, u_2\}$ is a dominating set of G. If in addition $y_2u_1 \in E(G)$, then $\{y_1, u_1\}$ is a dominating set of G. If in addition $y_1u_1 \in E(G)$, then $\{y_2, u_1\}$ is a dominating set of G.

Case 4.3.2.4.2.2. Assume that $y_3v_1, y_3u_1 \in E(G)$ and $y_3u_2 \notin E(G)$. If $y_2u_2 \in E(G)$, then $\{y_2, v_1\}$ is a dominating set of G. It remains the case that $y_2u_2 \notin E(G)$. This yields $y_1u_2 \in E(G)$ and thus $\{y_1, v_1\}$ is a dominating set of G.

Case 4.3.2.4.2.3. Assume that $y_3v_1, y_3u_2 \in E(G)$ and $y_3u_1 \notin E(G)$. If $y_2u_2 \in E(G)$, then $\{y_2, v_1\}$ is a dominating set of G. If $y_1u_2 \in E(G)$, then $\{y_1, v_1\}$ is a dominating set of G. It remains the case that $y_2u_2, y_1u_2 \notin E(G)$. If either $y_1u_1 \in E(G)$ or $y_2u_1 \in E(G)$, then we obtain F_5. In the case that both of these edges are in G, then we arrive at F_3.

Case 4.3.3. Let $\delta \geq 4$. It follows that u_2 and v_2 have a common neighbor in $N(x)$, say y_δ. If $y_1, y_2, \ldots, y_{\delta-1} \in N(v_1)$, then $\{v_1, y_\delta\}$ is a dominating set of G and we are done. Hence we assume, without loss of generality, that $y_2, y_3, \ldots, y_\delta \in N(v_1)$ and $v_1y_1 \notin E(G)$. If $y_1y_\delta \in E(G)$ or $u_1y_\delta \in E(G)$, then $\{v_1, y_\delta\}$ or $\{x, y_\delta\}$ is a dominating set of G, respectively. Thus we assume in the following that $u_1y_\delta, y_1y_\delta \notin E(G)$.

Case 4.3.3.1. Let $u_1y_1 \notin E(G)$. This yields $y_2, y_3, \ldots, y_{\delta-1} \in N(u_1)$. If $v_2y_1 \in E(G)$ and, without loss of generality, $y_3, y_4, \ldots, y_{\delta-2} \in N(v_2)$, then $\{v_2, y_{\delta-1}\}$ is a dominating set of G. If $v_2y_1 \notin E(G)$, then $y_2, y_3, \ldots, y_\delta \in N(v_2)$ and $y_1y_2 \in E(G)$. Thus $\{v_2, y_2\}$ is a dominating set of G.

Case 4.3.3.2. Let $u_1y_1 \in E(G)$. Then we can assume, without loss of generality, that $y_2, y_3, \ldots, y_{\delta-2} \in N(u_1)$. If $u_1y_{\delta-1} \in E(G)$ or $y_{\delta-1}y_\delta \in E(G)$, then $\{u_1, y_\delta\}$ is a dominating set of G. Thus assume in the following that $u_1y_{\delta-1}, y_{\delta-1}y_\delta \notin E(G)$.

Case 4.3.3.2.1. Let $v_2y_{\delta-1} \notin E(G)$. This implies that $y_1, y_2, \ldots, y_{\delta-2} \in N(v_2)$. If $y_1u_2 \in E(G)$, then $\{v_1, y_1\}$ is a dominating set of G. So let $y_1u_2 \notin E(G)$. If $y_iu_2 \in E(G)$ for any $2 \leq i \leq \delta - 2$, then $\{x, y_i\}$ is a dominating set of G. If not, then $\delta = 4$ and $y_3u_2 \in E(G)$. If $y_1y_2 \in E(G)$, then $\{y_2, u_2\}$ is a dominating set of G. In the remaining case that $y_1y_2 \notin E(G)$ it follows that $y_1y_3 \in E(G)$ and we arrive at the exceptional graph F_6. In the case that there are further edges y_2y_3 or y_2y_4, then $\{y_2, v_2\}$ or $\{y_2, y_3\}$ are dominating sets of G, respectively.

