Domination Parameters and Gallai-type Theorems for Directed Trees

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Abstract. The lower domination number of a digraph D, denoted by $\gamma(D)$, is the least number of vertices in a set S, such that $O[S] = V(D)$. A set S is irredundant if for all $x \in S$, $|O[x] - O[S - x]| \geq 1$. The lower irredundance number of a digraph, denoted $ir(D)$, is the least number of vertices in a maximal irredundant set. A Gallai-type theorem has the form $x(G) + y(G) = n$, where x and y are parameters defined on G, and n is the number of vertices in the graph. We characterize directed trees satisfying $\gamma(D) + \Delta_+(D) = n$ and directed trees satisfying $ir(D) + \Delta_+(D) = n$.

Key words. domination, irredundance, directed tree, Gallai theorem

All digraphs are assumed to be loopless and without multiarcs. Given a digraph D, $V(D)$ refers to the vertex set and n denotes $|V(D)|$. We say y is an *out-neighbor* of x if x has an arc to y. The set of all out-neighbors of x is denoted by $O(x)$. The *outdegree* of a vertex x, denoted $od(x)$, is $|O(x)|$. The maximum outdegree of a vertex in D is denoted by $\Delta_+(D)$. The set $O[x] = \{x\} \cup O(x)$. If S is a set, then $O(S) = \cup_{x \in S} O(x)$. The set $O[S] = O(S) \cup S$. The *indegree* of a vertex x, denoted $id(x)$, is the number of vertices that have an arc to x. We say y is an *in-neighbor* of x if y has an arc to x. The sets $I(x)$, $I[x]$, $I(S)$, and $I[S]$ are defined analogously.

A set of vertices, S, is *dominating* if $O[S] = V(D)$. The *lower domination number*, denoted by $\gamma(D)$, is the minimum size of a dominating set. A set S is *irredundant* if, for all $x \in S$, $|O[x]-O[S-x]| \geq 1$. If $y \in O[x]-O[S-x]$, we say that y is a *private neighbor* of x with respect to S. Observe that x may be its own private neighbor. The *lower irredundance number* of a digraph, denoted $ir(D)$, is the least number of vertices in a maximal irredundant set. Since every minimal dominating set is irredundant, $ir(D) \leq \gamma(D)$. These parameters have been extensively studied in a graph setting. For example, it is well known that $\gamma(G) + \Delta(G) \leq n$ (see Berge [1]), where G is a graph. Such a result, in which equality holds, is called a Gallai-type theorem, in reference to Gallai's result of 1959 [5] in which he showed that the independence and covering numbers of a graph sum to n. Subsequently, Gallai-type theorems have been of interest (for example, see Cockayne, et al. [2]). In [3], Domke, Dunbar, and Markus and in [4], Favaron and

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Mynhardt derive several relationships between domination parameters in graphs. For example, they characterize trees satisfying $\gamma(T) + \Delta(T) = n$. In general, domination in digraphs has been studied to a lesser extent than their graph theoretic counterparts. In [6], Ghoshal, Laskar, and Pillone consider related topics in digraphs and suggest further avenues of study. Gallai-type results for digraphs have been considered in [9]. Therein, it is observed that $\gamma(D) + \Delta_+(D) \leq n$ for any digraph.

As defined in [3], a *spider* is a tree with a single vertex of degree p, p pendant vertices, and p vertices with degree two, each of which is adjacent to a pendant vertex and the vertex of degree p, where $p \geq 1$. A *wounded spider* is a tree with a single vertex of degree p, p pendant vertices, and at most $p-1$ vertices of degree 2, each of which is adjacent to a pendant vertex and the vertex of degree p, where $p \geq 1$. In the directed case, we analogously define *(directed) spiders* and *(directed) wounded spiders*, the arcs are oriented as in a rooted tree so that a vertex of maximum degree becomes the root in the directed graph.

1. The Lower Domination Number.

In [3], Domke et al. determined that an undirected tree, G satisfies $\gamma(G) + \Delta(G) = n$ if and only if G is a wounded spider (or a single vertex). The analogous statement for directed trees is not true, as shown in Figure 1. To characterize directed trees satisfying $\gamma(T) + \Delta_+(T) = n$, we begin with some previously made observations. A set S of vertices from a digraph is *independent* if for all $x, y \in S$, (x, y) is not an arc.

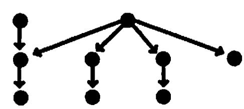

Figure 1. D is not a directed wounded spider, but $\gamma(D) + \Delta_+(D) = n$.

Lemma 1.1. [9] Let D be a digraph. If $\gamma(D) + \Delta_+(D) = n$ and $od(x) = \Delta_+(D)$, then $V(D) - O[x]$ is an independent set.

