Some Properties of Strong Domination Numbers of Hypergraphs

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Abstract

We introduce a new concept of strong domination and connected strong domination in hypergraphs. The relationships between strong domination number and other hypergraph parameters like domination, independence, strong independence and irredundant numbers of hypergraphs are considered. There are also some chains of inequalities generalizing the famous Cockayne, Hedetniemi and Miller chain for parameters of graphs. There are given some generalizations of well known theorems for graphs, namely Gallai type theorem generalizing Nieminen, Hedetniemi and Laskar theorems.

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1 Introduction and preliminaries

A hypergraph $H = (V, \mathcal{E})$ of *order* p and *size* q consists of a set of p *vertices* V together with a set of q *edges* $\mathcal{E}$, each element of which is a subset of V of cardinality at least two.

Two vertices u and v *are adjacent* in a hypergraph H if there exists an edge $E \in \mathcal{E}$ such that $u \in E$ and $v \in E$. For a vertex $v \in V$, we denote by $N(v)$ the set of vertices of H adjacent to v (*neighbours of* v) and for $A \subseteq V$, by $N(A)$ the set of neighbours of vertices of A. By $N[v]$, we denote $N(v) \cup \{v\}$ and $N[A] = N(A) \cup A$.

Let $H = (V, \mathcal{E})$ be a hypergraph. For a vertex v we denote the set of all edges containing v, by $\mathcal{E}_v$. The number $|\mathcal{E}_v|$ is called the *degree of* v and it is denoted by $deg(v)$. If $deg(v) = 0$, then v is called an *isolated vertex*.

Now we recall the definition of the 2-section of a hypergraph. If $H = (V, \mathcal{E})$ is a hypergraph then the couple $(V, \mathcal{E}_{(2)})$ where $\mathcal{E}_{(2)} = \{F \subseteq V : F \subseteq E \in \mathcal{E}$ and $|F| = 2\}$ is the *2-section of H* and we denote it by $(H)_2$.

Let $H = (V, \mathcal{E})$ be a hypergraph. *A spanning subhypergraph* of H is defined to be a hypergraph $H' = (V, \mathcal{F})$, where $\mathcal{F} \subseteq \mathcal{E}$.
Let $A \subseteq V$. The subhypergraph of H *induced by* A is denoted by $H[A]$ and it is defined to be the hypergraph $(A, \mathcal{E}(A))$ where $\mathcal{E}(A) = \{E \in \mathcal{E} : E \subseteq A\}$.

In a hypergraph $H = (V, E)$ a *chain (of length q)* is defined to be a sequence $(v_1, E_1, v_2, E_2, ..., E_q, v_{q+1})$ such that $v_1, v_2, .., v_{q+1}$ are distinct vertices of H, and $E_1, E_2, ..., E_q$ are distinct edges in H, and $\{v_k, v_{k+1}\} \subseteq E_k$ for $k = 1, 2..., q$. We also say that, the chain defined above, starts at the vertex v_1 and terminates at the vertex v_{q+1}.
Hypergraph $H = (V, \mathcal{E})$ is called *connected* if for any two vertices u and v there exists a chain that starts at vertex u and terminates at vertex v.

A set D of vertices of a hypergraph H is *a dominating set in H*, if each vertex v of $V - D$ is adjacent with an element of D, i.e., $N(v) \cap D \neq \emptyset$.
The minimum (maximum) cardinality of a minimal dominating set in H is called the *lower (upper) domination number* and it is denoted by $\gamma(H)(\Gamma(H))$, respectively.

A set $D \subseteq V$ is said to be *strong dominating set in H* if for each $v \in V - D$ there is an edge $E \in \mathcal{E}$ such that $v \in E$ and $(E - \{v\}) \subseteq D$.
The minimum (maximum) cardinality of a minimal strong dominating set in H is called the *lower (upper) strong domination number* and it is denoted by $\gamma_s(H)(\Gamma_s(H))$, respectively.

A set $S \subseteq V$ is said to be *independent* in H [2], if it contains no edges of H. The minimum (maximum) of the cardinalities of the maximal independent sets in H is called the *lower (upper) independence number* and it is denoted by $i(H)(\alpha(H))$, respectively.

A set $S \subseteq V$ is said to be *strong independent* in H [2], if for every $v \in S$, $N(v) \cap S = \emptyset$, that is, no two vertices in S are contained in an edge.
The minimum (maximum) cardinality of a maximal strong independent set in H is called the *lower (upper) strong independence number* and it is denoted by $i'(H)(\alpha'(H))$, respectively.

A set $I \subseteq V$ is said to be *irredundant* in H if for each vertex $v \in I$, $N[v] - N[I - \{v\}] \neq \emptyset$.
The minimum (maximum) cardinality of a maximal irredundant set in H is called the *lower (upper) irredundant number* and it is denoted by $ir(H)(IR(H))$, respectively.