Case 4.3.3.2.2. Let $v_2y_{\delta-1} \in E(G)$. If $v_2y_1 \in E(G)$ and, without loss of generality $y_2, y_3, \ldots, y_{\delta-3} \in N(v_2)$, then $\{v_2, y_{\delta-2}\}$ is a dominating set of G. Thus assume that $v_2y_1 \notin E(G)$. This leads to $y_2, y_3, \ldots, y_\delta \in N(v_2)$. The assumption $y_1u_2 \notin E(G)$ implies $y_2, y_3, \ldots, y_{\delta-1} \in N(y_1)$ and so $\{y_1, y_\delta\}$ is a dominating set of G. Hence let now $y_1u_2 \in E(G)$. If $y_iu_2 \in E(G)$ for any $2 \leq i \leq \delta - 2$, then $\{x, y_i\}$ is a dominating set of G. If not, then we deduce that $\delta \leq 5$. In the case that $\delta = 5$, Theorem 2.4 yields the desired result $\gamma \leq 2$.

It remains the case that $\delta = 4$. If $y_1y_2 \in E(G)$, then $\{y_2, v_2\}$ is a dominating set of G. If otherwise $y_1y_2 \notin E(G)$, then we see that $y_1y_3 \in E(G)$, and we arrive at F_7. In the case that there are further edges y_2y_3, y_2y_4 or y_3u_2, then $\{y_3, u_2\}$, $\{y_1, y_4\}$, or $\{y_2, u_2\}$ are dominating sets of G, respectively.

Case 4.4. Let $R = C_4 = u_1u_2u_3u_4u_1$. If $\delta = 2$, then it is easy to verify that (2) is valid or G is a member of the family $\mathcal{A}$. If $\delta \geq 9$, then there exist at least $4(\delta - 2) > 3\delta$ edges from R to $N(x)$. Hence there exists a vertex, say y_1, in $N(x)$ with four neighbors in R. Thus $\{x, y_1\}$ is a dominating set of G and we are done. In the cases that $\delta = 5$, $\delta = 7$, or $\delta = 8$, Theorem 2.4 shows that (2) is true.

Case 4.4.1 Let $\delta = 3$.

Case 4.4.1.1. Assume that y_1 has exactly three neighbors, say u_1, u_2, u_3, in R. Let, without loss of generality, $y_2u_4 \in E(G)$. If y_2y_3, y_1y_3, or y_3u_4 is an edge of G, then $\{y_1, y_2\}$, $\{y_1, y_2\}$, or $\{y_1, u_4\}$ is a dominating set of G, respectively. Thus assume in the following that $y_2y_3, y_1y_3, y_3u_4 \notin E(G)$. Now let, without loss of generality, $y_3u_1 \in E(G)$. If $y_1y_2 \in E(G)$, then $\{y_1, u_1\}$ is a dominating set of G. If $y_2u_1 \in E(G)$, then $\{y_1, u_1\}$ is a dominating set of G. If $y_2u_3 \in E(G)$, then $\{y_2, u_1\}$ is a dominating set of G. Hence assume next that $y_1y_2, y_2u_1, y_2u_3 \notin E(G)$. This yields $y_2u_2 \in E(G)$. If $y_3u_2 \in E(G)$, then $\{y_2, u_2\}$ is a dominating set of G. This leads to $y_3u_3 \in E(G)$ and we arrive at F_3.

Case 4.4.1.2. Assume that y_1 has exactly two adjacent neighbors, say u_1, u_2, in R. Let, without loss of generality, $y_2u_3 \in E(G)$.

