Panpositionable Hamiltonian Graphs

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Abstract

A hamiltonian graph G is *panpositionable* if for any two different vertices x and y of G and any integer k with $d_G(x,y) \leq k < |V(G)|/2$, there exists a hamiltonian cycle C of G with $d_C(x,y) = k$. A bipartite hamiltonian graph G is *bipanpositionable* if for any two different vertices x and y of G and for any integer k with $d_G(x,y) \leq k < |V(G)|/2$ and $(k - d_G(x,y))$ is even, there exists a hamiltonian cycle C of G such that $d_C(x,y) = k$. In this paper, we prove that the hypercube Q_n is bipanpositionable hamiltonian if and only if $n \geq 2$. The recursive circulant graph $G(n;1,3)$ is bipanpositionable hamiltonian if and only if $n \geq 6$ and n is even; $G(n;1,2)$ is panpositionable hamiltonian if and

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only if $n \in \{5, 6, 7, 8, 9, 11\}$, and $G(n; 1, 2, 3)$ is panpositionable hamiltonian if and only if $n \geq 5$.

Keywords: hamiltonian, pancyclic, panconnected.

1 Introduction

For the graph definitions and notations we follow [3]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set* of G. Two vertices u and v are *adjacent* if $(u, v) \in E$. A *path* is represented by $\langle v_0, v_1, v_2, \cdots, v_k \rangle$, where all vertices are distinct. The *length* of a path Q is the number of edges in Q. We also write the path $\langle v_0, v_1, v_2, \cdots, v_k \rangle$ as $\langle v_0, Q_1, v_i, v_{i+1}, \cdots, v_j, Q_2, v_t, \cdots, v_k \rangle$, where Q_1 is the path $\langle v_0, v_1, \cdots, v_{i-1}, v_i \rangle$ and Q_2 is the path $\langle v_j, v_{j+1}, \cdots, v_{t-1}, v_t \rangle$. Hence, it is possible to write a path $\langle v_0, v_1, Q, v_1, v_2, \cdots, v_k \rangle$ if the length of Q is zero. We use $d_G(u, v)$ to denote the distance between u and v in G, i.e., the length of the shortest path joining u and v in G. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. We use $d_C(u, v)$ to denote the distance between u and v in a hamiltonian cycle C of G, i.e., the length of the path joining u and v in C. A *hamiltonian graph* is a graph with a hamiltonian cycle.

Hamiltonian graphs is perhaps the most important outstanding materials in graph theory and has been defying solutions for more than a century. Further attempts at hamiltonian problems led researchers into the study of super-hamiltonian graphs, such as pancyclic graphs and panconnected graphs.

A graph is *pancyclic* if it contains a cycle of every length from 3 to $|V(G)|$ inclusive. The concept of pancyclic graphs is proposed by Bondy [2]. A graph $G = (V_0 \cup V_1, E)$ is *bipartite* if

$V(G) = V_0 \cup V_1$ and $E(G)$ is a subset of $\{(u,v) \mid u \in V_0, v \in V_1\}$. It is known that there is no odd cycle in any bipartite graph. Hence, any bipartite graph is not pancyclic. For this reason, the concept of bipancyclicity is proposed [8]. A bipartite graph is *bipancyclic* if it contains a cycle of every even length from 4 to $|V(G)|$ inclusive. It is proved that the hypercube is bipancyclic [5, 9].

A graph G is *panconnected* if there exists a path of length l joining any two different vertices x and y with $d_G(x,y) \leq l \leq |V(G)| - 1$. The concept of panconnected graphs is proposed by Alavi and Williamson [1]. It is obvious that any bipartite graph with at least 3 vertices is not panconnected. For this reason, we say a bipartite graph is *bipanconnected* if there exists a path of length l joining any two different vertices x and y with $d_G(x,y) \leq l \leq |V(G)| - 1$ and $(l - d_G(x,y))$ is even. It is proved that the hypercube is bipanconnected [5].

Here, we introduce a new concept, called panpositionable hamiltonian. A hamiltonian graph G is *panpositionable* if for any two different vertices x and y of G and any integer k with $d_G(x,y) \leq k < |V(G)|/2$, there exists a hamiltonian cycle C of G with $d_C(x,y) = k$. Obviously, the complete graph K_n with $n \geq 3$ is panpositionable. It is easy to see that the length of the shortest cycle for any panpositionable hamiltonian graph is 3. A hamiltonian bipartite graph G is *bipanpositionable* if for any two different vertices x and y of G and for any integer k with $d_G(x,y) \leq k < |V(G)|/2$ and $(k - d_G(x,y))$ is even, there exists a hamiltonian cycle C of G such that $d_C(x,y) = k$. Obviously, the complete bipartite graph $K_{n,n}$ with $n \geq 2$ is bipanpositionable.