2 Some properties of dominating, strong dominating, independent, strong independent and irredundant sets of hypergraphs

We begin our investigation with the following elementary results that will be useful.
Let $H = (V, \mathcal{E})$ be a hypergraph and $G = (H_2)$. By the definition of $(H)_2$ for each vertex $v \in V$ we have $N_H(v) = N_G(v)$.

Using the above we obtain the following statements:

Observation 1 *A set X of vertices is a dominating set in H if and only if X is a dominating set in $G = (H)_2$.*

Observation 2 *A set X of vertices if H is an irredundant set in H if and only if X is an irredundant set in $G = (H)_2$.*

The correspondence between strong independent sets in a hypergraph and independent sets in its 2-section is well known ([2]).

Observation 3 *A set X of vertices if H is a strong independent set in H if and only if X is a independent set in $G = (H)_2$.*

By Observations 1-3 we easily obtain some equalities :

$$\gamma(H) = \gamma((H)_2), \quad \Gamma(H) = \Gamma((H)_2) \tag{1}$$

$$i'(H) = i((H)_2), \quad \alpha'(H) = \alpha((H)_2) \tag{2}$$

$$ir(H) = ir((H)_2), \quad IR(H) = IR((H)_2) \tag{3}$$

Also Ore's [9] result can be presented in terms of properties of a dominating set of a hypergraph.

Theorem 4 *Let D be a dominating set of a hypergraph H. Then D is a minimal dominating set in H if and only if for each vertex $d \in D$, d has at least one of the following properties:*
(i) there exists a vertex $v \in V - D$ such that $N(v) \cap D = \{d\}$,
(ii) $N(d) \cap D = \emptyset$.

In the same way we can write the result of Bollobás and Cockayne [1]

Theorem 5 *If H is a hypergraph without isolated vertices, then there exists a minimum dominating set in which every vertex has the property (i).*

We will look on the relationship between these various parameters of hypergraphs.
Cockayne, Hedetniemi and Miller [4] introduced the following inequality chain.

For any graph G:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G). \tag{4}$$

By (4) and Observations 1-3 we have:

For any hypergraph H

$$ir(H) \leq \gamma(H) \leq i'(H) \leq \alpha'(H) \leq \Gamma(H) \leq IR(H). \tag{5}$$

From the definition of a strong dominating set in H it is easy to see that each minimal strong dominating set in H is a dominating set in H. Thus

$$\gamma(H) \leq \gamma_s(H) \tag{6}$$

for any hypergraph H.

Theorem 6 *If $X \subseteq V(H)$ is a maximal independent set in H, then X is a minimal strong dominating set in H.*

Proof. For each vertex $v \in V - X$, the set $X \cup \{v\}$ contains an edge E of H such that $E - \{v\} \subseteq X$. It implies that X is a strong dominating set in H. Suppose that $X - \{x\}$ is a strong dominating set in H. This implies that there exists an edge E' such that $x' \in E'$ and $E' - \{x\} \subseteq X - \{x\}$, so $E' \subseteq X$, a contradiction. Thus X is a minimal strong dominating set in H. □

From Theorem 6 we obtain:

For any hypergraph H

$$\gamma_s(H) \leq i(H) \leq \alpha(H) \leq \Gamma_s(H), \tag{7}$$

and from (5), (6) and (7)

$$ir(H) \leq \gamma(H) \leq \gamma_s(H) \leq i(H) \leq \alpha(H) \leq \Gamma_s(H). \tag{8}$$

Now we look at cases in which $\gamma(H)$ and $\gamma_s(H)$ have the same value and H is not a graph.

Theorem 7 *Let $H = (V, \mathcal{E})$ be a hypergraph. If $|E| \geq 3$ for each $E \in \mathcal{E}$ then $\gamma(H) < \gamma_s(H)$.*

Proof. As we stated above each strong dominating set is a dominating set in H. Suppose $H = (V, \mathcal{E})$ is a hypergraph with $|E| \geq 3$ for each $E \in \mathcal{E}$ for which the equality $\gamma(H) = \gamma_s(H)$ holds. Hence there exists a minimum dominating set D which is a minimum strong dominating set in H. By Theorem 4 each vertex $d \in D$ has the property *(i)* or *(ii)*.

