The Fractional Vertex Linear Arboricity of Graphs

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Abstract

The vertex linear arboricity $vla(G)$ of a graph G is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that each subset induces a subgraph whose connected components are paths. In this paper, we seek to convert vertex linear arboricity into its fractional analogues, i.e., the fractional vertex linear arboricity of graphs. Let Z_n denote the additive group of integers modulo n. Suppose that $C \subseteq Z_n \setminus 0$ has the additional property that it is closed under additive inverse, that is, $-c \in C$ if and only if $c \in C$. A *circulant graph* is the graph $G(Z_n, C)$ with the vertex set Z_n and i, j are adjacent if and only if $i - j \in C$. The fractional vertex linear arboricity of the complete n-partite graph, the cycle C_n, the integer distance graph $G(D)$ for $D = \{1, 2, \cdots, m\}$, $D = \{2, 3, \cdots, m\}$ and $D = P$ the set of all prime numbers, the Petersen graph and the circulant graph $G_{a,b} = G(Z_a, C)$ with $C = \{-a+b, \cdots, -b, b, \cdots, a-b\}$ $(a-2b \geq b-3 \geq 3)$ are determined, and an upper and a lower bounds of the fractional vertex linear arboricity of Mycielski graph are obtained.

Keywords: Fractional vertex linear arboricity; integer distance graph; complete $n-$partite graph; Petersen graph; circulant graph $G_{a,b}$

1 Introduction

In this paper, R and Z denote the set of all real numbers and all integers, respectively. For $x \in R$, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x; $\lceil x \rceil$ denotes the least integer not less than x. For a finite set S, $|S|$ denotes the cardinality of S. If H is a subgraph of G, then G is called a *supergraph* of H (see [3]).

A *k-coloring* of a graph G is a mapping f from $V(G)$ to $\{1, 2, ..., k\}$. With respect to a given $k-$coloring, V_i denotes the set of all vertices of G colored with i.

If V_i is an independent set for every $1 \leq i \leq k$, then f is called a proper $k-$coloring. The chromatic number $\chi(G)$ of a graph G is the minimum number k of colors for which G has a proper $k-$coloring. If V_i induces a subgraph whose connected components are paths, then f is called a *path $k-$coloring*. The *vertex linear arboricity* of a graph G, denoted by $vla(G)$, is the minimum number k of colors for which G has a path $k-$coloring.

Matsumoto [10] proved that for any finite graph G, $vla(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$, moreover, if $\Delta(G)$ is even, then $vla(G) = \lceil \frac{\Delta(G)+1}{2} \rceil$ if and only if G is the complete graph of order $\Delta(G) + 1$ or a cycle. Goddard [8] and Poh [11] proved that $vla(G) \leq 3$ for a planar graph G. Akiyama *et al.* [1] proved $vla(G) \leq 2$ if G is an outerplanar graph. Fang and Wu [7] determined the vertex linear arboricity of complete multipartite graphs and obtained an upper bound on the vertex linear arboricity of cartesian product of graphs. Alavi *et al.* [2] proved that $vla(G) + vla(\overline{G}) \leq 1 + \lceil \frac{n+1}{2} \rceil$ for any graph G of order n where $\overline{G}$ is the complement of G.

In this paper, we seek to convert the vertex linear arboricity into its fractional analogues.

2 Main results and their proofs

A hypergraph H is a pair $(V(H), \chi)$, where $V(H)$ is a finite set and χ is a family of subsets of $V(H)$. The set $V(H)$ is called the vertex set of the hypergraph and the elements of χ are called hyperedges or sometimes just edges. A covering of H is a collection of hyperedges $L_1, L_2, \cdots, L_j$ such that $V \subseteq L_1 \cup \cdots \cup L_j$.

A graph G whose connected components are pathes is called a *linear forest.*

For any finite graph G, let LF be the set of all subsets of V that induce linear forests of G and V be the vertex set of G, then $H = (V, LF)$ is a hypergraph and the elements of LF are hyperedges.

An automorphism of a hypergraph H is a bijection $\pi : V(H) \longrightarrow V(H)$ with the property that X is a hyperedge if and only if $\pi(X)$ is a hyperedge as well. The set of all automorphisms of a hypergraph forms a group under the operation of composition; this group is called the automorphism group of the hypergraph. A hypergraph H is called vertex-transitive provided for every pair of vertices $u, v \in V(H)$ there exists an automorphism of H with $\pi(u) = v$(see[12]).

The vertex linear arboricity of a finite graph G can be formulated as an integer program. To each set $L_i \in LF$ associate a $0,1-$ variable x_i. The vector X is an indicator of the sets we have selected for the covering. Let M be the vertex-linear forest incidence matrix of G, i.e., the $0,1-$ matrix whose rows are indexed by $V(G)$, whose columns are indexed by LF, and whose $i,j-$entry is exactly 1 when $v_i \in L_j$. The condition that the indicator vector X corresponds to a covering is simply $MX \geq \mathbf{1}$ (that is, every coordinate of MX is at least 1). Hence the vertex linear arboricity of G is the value of the integer program

$$\min \mathbf{1}'X$$
$$subject\ to \begin{cases} MX \geq \mathbf{1}, \\ x_i = 0 \text{ or } 1, \\ i = 1, 2, \cdots, |LF|. \end{cases} \qquad (1)$$

The relaxation of the integer program (1) is the following linear program

$$\min \mathbf{1}'X$$
$$subject\ to \begin{cases} MX \geq \mathbf{1}, \\ 0 \leq x_i \leq 1, \\ i = 1, 2, \cdots, |LF|, \end{cases} \qquad (2)$$

and the value of (2) is called the fractional vertex linear arboricity of G. In other word, we can define the fractional vertex linear arboricity $vla_f(G)$ of any graph G as followings.