Lemma 1.2. [9] If $\gamma(D) + \Delta_+(D) = n$, $od(x) = \Delta_+(D)$ and $y \in O(x)$, then $|O(y) - O[x]| \leq 1$.

Lemma 1.3. Let T be a directed tree. If $\gamma(T) + \Delta_+(T) = n$, then T has at most one vertex with outdegree greater than 1.

Proof. Suppose not. Let x be a vertex satisfying $od(x) = \Delta_+(T)$. Let y be another vertex satisfying $od(y) > 1$. If $y \notin O[x]$, then since T is a directed tree, y has at least one out-neighbor, z, such that $z \notin O[x]$. Thus, $V(T) - O[x]$ is not independent, a contradiction by Lemma 1.1. Thus, $y \in O[x]$. Since T is a directed tree, y has at least 2 out-neighbors not in $O[x]$. This contradicts Lemma 1.2. So there can be at most one vertex with outdegree greater than one. □

Lemma 1.4. Let T be a directed tree. Let z be a vertex with $id(z) = 0$, $od(z) = 1$ and let T' be T with z removed. If $\gamma(T) + \Delta_+(T) = n$, then $\gamma(T') + \Delta_+(T') = |V(T')| = n - 1$.

Proof. Assume that $\gamma(T) + \Delta_+(T) = n$. If $\Delta_+(T') < \Delta_+(T)$ then T' is a single vertex and the statement is true. So we can assume that $\Delta_+(T') = \Delta_+(T)$. Suppose $\gamma(T') < n - \Delta_+(T) - 1$. Let S' be a minimum dominating set of T'. Then $S = S' \cup \{z\}$ is a dominating set of T of size less than $n - \Delta_+(T)$, a contradiction. □

Using the convention that the vertices of a given directed tree are partitioned into levels so that each arc is directed from a vertex on level i to a vertex on level $i+1$, where i is an integer, we can refer to the height of the tree. Assume each vertex is an element of the least indexed level possible. The *height* of tree T, denoted $h(T)$, is the number of levels in the tree.

Theorem 1.5. Let T be a directed tree with $n \geq 2$, and vertex x satisfying $od(x) = \Delta_+(T)$. Then $\gamma(T) + \Delta_+(T) = n$ if and only if T is in class C_1 or C_2 defined as follows:

C_1: $I(x) = \emptyset$. For all y with $id(y) = 0$, $O(y) \subseteq O(x)$. Removal of all vertices with empty inset, other than x, leaves a spider or wounded spider.

C_2: $I(x) \neq \emptyset$. For all y with $id(y) = 0$, $O(y) \subseteq O[x]$. There exists $z \in O(x)$ such that $id(z) = 1$ and $od(z) = 0$. Removal of all vertices with empty inset leaves a wounded spider.

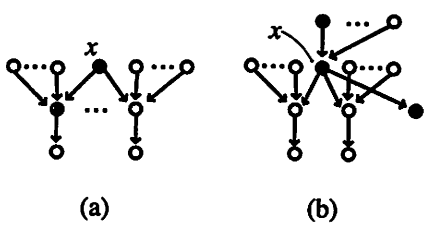

Figure 2. The digraphs described in Theorem 1.5: C_1 in (a) and C_2 in (b). The white vertices may or may not be present.

Proof. ($\Rightarrow$) Assume that $\gamma(T) + \Delta_+(T) = n$. Let x be a vertex with $od(x) = \Delta_+(T)$. By Lemma 1.3, there is at most one vertex with more than one out-neighbor. Thus, for all $y \neq x$, $od(y) \leq 1$. First, suppose that $id(x) = 0$. We claim that T satisfies C_1.

By Lemma 1.1, $V(T) - O[x]$ is independent, so $h(T) \leq 3$. Suppose y is a vertex, other than x, in T with $id(y) = 0$. Since T is a directed tree, we conclude that $od(y) = 1$. By Lemma 1.1, $O(y) \subseteq O(x)$.

Let T' be T with all vertices with empty inset, other than x, removed. Observe that T' is connected and has only one vertex with empty inset, namely x, since for all y in T with $id(y) = 0$, $O(y) \subseteq O(x)$. We claim that T' is a spider or wounded spider.