Suppose first that each vertex of D has the property *(ii)*. By strong domination of D for each vertex v of $V - D$ there is an edge $E \in \mathcal{E}$ such that $v \in E$ and $E - \{v\} \subseteq D$. It implies that $|E - \{v\}| = 1$, so $|E| = 2$, which contradicts our assumption.
From now on we may assume that there exists a non-empty set $Z \subseteq D$ and each vertex $d \in Z$ has the property *(i)* and does not have the property *(ii)*. Let $d \in Z$ and $v \in V - D$ such that $N(v) \cap D = \{d\}$. By our assumption each edge E containing v has $|E| \geq 3$ and $|D \cap E| = 1$, thus D is not a strong dominating set, a contradiction. □

An examination of the proof of Theorem 7 shows us that if there exists a minimum dominating set D such that for each vertex $v \in V - D$, there exists $v' \in D$ such that vv' is an edge of the hypergraph H, then $\gamma(H) = \gamma_s(H)$.

Theorem 8 *Let H be a hypergraph without isolated vertices. If $X \subseteq V(H)$ is a maximal strong independent set in H, then $V - X$ is a strong dominating set in H.*

Proof. If for each vertex $v \in X$, $N(v) \cap X = \emptyset$, then for each edge $E \in \mathcal{E}$ such that $v \in E$, it is obvious that $E - \{v\} \subseteq V - X$. Hence $V - X$ is a strong dominating set. □

From Theorem 8 we obtain:

If H is a hypergraph without isolated vertices then

$$\gamma_s(H) + \alpha'(H) \leq p. \tag{9}$$

As the consequences of (9) and (5) we obtain:

For any hypergraph without isolated vertices

$$\gamma_s(H) + i'(H) \leq p, \quad \gamma_s(H) + \gamma(H) \leq p, \quad \gamma(H) + ir(H) \leq p \tag{10}$$

and also by (8) and (9)

$$\gamma(H) + \alpha'(H) \leq p, \quad ir(H) + \alpha'(H) \leq p. \tag{11}$$

To prove the next theorem we need the following theorem due to Cockayne, Favaron, Payan and Thomason

Theorem ([3]) *If G is a graph without isolated vertices and $\gamma(G) + IR(G) = p$ then $IR(G) = \Gamma(G) = \alpha(G)$.*

Theorem 9 *If H is a hypergraph without isolated vertices and $\gamma(H) + IR(H) = p$ then $\gamma(H) = \gamma_s(H)$.*

Proof. Let H be a hypergraph without isolated vertices and $\gamma(H)+IR(H)=p$, $G=(H)_2$. By (1) and (3) we obtain $\gamma(G)+IR(G)=p$. Using the above result of [3] we obtain $IR(G)=\Gamma(G)=\alpha(G)$. By (1) and (2) we have $\gamma(H)+\alpha'(H)=p$ and by (6) and (9) we obtain the claimed equality. □

Corollary 10 *If H is a hypergraph without isolated vertices and $E| \geq 3$ for each $E \in \mathcal{E}$ then $\gamma(H)+IR(H)<p$.*

3 Gallai-type Theorems

In 1959 Gallai presented his now classical theorem:

Theorem (Gallai [5]) *For any nontrivial connected graph $G=(V,E)$ with p vertices, $\alpha_0+\beta_0=p$, $\alpha_1+\beta_1=p$*
where α_0 denotes vertex covering number, β_0 the vertex independence number, α_1 the edge covering number and β_1 the maximum size of a matching.

A large number of similar results and generalizations of this theorem have been obtained in subsequent years; they are called Gallai-type equalities. We present generalizations of two of them.

An edge of G is called *a pendant edge* if at least one of its vertices is of degree 1. By $\epsilon(G)$ is denoted the maximum number of pendant edges in a spanning forest of a graph G.

Theorem (Nieminen [8]). *Let G be a graph with p vertices. Then*

$$\gamma(G)+\epsilon(G)=p$$.

Hedetniemi and Laskar proved a similar equality as in Nieminen's Theorem, involving connectivity.

A set $D \subseteq V$ is called *connected dominating* in G if D is dominating and the subgraph of G induced by D is a connected graph. By $\gamma_c(G)$ is denoted the *connected domination number* i.e., the minimum number of vertices of a connected dominating set in G. By $\epsilon_c(G)$ is denoted maximum number of pendant edges in a spanning tree of G.

Theorem (Hedetniemi, Laskar [7]). *Let G be a connected graph of order p. Then*

$$\gamma_c(G)+\epsilon_c(G)=p$$.

Let $H=(V,\mathcal{E})$ be a hypergraph. An edge $E \in \mathcal{E}$ is called *pendant* if it contains a vertex of degree 1. Let $\epsilon(H)$ be the maximum number of pendant edges in a spanning subhypergraph of H. Note, that if $\mathcal{E} \neq \emptyset$ then $\epsilon(H)>0$

Theorem 11 *For any hypergraph on p vertices we have*

$$\gamma_s(H)+\epsilon(H)=p.$$

Proof. Assume that $\mathcal{E} = \emptyset$, thus $\gamma_s(H) = p$. Hence, $\mathcal{E} \neq \emptyset$. Let D be a minimal strong dominating set in H with $|D| = \gamma_s(H)$. For every $v \in V - D$ there exists $E \in \mathcal{E}$ such that $v \in E$ and $E - \{v\} \subseteq D$, by the definition of strong dominating set. Then for every $v \in V - D$ we choose exactly one such edge and denote it by E_v. $\mathcal{E}' = \{E_v : v \in V - D\}$ is a family of pendant edges in the spanning subhypergraph $H' = (V, \mathcal{E}')$, hence $\epsilon(H) \geq |V - D| = p - \gamma_s(H)$.