Definition 2.1. *A fractional path coloring of a graph G (can be infinite) is a mapping c from $LF(G)$, the set of all subsets of V that induce linear forests of G, to the interval $[0,1]$ such that $\sum_{x\in L\in LF(G)} c(L) \geq 1$ for all vertices x in G. The weight of a fractional path coloring is the sum of its values, and the fractional vertex linear arboricity of the graph G is the minimum possible weight of a fractional path coloring, that is,*

$$vla_f(G) = \min\{ \sum_{L\in LF(G)} c(L) \mid c \text{ is a fractional path coloring of } G\}.$$

Clearly, we have $vla_f(H) \leq vla_f(G)$ for any subgraph H of G.

If f is a path $vla(G)-$coloring of G and $V_i = \{v|v \in V(G),\ f(v) = i\}$ $(1 \leq i \leq vla(G))$, then we can give a mapping c: $LF \longrightarrow [0,1]$ by

$$c(L) = \begin{cases} 1, & \text{for } L = V_i,\ 1 \leq i \leq vla(G), \\ 0, & \text{otherwise}, \end{cases}$$

such that c is a fractional path coloring of G which has weight $vla(G)$. Therefore, it follows immediately that $vla_f(G) \leq vla(G)$.

Conversely, suppose that G has a $0,1-$valued fractional path coloring f of weight k. Then the support of f consists of k linear forests $V_1, V_2, \cdots, V_k$ whose union is $V(G)$. If we color a vertex v with the smallest i such that $v \in V_i$, then we have a path $k-$coloring of G. Thus the vertex linear arboricity of G is the minimum weight of a $0,1-$valued fractional path coloring.

The dual LP of (2) is the following linear program

$$\max \mathbf{1}'Y$$
$$\textit{subject to} \begin{cases} M'Y \leq \mathbf{1}, \\ 0 \leq y_i \leq 1, \\ i = 1, 2, \cdots, |V|. \end{cases} \qquad (3)$$

Thus if we define f to take the value $f(v)$ on each vertex of the vertex set V with $0 \le f(v) \le 1$ and $M'Y \le \mathbf{1}$ for $Y = (f(v_1), \cdots, f(v_n))'$ with $n = |V|$, then Y is a feasible solution of (3).

(2) and (3) form a dual pair. Suppose that ω is the value of the optimization problem (3), then $\omega \le vla_f(G)$ by the weak duality theorem from linear programming. Hence we have the following lemma.

Lemma 2.2. *Let G be a finite graph, $e = \max\{|X| : X \in LF\}$, then $vla_f(G) \ge \frac{|V(G)|}{e}$.*

Proof. If we assign each vertex of H weight $\frac{1}{e}$, then we have a feasible solution of (3). Thus $vla_f(G) \ge \frac{|V(G)|}{e}$. □

Therefore, $vla_f(G) \ge 1$ for any nonempty graph G.

Theorem 2.3. *For any complete $n-$partite graph $G = K(m_1, m_2, \cdots, m_n)$ $(n \ge 2)$,*

$$vla_f(G) = \begin{cases} n, & \text{for } m_1 = m_2 = \cdots = m_n = m \ge 3, \\ \frac{2n}{3}, & \text{for } m_1 = m_2 = \cdots = m_n = m = 2, \\ \frac{n}{2}, & \text{for } m_1 = m_2 = \cdots = m_n = 1, \\ n - \frac{2}{3}, & \text{for } m_1 = m_2 = \cdots = m_{n-1} = 3 \text{ and } m_n = 1, \\ n - \frac{1}{3}, & \text{for } m_1 = m_2 = \cdots = m_{n-1} = 3 \text{ and } m_n = 2. \end{cases}$$

Proof. Suppose that X_1, X_2, $\cdots$, X_n are n-partite of $V(G)$ such that $|X_i| = m_i$ for $1 \le i \le n$. Let $H = (V, LF)$ have $V = V(G)$ and LF the set of all subsets of V which induced linear forests of G.

(1) When $m \ge 3$, it is straight forward to verify that $e = \max\{|X| : X \in LF\} = m$. So $vla_f(G) \ge \frac{mn}{m} = n$ by Lemma 2.2. Define a mapping $h_1 : LF \longrightarrow [0, 1]$ by

$$h_1(X) = \begin{cases} 1, & \text{for } X = X_i,\ 1 \le i \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Then h_1 is a fractional path coloring of G which has weight n. So $vla_f(G) \le n$. Therefore $vla_f(G) = n$.

(2) When $m = 2$, it is straight forward to verify that $e = \max\{|X| : X \in LF\} = 3$. So $vla_f(G) \geq \frac{2n}{3}$. Define a mapping $h_2 : LF \to [0,1]$ by

$$h_2(X) = \begin{cases} \frac{1}{3(n-1)}, & \text{for } |X| = 3 \text{ and there are } (1 \leq) i < j (\leq n) \\ & \text{such that } X \subseteq X_i \cup X_j, \\ 0, & \text{otherwise.} \end{cases}$$

The number of all 3−linear forests that contain two elements of X_1 is $2(n-1)$ and the number of all 3−linear forests that contain one element of X_1 is $2(n-1)$. So there are $4(n-1)+4(n-2)+\cdots+8+4 = 2(n-1)n$ elements in LF that have value nonzero. Then h_2 is a fractional path coloring of G which has weight $\frac{1}{3(n-1)}2(n-1)n = \frac{2n}{3}$. Hence $vla_f(G) \leq \frac{2n}{3}$. Therefore $vla_f(G) = \frac{2n}{3}$.