Let y be a vertex in T', other than x. Suppose $y \in O(x)$. By Lemma 1.3, $od(y) \leq 1$. If $od(y) = 1$, then since T is a directed tree, $O(y) \cap O[x] = \emptyset$. Suppose $y \notin O(x)$. Since x is the only vertex in T' with empty inset, there is a vertex z such that (z, y) is an arc. Since $\gamma(T') + \Delta_+(T') = |V(T')|$ by Lemma 1.4, Lemma 1.1 implies that $V(T') - O[x]$ is independent. So $z \in O[x]$. Since $y \notin O(x)$, $z \in O(x)$. Since T is a directed tree and $V(T') - O[x]$ we conclude that $I(y) = \{z\}$. Since $h(T) \leq 3$, $O(y) = \emptyset$. Thus, T' is a spider or wounded spider. So T satisfies C_1.

Next, suppose that $id(x) \neq 0$. We claim that T satisfies C_2. Observe that since $V(T) - O[x]$ is an independent set, $h(T) \leq 4$. Let y be a vertex with $id(y) = 0$. Since T is a directed tree and $od(y) \leq 1$, we conclude that $od(y) = 1$. By Lemma 1.1, $O(y) \subseteq O[x]$.

Let T' be T with all vertices with empty inset removed. The same argument used in the case where $id(x) = 0$ will establish that T' is a spider or wounded spider. Suppose that, in T, for all $y \in O(x)$, $id(y) \geq 2$ or $od(y) = 1$. Since T' is a spider or wounded spider, we conclude that, in T, if $y \in O(x)$ and $id(y) \geq 2$, then every vertex in the inset of y, other than x, has empty inset. Then T has a dominating set of size $n - \Delta_+(T) - 1$, namely all vertices with indegree 0 together with all vertices y such that (y, z) is an arc, where $y \in O(x)$ and $I(y) = \{x\}$, and all vertices z such that (y, z) is an arc, where $y \in O(x)$ and $I(y) \neq \{x\}$, a contradiction. Thus, there is a vertex $y \in O(x)$ such that $id(y) = 1$ and $od(y) = 0$. This implies that T' is a wounded spider and so T satisfies C_2.

($\Leftarrow$) Conversely, suppose T is a directed tree as in C_1. Every dominating set must include all vertices with indegree zero, including x, the vertex with maximum outdegree. Furthermore, for each arc (y, z) where $y \in O(x)$, either y or z must be included in any dominating set. Thus, every dominating set contains at least $n - \Delta_+(T)$ vertices. Since it is always true that $\gamma(T) + \Delta_+(T) \leq n$, we conlcude that $\gamma(T) + \Delta_+(T) = n$.

Suppose T is a directed tree as in C_2. Again, every dominating set must include all vertices with indegree zero. For each arc (y, z) where $y \in O(x)$, either y or z must be included in any dominating set. Finally, there is an arc (x, y) where $I(y) = \{x\}$ and $O(y) = \emptyset$. Thus, x or y must be contained in any dominating set. Thus, every dominating set contains at least $n - \Delta_+(T)$ vertices. Thus $\gamma(T) + \Delta_+(T) = n$. □

We can use the previous theorem to characterize rooted trees with $\gamma(T) + \Delta_+(T) = n$. In a rooted tree there is only one vertex with indegree zero, namely the root.

Corollary 1.6. Let T be a directed rooted tree with $n \geq 2$. Then $\gamma(T) + \Delta_+(T) = n$ if and only if T is a wounded spider, a spider, or a rooted tree such that removal of the root leaves a wounded spider.

2. The Lower Irredundance Number.

In the undirected case, a tree satisfies $\gamma(G) + \Delta(G) = n$ if and only if $ir(G) + \Delta(G) = n$. This is not true for directed trees as shown in Figure 3.

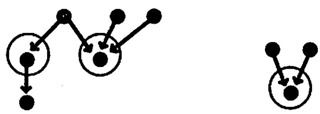

Figure 3. For each digraph, $\gamma(D) + \Delta_+(D) = n$, but $ir(D) + \Delta_+(D) \neq n$. The vertices in the minimum size irredundant set of each digraph are circled.

Theorem 2.1. Let T be a directed tree with vertex x satisfying $od(x) = \Delta_+(T)$. Let $Z = \{y \in O(x) : od(y) = 0\}$. Then $ir(T) + \Delta_+(T) = n$ if and only if $\gamma(T) + \Delta_+(T) = n$ and $|I(Z)| \leq |Z|$.

Proof. ($\Leftarrow$) Assume that $\gamma(T) + \Delta_+(T) = n$ and $|I(Z)| \leq |Z|$. The statement is clearly true if T is a single vertex, so assume $n \geq 2$. Then by Theorem 1.5, we conclude that T is a member of either class C_1 or C_2 of digraphs. Let S be a maximal irredundant set satisfying $|S| = ir(T)$.

If x and all $y \in O(x)$ are in S, we conclude that every vertex $y \in O(x)$ has an out-neighbor as a private neighbor and consequently T is a member of C_1. Thus, every in-neighbor of y is in S and each out-neighbor of y is not in S. Thus, $ir(T) + \Delta_+(T) = n$.