In addition, let $y_2u_4 \in E(G)$. If $y_2y_3 \in E(G)$ or $y_1y_3 \in E(G)$, then $\{y_1, y_2\}$ is a dominating set of G. If $y_3u_1 \in E(G)$, then $\{y_2, u_1\}$ is a dominating set of G. If $y_3u_2 \in E(G)$, then $\{y_2, u_2\}$ is a dominating set of G. It remains the case that $y_3u_3, y_3u_4 \in E(G)$. This implies that $\{y_1u_4\}$ is a dominating set of G.

Now assume that $y_2u_4 \notin E(G)$. This yields $y_3u_4 \in E(G)$. If $y_2y_3 \in E(G)$, then $\{y_2, u_1\}$ is a dominating set of G. If $y_1y_2 \in E(G)$, then $\{y_1, u_4\}$ is a dominating set of G. If $y_3u_3 \in E(G)$, then $\{y_1, u_3\}$ is a dominating set of G. If $y_3u_1 \in E(G)$, then $\{y_2, u_1\}$ is a dominating set of G. If

$y_1y_3 \in E(G)$, then $\{y_1, u_3\}$ is a dominating set of G. In the remaining case, we deduce that $y_3u_2 \in E(G)$. If $y_2u_2 \in E(G)$, then $\{y_3, u_2\}$ is a dominating set of G. If not, then $y_2u_1 \in E(G)$, and we have the exceptional graph F_5.

Case 4.4.1.3. Assume that y_1 has exactly two non-adjacent neighbors, say u_1, u_3, in R. Let, without loss of generality, $y_2u_2 \in E(G)$.

In addition, let $y_2u_4 \in E(G)$. If $y_1y_2 \in E(G)$, $y_1y_3 \in E(G)$ or $y_2y_3 \in E(G)$, then it is easy to see that $\gamma \leq 2$. If not, then by the cases above, we only have to consider the case that y_3 has two non-adjacent neighbors, say u_1, u_3, in R. This yields F_4.

Now assume that $y_2u_4 \notin E(G)$. This implies that $y_3u_4 \in E(G)$. If there is a further edge between $N(x)$ and R, then we arrive at a case above. If not, then $y_2y_3 \in E(G)$, and this is F_1 or there are at least two edges in $G[N(x)]$, and we obtain $\gamma \leq 2$.

Case 4.4.2. Let $\delta = 4$.

Case 4.4.2.1. Assume that y_1 has exactly three neighbors, say u_1, u_2, u_3, in R. Let, without loss of generality, $y_2u_4, y_3u_4 \in E(G)$. If $y_4u_4 \in E(G)$ or $y_1y_4 \in E(G)$, then $\{y_1, u_4\}$ is a dominating set of G. Thus we consider in the following the case $y_4u_4, y_1y_4 \notin E(G)$.

Case 4.4.2.1.1. Let y_3y_4 or y_2y_4, say y_3y_4, an edge of G. If $y_2y_3 \in E(G)$ or $y_1y_2 \in E(G)$, then $\{y_1, y_3\}$ is a dominating set of G. Hence we assume next that $y_2y_3, y_1y_2 \notin E(G)$.

Case 4.4.2.1.1.1. Let y_4u_3 or y_4u_1, say y_4u_3, an edge of G. If $y_2u_1 \in E(G)$ or $y_2y_4 \in E(G)$, then $\{y_4, u_1\}$ is a dominating set of G. Otherwise, we conclude that $y_2u_2, y_2u_3 \in E(G)$ and $\{y_3, u_2\}$ is a dominating set of G.

Case 4.4.2.1.1.2. Assume that $y_4u_2 \in E(G)$ and $y_4u_1, y_4u_3 \notin E(G)$. This implies that $y_2y_4 \in E(G)$. If $y_2u_2 \in E(G)$, then $\{y_3, u_2\}$ is a dominating set of G. So assume that $y_2u_2 \notin E(G)$.

If $y_2u_1 \notin E(G)$, then $y_3u_1, y_2u_3 \in E(G)$ and $\{y_3, u_3\}$ is a dominating set of G.