(3) For $m_1 = m_2 = \cdots = m_n = 1$, define a mapping $h_3 : LF \to [0,1]$ by

$$h_3(X) = \begin{cases} \frac{1}{n-1}, & \text{if } |L| = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then h_3 is a fractional path coloring of G which has weight $\frac{n}{2}$. Thus $vla_f(G) \leq \frac{n}{2}$. It is straight forward to verify that $e = \max\{|X| : X \in LF\} = 2$, so $vla_f(G) \geq \frac{|V(G)|}{e} = \frac{n}{2}$. Hence, $vla_f(G) = \frac{n}{2}$.

(4) For $m_1 = \cdots = m_{n-1} = 3$ and $m_n = 1$, it is easy to prove that $e = \max\{|X| : X \in LF\} = 3$, then $vla_f(G) \geq \frac{|V(G)|}{e} = n - 1 + \frac{1}{3} = n - \frac{2}{3}$. Let $X_n = \{v\}$. There are $C_3^2(n-1) = 3(n-1)$ members in LF, assuming them to form T_1, that contain v and have cardinality 3, and $1 + C_3^2 3(n-2) + 3(n-2)C_3^2 + 1 + C_3^2 3(n-3) + 3(n-3)C_3^2 + \cdots + 1 + C_3^2 3 + 3C_3^2 + 1 = 1 + 18(n-2) + 1 + 18(n-3) + 1 + \cdots + 18 + 1 = (n-1)(9n-17)$ members in LF, assuming them to form T_2, that have cardinality 3 and do not contain v. Every vertex of $X_i (1 \leq i \leq n-1)$ is contained in two members of T_1 and $C_3^2(n-2) + 2C_3^1(n-2) + 1 = 9(n-2) + 1$ members of T_2 (the first part in the sum is the number of members that contain one element of X_i and the second part in the sum is the number of members that contain two

elements of X_i). Define a mapping $h_4 : LF \rightarrow [0,1]$ by

$$h_4(X) = \begin{cases} \frac{1}{3(n-1)}, & \text{when } X_n \subseteq X \text{ and } |X| = 3, \\ \frac{3n-5}{3(n-1)[9(n-2)+1]}, & \text{when } X_n \bigcap X = \phi \text{ and } |X| = 3, \\ 0, & \text{else.} \end{cases}$$

Then h_4 is a fractional path coloring of G which has weight $3(n-1)\frac{1}{3(n-1)} + (n-1)(9n-17)\frac{3n-5}{3(n-1)(9n-17)} = 1 + \frac{3n-5}{3} = n-1+\frac{1}{3}$, so $vla_f(G) \leq n-1+\frac{1}{3}$. Hence $vla_f(G) = n-1+\frac{1}{3} = n-\frac{2}{3}$.

(5) Let $|X_n| = 2$. There are $C_3^1(n-1) + 2C_3^2(n-1) = 9(n-1)$ members of LF, assuming them to form H_1, that contain vertices of X_n and have cardinality 3, and $C_3^1C_3^2(n-2) + C_3^2C_3^1(n-2) + 1 + C_3^1C_3^2(n-3) + C_3^2C_3^1(n-3) + 1 + \cdots + C_3^1C_3^2 + C_3^2C_3^1 + 1 + 1 = 18(n-2) + 1 + 18(n-3) + 1 + \cdots + 18 + 1 + 1 = (n-1)(9n-17)$ members of LF, assuming them to form H_2, that do not contain vertices of X_n and have cardinality 3. Every vertex of X_n is contained in $C_3^1(n-1) + C_3^2(n-1) = 6(n-1)$ members of H_1 and every vertex of $X_i (1 \leq i \leq n-1)$ is contained in $1+2+2 = 5$ members of H_1 and $C_3^2(n-2) + 2C_3^1(n-2) + 1 = 9(n-2) + 1$ members in H_2. Define a mapping h_5 by

$$h_5(X) = \begin{cases} \frac{1}{6(n-1)}, & \text{for } X \in H_1, \\ \frac{6n-11}{6(n-1)[9(n-2)+1]}, & \text{for } X \in H_2, \\ 0, & \text{else.} \end{cases}$$

Then h_5 is a fractional path coloring of G which has weight $9(n-1)\frac{1}{6(n-1)} + (n-1)(9n-17)\frac{6n-11}{6(n-1)[9(n-2)+1]} = n - \frac{1}{3}$. Thus $vla_f(G) \leq n - \frac{1}{3}$. It is obvious that $e = \max\{|X| : X \in LF\} = 3$, and then $vla_f(G) \geq \frac{|V(G)|}{e} = n - \frac{1}{3}$. Therefore $vla_f(G) = n - \frac{1}{3}$. □

In these cases, we have $vla(G) = \lceil vla_f(G) \rceil$. For example, in (2) of Theorem 2.3, any four vertices induce a cycle, so that $vla(G) = \lceil \frac{2n}{3} \rceil = \lceil vla_f(G) \rceil$. In (5) of Theorem 2.3, it is obvious that $vla(G) = n$ since any four vertices induce a $K_{1,3}$ or a cycle, so that $vla(G) = n = \lceil n - \frac{1}{3} \rceil = \lceil vla_f(G) \rceil$.