On the other hand, let H' be a spanning subhypergraph of H, and $\mathcal{F} = \{E_{i_1}, E_{i_2}, ..., E_{i_k}\}$ be a family of pendant edges of H' where $1 \leq k = \epsilon(H)$. The family $\mathcal{F}$ has a system of distinct representatives. Let Y be one of them containing pendant vertices of $(V, \mathcal{F})$. Then for every $v \in Y$ there is an edge $E \in \mathcal{F}$ such that $v \in E$ and $E - \{v\} \subseteq V - Y$. Consequently $V - Y$ is a strong dominating set. Hence $\gamma_s(H) \leq |V - Y| = p - \epsilon(H)$.
Finally, $\gamma_s(H) + \epsilon(H) = p$. □

Consider the inequality (9) and Theorem 11 we obtain for a hypergraph without isolated vertices

$$\epsilon(H) \geq \alpha'(H). \tag{12}$$

Let $H = (V, \mathcal{E})$ be a connected hypergraph. If $D \subseteq V$ is a strong dominating set and $H[D]$ is a connected hypergraph then D is said to be a *connected strong dominating set.*
The minimum cardinality of a minimal connected strong dominating set in H is called the *connected strong domination number* and it is denoted by $\gamma_{sc}(H)$. Note that, $0 < \gamma_s(H) \leq \gamma_{sc}(H) \leq p$ for any connected hypergraph H.

For a connected hypergraph H, a set $\mathcal{E}'$ of pendant edges is called a *set of proper pendant edges* if each edge of $\mathcal{E}'$ contains exactly one pendant vertex and $H[V - V']$ is a connected hypergraph, where V' is the set of pendant vertices of $\mathcal{E}'$. By $\epsilon_c(H)$ we denote the maximum order of a set of proper pendant edges in a connected spanning subhypergraph of H.

Theorem 12 *For a connected hypergraph on p vertices we have*

$$\gamma_{sc}(H) + \epsilon_c(H) = p.$$

Proof. First, we assume that $\gamma_{sc}(H) = p$. By the connectivity of H the set $V - \{v\}$ is strong dominating set for each $v \in V$. If $H[V - \{v\}]$ is connected for some vertex $v \in V$, then $\gamma_{sc}(H) < p$, a contradiction. Thus $H[V - \{v\}]$ is not connected for each $v \in V$ and it implies that the set of proper pendant edges is the empty set.
Now assume that $\gamma_{sc}(H) < p$. Let D be a minimal connected strong dominating set in H with $|D| = \gamma_{sc}(H)$. For every $v \in V - D$ there exists $E \in \mathcal{E}$ such that $v \in E$ and $E - \{v\} \subseteq D$. Then for every $v \in V - D$ we choose exactly one such edge and denote it by E_v. The set $\mathcal{E}' = \{E_v : v \in V - D\}$, is the set of proper pendant edges in the connected spanning subhypergraph $H' = (V, \mathcal{E}' \cup \mathcal{E}(D))$, hence $\epsilon_c(H) \geq |V - D| = p - \gamma_{sc}(H)$.

On the other hand, let H' be a connected spanning subhypergraph of H, and $\mathcal{F} = \{E_{i_1}, E_{i_2}, ..., E_{i_k}\}$ be a set of proper pendant edges of H' where $k = \epsilon_c(H)$. Assume that $k = 0$. It implies that for each vertex $v \in V(H)$ the induced

subhypergraph $H[V - \{v\}]$ is not connected, so the unique connected strong dominating set is the set V. If $k > 0$, then for each E_{i_j} we denote its pendant vertex by v_j, $1 \leq j \leq k$. Let $Y = \{v_1, v_2, ..., v_k\}$.
Then for every $v \in Y$ there is an edge $E_{i_j} \in \mathcal{F}$ such that $v \in E_{i_j}$ and $E_{i_j} - \{v\} \subseteq V - Y$ and $H[V - Y]$ is a connected hypergraph. Consequently $V - Y$ is a connected strong dominating set. Hence $\gamma_{sc}(H) \leq |V - Y| = p - \epsilon_c(H)$.
Finally, $\gamma_{sc}(H) + \epsilon_c(H) = p$. □

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