Let now $y_2u_1 \in E(G)$. If $y_3u_3 \in E(G)$, then $\{y_3, u_1\}$ is a dominating of G. If not, then it follows that $y_2u_3 \in E(G)$. If $y_1y_3 \in E(G)$, then $\{y_1, y_2\}$ is a dominating of G. If $y_3u_1 \in E(G)$, then $\{y_2, u_1\}$ is a dominating of G. If $y_1y_3, y_3u_1 \notin E(G)$, then we conclude that $y_3u_2 \in E(G)$ and $\{y_2, u_2\}$ is a dominating set of G.

Case 4.4.2.1.2. Assume that $y_3y_4, y_2y_4 \notin E(G)$. This yields that $y_4u_1, y_4u_2, y_4u_3 \in E(G)$. If $y_2u_2 \in E(G)$, then $\{y_3, u_2\}$ is a dominating of G. If $y_3u_2 \in E(G)$, then $\{y_2, u_2\}$ is a dominating of G. Hence we consider now the case that $y_2u_2, y_3u_2 \notin E(G)$.

Case 4.4.2.1.2.1. Let $y_2u_1 \in E(G)$. If $y_3u_1 \in E(G)$, then $\{y_4, u_1\}$ is a dominating of G. If $y_3u_3 \in E(G)$, then $\{y_2, u_3\}$ is a dominating of G. If not, then $y_3y_2, y_3y_1 \in E(G)$ and $\{y_3, y_4\}$ is a dominating set of G.

Case 4.4.2.1.2.2. Assume that $y_2u_1 \notin E(G)$. If $y_1y_2 \in E(G)$ and $y_2y_3 \in E(G)$, then $\{y_2, u_2\}$ is a dominating set of G. If not, then we see that $y_2u_3 \in E(G)$. If $y_2y_3 \in E(G)$, then $\{y_2, u_1\}$ is a dominating of G. Thus assume that $y_2y_3 \notin E(G)$. This leads to $y_1y_2 \in E(G)$. If $y_3u_3 \in E(G)$, then $\{y_1, u_3\}$ is a dominating of G. If $y_3u_3 \notin E(G)$, then $y_1y_3 \in E(G)$ and $\{y_1, u_1\}$ is a dominating of G.

Case 4.4.2.2. Assume that y_i has at most two neighbors in R for $i = 1, 2, 3, 4$. This implies that each y_i has exactly two neighbors in R for $i = 1, 2, 3, 4$ and that each u_j has exactly two neighbors in $N(x)$ for $j = 1, 2, 3, 4$. Hence there are at least two edges in $G[N(x)]$. If a vertex y_i, say y_1, is adjacent to y_2, y_3, y_4 and, without loss of generality, $y_1u_1 \in E(G)$, then $\{y_1, u_3\}$ is a dominating set of G. Otherwise there exist two independent edges, say y_1y_2 and y_3y_4, in $G[N(x)]$. If the neighbors of y_i in R are all independent for $i = 1, 2, 3, 4$, then it is easy to that (2) is valid. Thus assume, without loss of generality, that $y_1u_1, y_1u_2 \in E(G)$. If $y_2u_i \in E(G)$ for any $i = 1, 2$, then $y_3u_3, y_3u_4 \in E(G)$ or $y_4u_3, y_4u_4 \in E(G)$, say $y_3u_3, y_3u_4 \in E(G)$. It follows that $\{y_1, y_3\}$ is a dominating set of G. Therefore it remains the case that, without loss of generality, $y_2u_3, y_2u_4 \in E(G)$. If $y_3u_1, y_3u_2 \in E(G)$, then $\{y_2, y_3\}$ is a dominating of G. If $y_3u_3, y_3u_4 \in E(G)$, then $\{y_1, y_3\}$ is a dominating of G. In the remaining cases that, without loss of generality, $y_3u_1, y_3u_4, y_4u_2, y_4u_3 \in E(G)$ or $y_3u_1, y_3u_3, y_4u_2, y_4u_4 \in E(G)$, we arrive at F_8 or F_7, respectively. If there exists a further edge, say y_2y_3 in $G[H(x)]$, then $\gamma = 2$.