Theorem 2.4. $vla_f(C_n) = \frac{n}{n-1}$.

Proof. Suppose that $C_n = a_1a_2\cdots a_na_1$. Let $L_i = a_ia_{i+1}\cdots a_{i+n-2}$ which subscripts with addition modulo n and $1 \le i \le n$. It is obvious that every a_j is contained in exactly $n-1$ paths $L_1, \cdots, L_j, L_{j+2}, \cdots, L_n$. Define a mapping $c : LF \to [0,1]$ by

$$c(L) = \begin{cases} \frac{1}{n-1}, & \text{if } L = L_j, j = 0, 1, \cdots, n, \\ 0, & \text{otherwise.} \end{cases}$$

Then c is a fractional path coloring of C_n which has weight $\Sigma_{L\in LF(C_n)}c(L) = \frac{n}{n-1}$, so $vla_f(C_n) \le \frac{n}{n-1}$. Clearly, the length of the longest induced path in C_n is $n-1$, hence $vla_f(C_n) \ge \frac{n}{n-1}$. Therefore $vla_f(C_n) = \frac{n}{n-1}$. □

Clearly, $vla(C_n) = 2 = \lceil vla_f(C_n) \rceil$.

If S is a subset of the set of real numbers and D is a subset of the set of positive real numbers, then the *distance graph* $G(S, D)$ is defined by the graph G with vertex set $V(G) = S$ and two vertices x and y are adjacent if and only if $|x - y| \in D$ where the set D is called the *distance set*. In particular, if all elements of D are positive integers and $S = Z$, the set of all integers, then the graph $G(Z, D) = G(D)$ is called *integer distance graph* and the set D is called the integer distance set of the graph. For the vertex linear arboricity of distance graphs, Zuo, Wu and Liu [14] obtained that $vla(G(R, D)) = n + 1$ if D is an interval between 1 and δ when $1 \le n - 1 < \delta \le n$, $vla(G(D)) = 2$ if $|D| \ge 2$ and D has at most one even number and $vla(G(D)) \le k$ if there is unique multiple of k in D. Moreover, $vla(G(P)) = 2$ where P is the set of all prime numbers.

It was proved that $vla(G(D)) = \lceil \frac{m+1}{2} \rceil$ for $D = \{1, 2, \cdots, m\}$ in [14] and $vla(G(D_{m,1})) = \lceil \frac{m}{4} \rceil + 1$ for $D_{m,1} = \{2, \cdots, m\}$ and $m \ge 3$ in [15].

Now we study the fractional vertex linear arboricity of $G(D)$ for $D_1 = \{1, 2, \cdots, m\}$, $D_2 = D_{m,1}$ and $D_3 = P$ the set of all prime numbers, respectively.

Theorem 2.5. (1) *For* $D_1 = \{1, 2, \cdots, m\}$, $vla_f(G(D_1)) = \frac{m+1}{2}$.

(2) *For* $D_{m,1} = \{2, 3, \cdots, m\}$ *and* $m \geq 5$, $\frac{m+3}{4} \leq vla_f(G(D_{m,1})) \leq \frac{m}{4} + 1$.

(3) $vla_f(G(P)) = 2$ *where* P *is the set of all prime numbers.*

Proof. (1) Let

$$\begin{aligned}
L_0 &= \{\cdots, 0, 1, m+1, m+2, 2(m+1), 2(m+1)+1, \cdots\}, \\
L_1 &= \{\cdots, 1, 2, m+2, m+3, 2(m+1)+1, 2(m+1)+2, \cdots\}, \\
L_2 &= \{\cdots, 2, 3, m+3, m+4, 2(m+1)+2, 2(m+1)+3, \cdots\}, \\
&\vdots \\
L_{m-1} &= \{\cdots, -2, -1, m-1, m, 2m, 2m+1, 3m+1, 3m+2, \cdots\}, \\
L_m &= \{\cdots, -1, 0, m, m+1, 2m+1, 2m+2, 2(m+1)+m, 3(m+1), \cdots\}.
\end{aligned}$$

Then each of $L_0, L_1, \cdots, L_m$ induces a linear forest and every $i \in Z$ is contained in exactly two $L_j (0 \leq j \leq m)$. Define a mapping $c : LF \to [0, 1]$ by

$$c(L) = \begin{cases} \frac{1}{2}, & \text{if } L = L_j, j = 0, 1, \cdots, m, \\ 0, & \text{otherwise.} \end{cases}$$

Then c is a fractional path coloring of $G(D_1)$ which has weight

$$\Sigma_{L \in LF(G(D_1))} c(L) = \frac{m+1}{2},$$

so that $vla_f(G(D_1)) \leq \frac{m+1}{2}$.

Let H be a subgraph induced by vertices $0, 1, \cdots, m$. Then $H = K_{m+1}$ is a complete graph and so that $vla_f(G(D_1)) \geq vla_f(H) = \frac{m+1}{2}$ by Theorem 2.3. Therefore, $vla_f(G(D_1)) = \frac{m+1}{2}$.