Suppose $x \in S$, but at least one $y \in O(x)$ is not. Observe that $S \cap Z = \emptyset$. Suppose $Z = \emptyset$. Then $T \in C_1$. For each $y \in O(x)$, exactly one of y or its out-neighbor must be in S, since if $y \notin S$, the out-neighbor of y serves as its own private neighbor and if $y \in S$, its out-neighbor has no private neighbor. Also every in-neighbor of y serves as its own private neighbor and is, therefore, in S. Thus, $ir(T) + \Delta_+(T) = n$.

Suppose $|Z| \geq 1$. At least one vertex $w \in Z$ has $id(w) = 1$, otherwise $|I(Z)| > |Z|$. Thus, w is a private neighbor of x. So, for each $y \in O(x) - w$, $I(y) \subseteq S$. Since x has a private neighbor other than itself, $I(x) \subseteq S$. Thus, there are exactly $\Delta_+(T)$ vertices not in S: the vertices in Z and one vertex for each $y \in O(x) - Z$, either y or its out-neighbor. Thus, $ir(T) + \Delta_+(T) = n$.

Finally, suppose $x \notin S$. If $Z = \emptyset$, then every $y \in O(x)$ has an out-neighbor. So $T \in C_1$, thus x may be added to S, a contradiction. So, $|Z| \geq 1$. Since $|I(Z)| \leq |Z|$, at least one vertex $w \in Z$ has $id(w) = 1$. Since w would be a private neighbor for x, we conclude that $w \in S$. Since $x \notin S$, $I(x) \subseteq S$. Also there are $\Delta_+(T) - |Z|$ vertices not in S, one for each $y \in O(x) - Z$: either y or its out-neighbor. Let Z' denote the vertices in $Z \cap S$. So $w \in Z'$. Observe that every vertex in $Z - Z'$ has at least one in-neighbor, other than x, otherwise S is not maximal. All such in-neighbors are in S.

Consider the vertices in Z'. Each vertex, except w, must have indegree at least 2. Otherwise, a smaller maximal irredundant set can be obtained by exchanging Z' for $I(Z')$. However, each vertex must have indegree at most 2, otherwise $|I(Z)| > |Z|$. Thus, $|I(Z')| = |Z'|$ and since $O(Z') = \emptyset$, none of these $|Z'|$ vertices is in S.

So, there are $\Delta_+(T) - |Z| + |Z'| + (|Z| - |Z'|)$ vertices not in S. So $ir(T) = n - \Delta_+(T)$.

($\Rightarrow$) Conversely, assume $ir(T) + \Delta_+(T) = n$. Then $ir(T) \leq \gamma(T)$ implies that $\gamma(T) + \Delta_+(T) \geq n$. Since it is always true that $\gamma(T) + \Delta_+(T) \leq n$, we conclude that $\gamma(T) + \Delta_+(T) = n$.

Suppose that $|I(Z)| > |Z|$. Let $W = O(x) - Z$. Let $S = O(W) \cup (I(W) - x) \cup Z \cup I(x)$. Since each vertex in S is its own private neighbor, S is irredundant. We claim that S is maximally irredundant. First, neither x, nor any other vertex in $I(Z)$ can be added, because each $y \in Z$ is its own, and only, private neighbor. No vertex in W can be added because $O(W) \subseteq S$. So, S is maximally irredundant.

But,

$$\begin{aligned} ir(T) \leq |S| &= n - (\Delta_+(T) - |Z|) - |I(Z)| \\ &< n - (\Delta_+(T) - |Z|) - |Z| = n - \Delta_+(T), \end{aligned}$$

a contradiction. Thus, $|I(Z)| \leq |Z|$. □

3. A Note on the Upper Parameters and Directed Trees.

The maximum size of a minimal dominating set in D, denoted $\Gamma(D)$, is called the *upper domination number*. The maximum size of a irredundant set in D, denoted $IR(D)$, is called the *upper irredundance number*. In [3] Domke et al. prove that $\Gamma(G) + \delta(G) = n$ if and only if $IR(G) + \delta(G) = n$. In [9], analogous statements in a digraph setting involving both $\delta_-(D)$ and $\delta_+(D)$ are proven. Observe that in a directed tree, $\delta_+(D) = \delta_-(D) = 0$ and the only directed tree satisfying $\Gamma(D) = n$ or $IR(D) = n$ is a single vertex. Thus, the only directed trees satisfying $\Gamma(D) + \delta_\pm(D) = n$ or $IR(D) + \delta_\pm(D) = n$ are trivial.

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