Case 4.4.3. Let $\delta = 6$. If $V(R) \subseteq N(y_i)$ for any $1 \leq i \leq 6$, then $\{x, y_i\}$ is a dominating set of G and we are done. Thus we only discuss in the following the case that $1 \leq |N(y_i) \cap V(R)| \leq 3$ for all $1 \leq i \leq 6$.

Case 4.4.3.1. Assume that there exists a vertex y_i, say y_1, such that $|N(y_1) \cap V(R)| = 1$. This implies that y_1 has exactly one neighbor, say u_1, in R and at least for neighbors, say y_2, y_3, y_4, y_5, in $N(x)$. In addition, we observe that $|N(y_i) \cap V(R)| = 3$ for all $2 \leq i \leq 6$ and $|N(u_j) \cap N(x)| = 4$ for all $1 \leq j \leq 4$. If $y_1y_6 \in E(G)$ or $y_6u_3 \in E(G)$, then $\{y_1, u_3\}$ is a dominating set of G. So assume that $y_1y_6, y_6u_3 \notin E(G)$. This leads to $u_1, u_2, u_4 \in N(y_6)$ and $y_2, y_3, y_4, y_5 \in N(u_3)$. Now let, without loss of generality, $y_2u_1, y_3u_1 \in E(G)$. We conclude that $N(y_i) \cap V(R) = \{u_2, u_3, u_4\}$ for $i = 4, 5$. If $y_2y_6 \in E(G)$, then $\{y_2, u_3\}$ is a dominating set of G. If $y_3y_6 \in E(G)$, then $\{y_3, u_3\}$ is a dominating set of G. If not, then it follows that $y_4y_6 \in E(G)$ and hence $\{y_1, y_4\}$ a dominating set of G.

Case 4.4.3.2. Assume that $|N(y_i) \cap V(R)| \geq 2$ for all $1 \leq i \leq 6$. Because of the investigations above, we only consider the case that four vertices of $N(x)$ have exactly three neighbors and two vertices of $N(x)$ have exactly two neighbors in R. Consequently, each vertex of R has exactly four neighbors in $N(x)$ and $H = G[N(x)]$ is a graph with $\delta(H) \geq 2$, and there are at least two vertices of degree at least three in H. Hence H is connected and not a tree. In the following we investigate the cases that the longest cycle in H has length 6, 5, 4, or 3

Case 4.4.3.2.1. Assume that the subgraph H has the Hamiltonian cycle $y_1y_2y_3y_4y_4y_5y_6y_1$. Now we distinguish the two possibilities that $y_1y_3 \in E(G)$ or $y_1y_4 \in E(G)$.

Case 4.4.3.2.1.1. Let $y_1y_3 \in E(G)$.

Firstly, assume that y_1 has exactly two adjacent neighbors, say u_1 and u_2, in R. If $\{u_3, u_4\} \subseteq N(y_i)$ for $i = 4, 5$, then $\{y_1, y_i\}$ is a dominating set of G. Hence, we can assume that $y_4u_1, y_4u_2, y_5u_1, y_5u_2 \in E(G)$ and, without loss of generality, that $y_4u_4 \in E(G)$ and $y_4u_3 \notin E(G)$. This leads to $y_5u_3 \in E(G)$ and $y_5u_4 \notin E(G)$. Therefore $y_3u_4 \in E(G)$ and $\{y_3, y_5\}$ is a dominating set of G.

Secondly, assume that y_1 has exactly two non-adjacent neighbors, say u_1 and u_3, in R. If $\{u_2, u_4\} \subseteq N(y_i)$ for $i = 4, 5$, then $\{y_1, y_i\}$ is a dominating set of G. Hence, we can assume that $y_4u_1, y_4u_3, y_5u_1, y_5u_3 \in E(G)$ and, without loss of generality, $y_4u_2 \in E(G)$. This leads to $y_5u_4 \in E(G)$. If $y_3u_2 \in E(G)$, then $\{y_3, y_5\}$ is a dominating set of G. If not, then $y_2, y_4, y_5, y_6 \in N(u_2)$, and $\{x, y_5\}$ is a dominating set of G.