(2) For any i with $0 \leq i \leq m+3$, let

$$L'_i = \{j \in Z : j - i \equiv x (\text{mod } m+4), 0 \leq x \leq 3\}.$$

It is straightforward to verify that L'_i induces a linear forest in $G(D_{m,1})$. It is not difficult to verify that any integer is contained in exactly four such linear forests. Define a mapping $h : LF(G(D_{m,1})) \to [0, 1]$ by

$$h(L) = \begin{cases} \frac{1}{4}, & \text{if } L = L'_j, 0 \leq j \leq m+3, \\ 0, & \text{otherwise.} \end{cases}$$

Then h is a fractional path coloring of $G(D_{m,1})$ which has weight $\frac{m+4}{4} = \frac{m}{4}+1$. Thus, $vla_f(G(D_{m,1}) \leq \frac{m}{4}+1$.

Let G be the subgraph of $G(D_{m,1})$ induced by the vertices $\{0, 1, \cdots, m+2\}$. Then $vla_f(G(D_{m,1})) \geq vla_f(G)$. If there are five vertices $0 \leq a_0 < a_1 < \cdots < a_4 \leq m+2$ in an $L \in LF(G)$, then $a_3 - a_0 > m$ and $a_4 - a_1 > m$ by the proof of Theorem 2.2 in [15]. Thus $a_0 = 0$, $a_1 = 1$, $a_3 = m+1$ and $a_4 = m+2$. Clearly, $a_0a_2, a_2a_4 \in E(H)$, so $a_1a_2, a_2a_3 \notin E(H)$, i.e., $a_3 - a_2 = a_2 - a_1 = 1$, and then $a_3 - a_1 = m = 2$ which is contrary to the assumption. Hence, $e = \max\{|L| : L \subseteq V(G)$ and L induces a linear forest of $G\} = 4$. Therefore, by Lemma 2.2, $vla_f(G(D_{m,1})) \geq \frac{m+3}{4}$.

(3) Let $\overline{L_i} = \{n | n \equiv i(\text{mod}2), n \in Z\}, i = 0, 1$, then $\overline{L_i}$ induces a linear forest. It is obvious that every integer is contained in exactly one of these linear forests. Define a mapping $c : LF \to [0, 1]$ by

$$c(L) = \begin{cases} 1, & \text{if } L = \overline{L_i}, i = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then c is a fractional path coloring which has weight 2. So $vla_f(G(P)) \leq 2$. Suppose that H is the subgraph induced by vertices $0, 1, 2, \cdots, 7$. It is straightforward to verify that

$$e = \max\{|L| : L \subseteq V(H) \text{ and } L \text{ induces a linear forest of } H\} = 4$$

and the vertex subset $\{0, 2, 4, 6\}$ induces a path. So $vla_f(H) \geq \frac{8}{4} = 2$. Hence $vla_f(G(P)) = 2$. □

Clearly, $vla(G(D_1)) = \lceil vla_f(G(D_1)) \rceil$ by [14] and $\lceil vla_f(G(D_{m,1})) \rceil = vla(G(D_{m,1})$ when $m \equiv i(\text{mod}4)$ for $i \neq 1$ by [15].

Mycielski graph is an important graph in vertex coloring. Given a graph G, define the graph $Y(G)$ as follows: $V(Y(G)) = (V(G) \times \{1, 2\}) \bigcup \{z\}$ and with an edge between two vertices of $Y(G)$ if and only if

(1) one of them is z and the other is $(v, 2)$ for some $v \in V(G)$, or

(2) one of them is $(v, 1)$ and the other is $(w, 1)$ where $vw \in E(G)$, or

(3) one of them is $(v, 1)$ and the other is $(w, 2)$ where $vw \in E(G)$.

The Grötzsch graph is $Y(C_5)$ and $C_5 = Y(K_2)$. Mycielski proved that $\chi(Y(G)) = \chi(G) + 1$ for any graph G with at least one edge. For the (fractional) vertex linear arboricity, we have the following result.

Theorem 2.6. *If G is a graph with at least one edge, then*

(1) $vla(G) \leq vla(Y(G)) \leq vla(G) + 1$. *In particular,* $vla(Y(C_5)) = vla(C_5)$ *and* $vla(Y(K_2)) = vla(K_2) + 1$.

(2) $vla_f(G) \leq vla_f(Y(G)) \leq vla_f(G) + 1$.

Proof. (1) The first inequality is trivial. Suppose that $vla(G) = m$ and $V_i (1 \leq i \leq m)$ is a linear forest partition of G. Let $W_{m+1} = \{(v, 2) | v \in V(G)\}$, $W_1 = \{z\} \bigcup \{(v, 1) | v \in V_1\}$ and $W_i = \{(v, 1) | v \in V_i\}$ for $2 \leq i \leq m$. It is clear that every $W_i (1 \leq i \leq m + 1)$ induces a linear forest. So that $vla(Y(G)) \leq vla(G) + 1$.

It is obvious that $vla(Y(K_2)) = vla(K_2) + 1$ because of $C_5 = Y(K_2)$. Let $V(C_5) = \{v_1, v_2, v_3, v_4, v_5\}$, $U_1 = \{(v_i, 1) | 1 \leq i \leq 4\} \bigcup \{z\}$ and $U_2 = \{(v_i, 2) | 1 \leq i \leq 5\} \bigcup \{(v_5, 1)\}$. It is not difficult to verify that $U_i (i = 1, 2)$ induce linear forests. So $vla(Y(C_5)) = vla(C_5) = 2$.

(2) The first inequality is trivial, too. Suppose that c is a fractional path coloring of G. Let $c_1 : LF(Y(G)) \to [0, 1]$ such that

$$c_1(L) = \begin{cases} c(L_1), & \text{for } L = \{(v, 1) | v \in L_1 \in LF(G)\} \cup \{z\} \\ 1, & \text{for } L = \{(v, 2) | v \in V(G)\} \\ 0, & \text{else.} \end{cases}$$

Then c_1 is a fractional path coloring of $Y(G)$ which has weight $vla_f(G) + 1$, so $vla_f(Y(G)) \leq vla_f(G) + 1$. □

The Petersen graph P is a graph with vertex set $V = \{a, b, c, d, e, a_1, b_1, c_1, d_1, e_1\}$ and the edge set $E = \{ab, bc, cd, de, ea, aa_1, bb_1, cc_1, dd_1, ee_1, a_1c_1, a_1d_1, b_1d_1, b_1e_1, c_1e_1\}$. We have the following result.

Theorem 2.7. $vla_f(P) = \frac{5}{3}$.

Proof. It is not difficult to verify that $\max\{|X| : X \in LF\} = 6$. Then $vla_f(\mathrm{P}) \geq \frac{10}{6} = \frac{5}{3}$ by Lemma 2.2.

Let

$$\begin{array}{l} L_1 = \{a,b,c,d,d_1,e_1\},\ L_2 = \{b,c,d,e,e_1,a_1\}, \\ L_3 = \{c,d,e,a,a_1,b_1\},\ L_4 = \{d,e,a,b,b_1,c_1\}, \\ L_5 = \{e,a,b,c,c_1,d_1\},\ L_6 = \{a,a_1,c_1,e_1,b_1,d\}, \\ L_7 = \{b,b_1,d_1,a_1,c_1,e\},\ L_8 = \{c,c_1,e_1,b_1,d_1,a\}, \\ L_9 = \{d,d_1,a_1,c_1,e_1,b\},\ L_{10} = \{e,e_1,b_1,d_1,a_1,c\}. \end{array}$$

Clearly, every vertex is contained in exactly six such linear forests. Define a mapping c by

$$c(L) = \begin{cases} \frac{1}{6}, & \text{if } L = L_i, 1 \leq i \leq 10, \\ 0, & \text{otherwise,} \end{cases}$$

then c is a fractional path coloring which has weight $\frac{10}{6} = \frac{5}{3}$. Hence, $vla_f(\mathrm{P}) \leq \frac{5}{3}$ and then $vla_f(\mathrm{P}) = \frac{5}{3}$. □

If let $h(L_1) = 1$ and $h(L_{11}) = 2$ for $L_{11} = \{a_1, c_1, b_1, e\}$, and $h(L) = 0$ for the other $L \in LF$, then h is a path coloring of P, so $vla(\mathrm{P}) = 2 = \lceil vla_f(\mathrm{P}) \rceil$ since the Petersen graph has cycles.

The following graph plays an important role in fractional vertex coloring. Let Z_n denote the additive group of integers modulo n. Suppose that $C \subseteq Z_n \setminus 0$ has the additional property that it is closed under additive inverse, that is, $-c \in C$ if and only if $c \in C$. A *circulant graph* is the graph $G(Z_n, C)$ with the vertex set Z_n and i, j are adjacent if and only if $i - j \in C$. Next we consider the circulant graph $G_{a,b} = G(Z_a, C)$ with $C = \{-a+b, \cdots, -b, b, \cdots, a-b\}$ $(a > 2b)$.

Theorem 2.8. *Let a and b be positive integers with $a \geq 2b$. The circulant graph $G_{a,b}$ is the graph with vertex set $V(G) = \{0, 1, \cdots, a-1\}$. The neighbors of vertex v are $\{v+b, v+b+1, \cdots, v+a-b\}$ with addition* *modulo a. Then* $vla_f(G_{a,b}) = \frac{a}{b+2}$ *and* $vla(G_{a,b}) = \lceil \frac{a}{b+2} \rceil = \lceil vla_f(G_{a,b}) \rceil$ *for* $a - 2b \geq b - 3 \geq 3$.

Proof. Let $a - 2b \geq b - 3 \geq 3$. Think of the vertices of $G_{a,b}$ as equally spaced points around a cycle with an edge between two vertices if they are not too near each other. Note that $G_{a,b}$ has a vertices and is vertex-transitive. Since $\{v, v+1, \cdots, v+b+1\}$ induces a linear forest for each $v \in V(G_{a,b})$, $e = \max\{|X| : X \in LF\} \geq b+2$.

Claim. The cardinality of the maximum linear forest of $G_{a,b}$ is $b+2$, i.e., $e = \max\{|X| : X \in LF\} = b+2$.

Assume, on the contrary, that there are $b+3$ vertices $0 \leq v_1 < v_2 < \cdots < v_{b+3} \leq a-1$ such that $\{v_1, v_2, \cdots, v_{b+3}\}$ induces a linear forest. Clearly, $v_{b+3} - v_1 \geq b+2$. We can suppose that

$$(v_1 - v_{b+3}) (\text{mod } a) \geq max\{v_{i+1} - v_i \mid for\ 1 \leq i \leq b+2\} \quad (*)$$

since $G_{a,b}$ is vertex-transitive. If $(v_1 - v_{b+3})(\text{mod } a) \geq b$, then v_1 is adjacent to vertices v_{b+1}, v_{b+2} and v_{b+3}, a contradiction. Hence, $(v_1 - v_{b+3})(\text{mod } a) < b$.