Case 4.4.3.2.1.1. Let $y_1y_4 \in E(G)$.

Firstly, assume that y_1 has exactly two adjacent neighbors, say u_1 and u_2, in R. If $\{u_3, u_4\} \subset N(y_4)$, then $\{y_1, y_4\}$ is a dominating set of G. Hence, we can assume, without loss of generality, that $y_4u_3 \notin E(G)$. This leads to $y_2, y_3, y_5, y_6 \in N(u_3)$, and $\{y_1, u_3\}$ is a dominating set of G.

Secondly, assume that y_1 has exactly two non-adjacent neighbors, say u_1 and u_3, in R. If $\{u_2, u_4\} \subset N(y_4)$, then $\{y_1, y_4\}$ is a dominating set of G. Hence, we can assume, without loss of generality, that $y_4u_2 \notin E(G)$. This leads to $y_2, y_3, y_5, y_6 \in N(u_2)$. If $y_4u_4 \in E(G)$, then $\{y_4, u_2\}$ is a dominating set of G. If not, then we deduce that $y_2, y_3, y_5, y_6 \in N(u_4)$ and $u_1, u_3 \in N(y_4)$. Next assume, without loss of generality, that $y_2u_1 \in E(G)$. If $y_5u_3 \in E(G)$, then $\{y_2, y_5\}$ is a dominating set of G. Therefore it remains the case that $y_5u_1 \in E(G)$ and thus $y_3u_3, y_6u_3 \in E(G)$. However, this is the exceptional graph F_9.

Case 4.4.3.2.2. Assume that the longest cycle of H has length 5. Then H consists, without loss of generality, of the path $y_1y_2y_3y_4y_5y_6$ and the edges y_1y_5 and y_3y_6.

Firstly, assume that y_3 has exactly two adjacent neighbors, say u_1 and u_2, in R. If $\{u_3, u_4\} \subseteq N(y_1)$, then $\{y_1, y_3\}$ is a dominating set of G. Hence, we can assume, without loss of generality, that $y_1u_3 \notin E(G)$. This leads to $y_2, y_4, y_5, y_6 \in N(u_3)$ and $u_1, u_2, u_4 \in N(y_1)$. If $y_2u_1 \in E(G)$, then $\{y_5, u_1\}$ is a dominating set of G. If not, then we observe that $u_2, u_3, u_4 \in N(y_2)$. If $y_5u_1 \in E(G)$, then $\{y_2, y_5\}$ is a dominating set of G. If not, then we have $y_1, y_3, y_4, y_5 \in N(u_1)$. If $y_5u_4 \in E(G)$, then $\{y_5, u_2\}$ is a dominating set of G. If not, then we have $y_1, y_2, y_4, y_6 \in N(u_4)$. This implies that $y_5u_2 \in E(G)$, and this is again F_9.

Secondly, assume that y_3 has exactly two non-adjacent neighbors, say u_1 and u_3, in R. If $\{u_2, u_4\} \subset N(y_1)$, then $\{y_1, y_3\}$ is a dominating set of G. Hence, we can assume, without loss of generality, that $y_1u_4 \notin E(G)$. This leads to $y_2, y_4, y_5, y_6 \in N(u_4)$. If $y_2u_2 \in E(G)$, then $\{y_2, u_4\}$ is a dominating set of G. If not, then we deduce that $y_1, y_4, y_5, y_6 \in N(u_2)$ and $\{y_2, u_2\}$ is a dominating set of G.