Suppose that $v_i - v_1 \leq b-1$ and $v_{i+1} - v_1 \geq b$ for some i with $1 < i \leq b$. If $(v_1 - v_{i+1})(\text{mod } a) < b$, then $v_{i+1} - v_i \geq a - (2b-2) = a - 2b + 2 \geq b-1$ and $(v_1 - v_{b+3})(\text{mod } a) \leq (v_1 - v_{i+1})(\text{mod } a) - 2 < b-2$ that contradicts $(*)$. So $v_1 v_{i+1} \in E(G_{a,b})$. If $(v_1 - v_{i+3})(\text{mod } a) \geq b$, then v_1 is adjacent to v_{i+1}, v_{i+2} and v_{i+3}, a contradiction. Thus, $(v_1 - v_{i+3})(\text{mod } a) < b$. Let j be the least integer such that $(v_1 - v_j)(\text{mod } a) < b$. Then $(v_1 - v_k)(\text{mod } a) < b$ for $j \leq k \leq b+3$ and $i+2 \leq j \leq i+3$.

Case 1. v_i is adjacent to v_k for all $j \leq k \leq b+3$.

Then $j \geq b+2$ (otherwise, $j \leq b+1$, then v_i, v_{b+1}, v_{b+2} and v_{b+3} induce a $K_{1,3}$, a contradiction), and $i \geq j - 3 \geq b-1$.

Subcase 1.1. If $j = b+3$, then $i = j - 3 = b$.

So that $v_b = v_{b-1} + 1 = \cdots = v_1 + b - 1$ and $v_1v_{b+1}, v_1v_{b+2}, v_2v_{b+2} \in E(G_{a,b})$, and then $v_2v_{b+1} \notin E(G_{a,b})$, that is, $v_{b+1} - v_2 \leq b - 1$. Hence, $v_{b+1} - v_2 = b - 1$ and then $v_{b+1} - v_1 = v_{b+1} - v_b + v_b - v_1 = 1 + b - 1 = b$. So that $v_{b+2} - v_1 = b + 1$ (otherwise, if $v_{b+2} - v_1 \geq b + 2$, then v_{b+2} is adjacent to v_1, v_2 and v_3, a contradiction). Thus, v_{b+3} is adjacent to v_{b+1-t}, v_{b+2-t} and v_{b+3-t} for $(v_1 - v_{b+3}) (\text{mod } a) = t < b$, a contradiction, too.

Subcase 1.2. If $j = b + 2$, then $i \geq j - 3 = b - 1$.

(1) If $i = b$, then $v_b - v_1 = b - 1$, so that v_{b+2} is adjacent to vertices $v_{b+1-t_1}, v_{b+2-t_1}, v_{b+3-t_1}$ when $(v_1 - v_{b+2}) (\text{mod } a) = t_1 \geq 3$, a contradiction. Thus, $(v_1 - v_{b+2}) (\text{mod } a) = t_1 \leq 2$, i.e., $v_{b+3} = v_{b+2} + 1 = a - 1$ and $v_1 = 0$ that contradict $(*)$.

(2) If $i = b - 1$, then $v_{b-1} - v_1 \leq b - 1$ and $v_b - v_1 \geq b$, so v_b and v_{b+1} are all adjacent to v_1, and then $v_b - v_2 \leq b - 1$ (otherwise, vertices v_b, v_{b+1}, v_1 and v_2 induce a cycle, a contradiction). (2.1) If $v_b - v_2 = b - 2$, then $v_2 - v_1 = 2$ (otherwise, if $v_2 - v_1 \geq 3$, then $v_{b-1} - v_1 \geq b$, a contradiction; if $v_2 - v_1 = 1$, then $v_b - v_1 = b - 1$, a contradiction, too). So $v_b = v_{b-1} + 1 = \cdots = v_2 + b - 2 = v_1 + b$. Thus v_b is adjacent to v_{b+2}, v_{b+3} and v_1 when $(v_1 - v_{b+2}) (\text{mod } a) = t_1 \leq a - 2b$, a contradiction. Hence, $(v_1 - v_{b+2}) (\text{mod } a) = t_1 > a - 2b \geq 3$ and then v_{b+2} is adjacent to v_{b+1-t_1}, v_{b+2-t_1} and v_{b+3-t_1}, a contradiction. (2.2) If $v_b - v_2 = b - 1$ and $v_2 - v_1 = 1$, then $v_b - v_1 = b$, we can get a contradiction similarly as (2.1). (2.3) If $v_b - v_2 = b - 1$ and $v_2 - v_1 = 2$, then $v_b - v_1 = b + 1$. Thus, $v_b - v_{b-1} = 2$ since $v_{b-1} - v_1 \leq b - 1$. So that $v_b = v_{b-1} + 2 = v_{b-2} + 3 = \cdots = v_2 + b - 1 = v_1 + b + 1$ and then $v_{b+1} - v_1 = b + 2$ (otherwise, $v_{b+1} - v_1 > b + 2$, then v_{b+1} is adjacent to vertices v_1, v_2 and v_3, a contradiction). Therefore, v_{b+2} is adjacent to vertices v_{b+2-t_1}, v_{b+1-t_1} and v_{b-t_1} when $(v_1 - v_{b+2}) (\text{mod } a) = t_1 \geq 3$, a contradiction. So that $(v_1 - v_{b+2}) (\text{mod } a) = t_1 \leq 2$ and then $(v_1 - v_{b+3}) (\text{mod } a) = 1$ which

contradicts (*). (2.4) If $v_b - v_2 = b-1$ and $v_2 - v_1 \geq 3$, then $v_{b-1} - v_1 = v_{b-1} - v_2 + v_2 - v_1 \geq b - 3 + 3 = b$, a contradiction, too.

Case 2. v_i is not adjacent to v_k for some $(j \leq)k(\leq b+3)$.

If $j = b+3$, then $i \geq j - 3 = b$, so $i = b$ and $v_b = v_1 + b - 1$. We can get a contradiction as Subcase 1.1 similarly.

If $j = b+2$, then $i \geq j - 3 = b - 1$. We can get a contradiction as Subcase 1.2 similarly.

Suppose that $j \leq b+1$ in the following. If $(v_i - v_k)(\text{mod } a) < b$, then $v_{i+1} - v_i > (v_1 - v_k)(\text{mod } a)$ since $(v_{i+1} - v_1) \geq b$, contrary to (*). So $v_k - v_i < b$. If $k \geq j+1$, then $(v_1 - v_k)(\text{mod } a) < b - 1$, so $v_i - v_1 \geq a - (b-1+b-2) = a - 2b + 3 \geq b$, a contradiction. Hence, $k \leq j$ and then $k = j$. Moreover, $j \geq b+1$ and then $j = b+1$ since v_i is adjacent to v_l for $j+1 \leq l \leq b+3$. So that $b-2 \leq i \leq b-1$.

Since $v_i - v_1 \leq b-1$ and $(v_1 - v_{b+1})(\text{mod } a) \leq b-1$, $v_{b+1} - v_i \geq a - (2b-2) = a - 2b + 2 \geq b-1$ and then $v_{b+1} - v_i = b-1$. Thus, v_{b+1} is adjacent to vertices v_{i-1}, v_{i-2} and v_{i-3} when $v_i - v_{i-3} \leq 4$, a contradiction. Hence $v_i - v_{i-3} > 4$. But $v_i - v_{i-3} = v_i - v_1 - (v_{i-3} - v_1) \leq b - 1 - (i-4) = b - i + 3 \leq b - (b-2) + 3 = 5$ since $v_i - v_1 \leq b-1$ and $i \geq b-2$, so that $v_i - v_{i-3} = 5$, $v_{i-3} - v_1 = i - 4$ and then $a - 2b = b - 3 = 3$, $v_i - v_{i-1} = 1$, $i = b-2$, $b = 6$ and $a = 15$. Clearly, $(v_1 - v_{b+1})(\text{mod } a) = 5$ since $v_{b+1} - v_{b-2} = b - 1 = 5$ and $v_{b-2} - v_1 = 5$. So that $(v_1 - v_{b+3})(\text{mod } a) = t \leq 3$. If $t = 3$, then $v_{b+3} - v_{b+2} = v_{b+2} - v_{b+1} = 1$ and then v_3 is adjacent to vertices v_7, v_8 and v_9, a contradiction. Hence, $t = 2$, and then $v_2 - v_1 = v_3 - v_2 = 2$ by (*). Therefore, vertices v_2, v_3, v_7 and v_8 induce a cycle when $v_9 - v_8 = 2$, and vertices v_3, v_7, v_8 and v_9 induce a $K_{1,3}$ when $v_9 - v_8 = 1$, a contradiction.

Therefore, the Claim is proved.

Hence, $e = \max\{|X| : X \in LF\} = b+2$, and then $vla_f(G) \geq \frac{|V(G)|}{e} =$

$\frac{a}{b+2}$. Define a mapping $f : LF \rightarrow [0,1]$ by

$$f(X) = \begin{cases} \frac{1}{b+2}, & \text{for } X = \{v, v+1, \cdots, v+b+1\} \text{ and } 0 \leq v \leq a-1, \\ 0, & \text{otherwise.} \end{cases}$$

Then f is a fractional path coloring of $G_{a,b}$ which has weight $a\frac{1}{b+2} = \frac{a}{b+2}$. Hence, $vla_f(G) \leq \frac{a}{b+2}$, and then $vla_f(G) = \frac{a}{b+2}$.

Therefore $vla(G_{a,b}) \geq \lceil \frac{a}{b+2} \rceil$. Let $\{i(b+2), i(b+2)+1, \cdots, i(b+2)+b+1\}$ be colored with i for $0 \leq i < \lceil \frac{a}{b+2} \rceil - 1$ and $\{(\lceil \frac{a}{b+2} \rceil - 1)(b+2), (\lceil \frac{a}{b+2} \rceil - 1)(b+2)+1, \cdots, a-1\}$ be colored with $\lceil \frac{a}{b+2} \rceil - 1$. This is a path coloring of $G_{a,b}$, so that $vla(G_{a,b}) \leq \lceil \frac{a}{b+2} \rceil$. Hence $vla(G_{a,b}) = \lceil \frac{a}{b+2} \rceil = \lceil vla_f(G_{a,b}) \rceil$. □

Remarks: 1. We conjecture: the Claim of Theorem 2.8 holds for any $a \geq 2b+2$. So Theorem 2.8 holds in this case.

2. We only discussed several cases of complete $n-$partite graphs in Theorem 2.3, the other cases can be studied similarly.

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