Case 4.4.3.2.3. Assume that the longest cycle of H has length 3 or 4. Since length 4 is impossible, it remains the case that the longest cycle of H has length 3. Then H consists, without loss of generality, of the path $y_1y_2y_3y_4y_5y_6$ and the edges y_1y_3 and y_4y_6. Assume, without loss of generality, that y_1 has exactly the neighbors u_1, u_2, u_3 in R. It follows that $y_4u_4 \in E(G)$ or $y_5u_4 \in E(G)$, say $y_5u_4 \in E(G)$. This implies that $\{y_1, y_5\}$ is a dominating set of G.

Case 5. Let $|I| = 0$ and assume that R consists of a least two components. If $\gamma(R) \leq (|V(R)| - 1)/2$, then Proposition 2.1 leads to (2). If not, then, in view of Theorem 2.2, the components of G consist of C_4 or the corona graph $H \circ K_1$ for any connected graph H. If one of the components is of the form $H \circ K_1$ with $|V(H)| \geq 3$, then Case 4.1 together with Proposition 2.1 show the desired inequality (2).

Case 5.1. The graph R contains the components $K_2 = w_1w_2$ and the path $v_1u_1u_2v_2$. We define the subgraph

$$Q = G[N[x] \cup \{u_1, u_2, v_1, v_2, w_1, w_2\}].$$

Because of Proposition 2.1, it is enough to show that $\gamma(Q) \leq 3$.

In the case $\delta \geq 3$, the vertices w_1 and w_2 have a common neighbor, say y_1, in $N(x)$. If there exists an edge y_1v_1 in G, then $\{x, y_1, u_2\}$ is a dominating set of Q. Otherwise, $y_2, y_3, \ldots, y_\delta \in N(v_1)$ and $\{y_1, v_1, v_2\}$ is a dominating set of Q. If $\delta = 2$, then it is easy to see that $\gamma(Q) \leq 3$.

Case 5.2. The graph R contains the components $K_2 = w_1w_2$ and the cycle $u_1u_2u_3u_4u_1$. If $Q = G[N[x] \cup \{u_1, u_2, u_3, u_4, w_1, w_2\}]$, then it is enough to show that $\gamma(Q) \leq 3$.

In the case $\delta \geq 3$, the vertices w_1 and w_2 have at least $\delta - 2$ common neighbors, say $y_1, y_2, \ldots, y_{\delta-2}$, in $N(x)$. If there exists an edge y_iu_j for $1 \leq i \leq \delta - 2$ and $1 \leq j \leq 4$, say y_1u_1 in G, then $\{x, y_1, u_3\}$ is a dominating set of Q. Otherwise, we deduce that $\delta \leq 4$. If $\delta = 4$, then $u_1, u_2, u_3, u_4 \in N(y_4)$ and $\{x, y_1, y_4\}$ is a dominating set of Q. If $\delta = 3$, then $\{y_1, y_2, y_3\}$ is a dominating set of Q. If $\delta = 2$, then it is easy to see that $\gamma(Q) \leq 3$.

Case 5.3. The graph R contains the components $K_2 = w_1w_2$ and $K_2 = u_1u_2$. If $Q = G[N[x] \cup \{u_1, u_2, w_1, w_2\}]$, then it is enough to show that $\gamma(Q) \leq 2$. If $\delta \geq 5$, then there exists a vertex $y_i \in N(x)$ such that $u_1, u_2, w_1, w_2 \subset N(y_i)$ and $\{x, y_i\}$ is a dominating set of Q.

Next let $\delta = 4$. If there exists a vertex $y_i \in N(x)$ with the property that $u_1, u_2, w_1, w_2 \subset N(y_i)$, then $\{x, y_i\}$ is a dominating set of Q. If not, then $|N(y_i) \cap \{u_1, u_2, w_1, w_2\}| = 3$ for every $1 \leq i \leq 4$. Let, without loss of generality $u_1, u_2, w_1 \in N(y_1)$. Since $y_1 w_2 \notin E(G)$, it follows that $y_2, y_3, y_4 \in N(w_2)$ and $\{y_1, w_2\}$ is a dominating set of Q. If $2 \leq \delta \leq 3$, then it is straightforward to verify that $\gamma(Q) \leq 2$.

Case 5.4. The graph R contains two paths $P_1 = u_1u_2u_3u_4$ and $P_2 = v_1v_2v_3v_4$ as components. If $Q = G[N[x] \cup V(P_1) \cup V(P_2)]$, then it is enough to show that $\gamma(Q) \leq 4$. If $\delta \geq 3$, then u_1 and v_1 have a common neighbor, say y_1, in $N(x)$, and $\{x, y_1, u_3, v_3\}$ is a dominating set of Q. If $\delta = 2$, then it is easy to see that $\gamma(Q) \leq 4$.

Case 5.5. The graph R contains a path $P = u_1u_2u_3u_4$ and a cycle $C = v_1v_2v_3v_4v_1$ as components. If $Q = G[N[x] \cup V(P) \cup V(C)]$, then it is enough to show that $\gamma(Q) \leq 4$. If $\delta \geq 4$, then u_1 and v_1 have a common neighbor, say y_1, in $N(x)$, and $\{x, y_1, u_3, v_3\}$ is a dominating set of Q. If $\delta = 3$, then $\{y_1, y_2, y_3\}$ is a dominating of Q. If $\delta = 2$, then it is easy to see that $\gamma(Q) \leq 4$.

Case 5.6. The graph R contains two cycles $C = u_1u_2u_3u_4u_1$ and $C' = v_1v_2v_3v_4v_1$ as components. If $Q = G[N[x] \cup V(C) \cup V(C')]$, then it is enough to show that $\gamma(Q) \leq 4$. If any u_i, say u_1, and any v_j, say v_1, have a common neighbor, say y_1, in $N(x)$, then $\{x, y_1, u_3, v_3\}$ is a dominating set of Q. Otherwise, we conclude that $\delta \leq 4$. However, if $\delta = 4$ or $\delta = 3$, then $\{y_1, y_2, y_3, y_4\}$ or $\{y_1, y_2, y_3\}$ is a dominating set of Q, respectively. If $\delta = 2$, then it is easy to see that $\gamma(Q) \leq 4$. □

Proposition 2.1 and Theorem 2.2 show that the corresponding result to Theorem 4.1 for $\delta = 1$ has the following form.

Observation 4.2 Let G be a connected graph of order n, minimum degree $\delta = 1$, and domination number γ. Then

$$\gamma \leq \frac{n - \delta}{2} = \frac{n - 1}{2},$$

with exception that G is a corona graph $H \circ K_1$ for any connected graph H.

References

[1] W.E. Clark and L.A. Dunning, Tight upper bounds for the domination numbers of graphs with given order and minimum degree, *Elec. J. Combin.* 4 1997, No. R26.

[2] J.F. Fink, M.S. Jacobson, L.F. Kinch and J. Roberts, On graphs having domination number half their order, *Period. Math. Hungar.* **16** (1985), 287 – 293.

[3] P. Flach and L. Volkmann, Estimations for the domination number of a graph, *Discrete Math.* **80** (1990), 145-151.

[4] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York, 1998.

[5] O. Ore, Theory of Graphs, Amer. Math. Soc. Colloq. Publ. **38**, 1962.

[6] C. Payan, Sur le nombre d'absorption d'un graph simple, *Cahiers Centre Études Rech. Opér.* **17** (1975), 307-317.

[7] C. Payan and N.H. Xuong, Domination-balanced graphs. *J. Graph Theory* **6** (1982), 23 – 32.

[8] B. Randerath and L. Volkmann, Characterization of graphs with equal domination and covering number, *Discrete Math.* **191** (1998), 159-169.

[9] L. Volkmann, Foundations of Graph Theory, Springer-Verlag, Wien New York, 1996 (in German).

[10] B. Xu, E.J. Cockayne, T.W. Haynes, S.T. Hedetniemi, and S. Zhou, Extremal graphs for inequalities involving domination parameters, *Discrete Math.* **216** (2000), 1-10.