Let $\mathbf{u} = u_{n-1}u_{n-2}\ldots u_1u_0$ and $\mathbf{v} = v_{n-1}v_{n-2}\ldots v_1v_0$ be two n-bit binary strings. The Hamming distance $h(\mathbf{u},\mathbf{v})$ between two vertices $\mathbf{u}$ and $\mathbf{v}$ is the number of different bits in the corresponding strings of both vertices. The *n-dimensional hypercube*, Q_n, consists of all n-bit binary strings as its vertices and two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent if and only if $h(\mathbf{u},\mathbf{v}) = 1$. Let Q_n^i be the subgraph of Q_n induced by $\{u_{n-1}u_{n-2}\dots u_1u_0 \mid u_{n-1} = i\}$ for $i = 0, 1$. Obviously, Q_n can be constructed recursively by taking two copies of Q_{n-1}, Q_{n-1}^0 and Q_{n-1}^1, and adding a perfect matching. We will prove that Q_n is bipanpositionable hamiltonian.

Assume that $n, s_1, s_2, \dots, s_r$ are integers with $1 \leq s_1 < s_2 < \dots < s_r \leq \frac{n}{2}$. The circulant graph $G(n; s_1, s_2, \dots, s_r)$ is the graph with the vertex set $\{0, 1, \dots, n-1\}$. Two vertices i and j are adjacent if and only if $i - j = \pm s_k \pmod{n}$ for some k where $1 \leq k \leq r$. We will prove that $G(n; 1, 3)$ is bipanpositionable for any even integer with $n \geq 6$, and $G(n; 1, 2)$ is panpositionable if and only if $n \in \{5, 6, 7, 8, 9, 11\}$. Moreover, $G(n; 1, 2, 3)$ is panpositionable for $n \geq 6$.

2 Some bipanpositionable hamiltonian graphs

Theorem 1 *Q_n is bipanpositionable hamiltonian for $n \geq 2$.*

Proof. Obviously, the theorem is true for Q_2. Now, we assume that the theorem is true for Q_{n-1} for some $n \geq 3$. Let $\mathbf{u}$ and $\mathbf{v}$ be two distinct vertices of Q_n with $h(\mathbf{u},\mathbf{v}) = r$. It is known that $h(\mathbf{u},\mathbf{v}) = d_{Q_n}(\mathbf{u},\mathbf{v})$. We need to show that for any integer i with $r \leq i \leq 2^{n-1} - 1$ and $i - r$ is even, there exists a hamiltonian cycle C of Q_n such that $d_C(\mathbf{u},\mathbf{v}) = i$. Since Q_n is edge symmetric, Q_n can be split into Q_{n-1}^0 and Q_{n-1}^1 such that $\mathbf{u} \in Q_{n-1}^0$ and $\mathbf{v} \in Q_{n-1}^1$. Let $\mathbf{y} = y_{n-1}y_{n-2}\cdots y_1y_0 \in V(Q_n)$. We use $\mathbf{y}^k$ to denote the vertex $y_{n-1}y_{n-2}\cdots\overline{y_k}\cdots y_1y_0$ for some $0 \leq k \leq n-1$. Let $\mathbf{z} = \mathbf{v}^{n-1}$. Obviously, $d_{Q_{n-1}^0}(\mathbf{u},\mathbf{z}) = r - 1$ and $\mathbf{z} = \mathbf{u}$ if $d_{Q_n}(\mathbf{u},\mathbf{v}) = 1$. By induction assumption, there exists a hamiltonian cycle $C = \langle \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2^{n-1}}, \mathbf{x}_1\rangle$ of Q_{n-1}^0 such that $d_C(\mathbf{u},\mathbf{z}) = r - 1$. Without loss of generality, we assume that $\mathbf{x}_1 = \mathbf{u}$ and $\mathbf{x}_r = \mathbf{z}$. Note that $r \leq \frac{i+r}{2}$ and $\mathbf{x}_r^{n-1} = \mathbf{v}$. Let

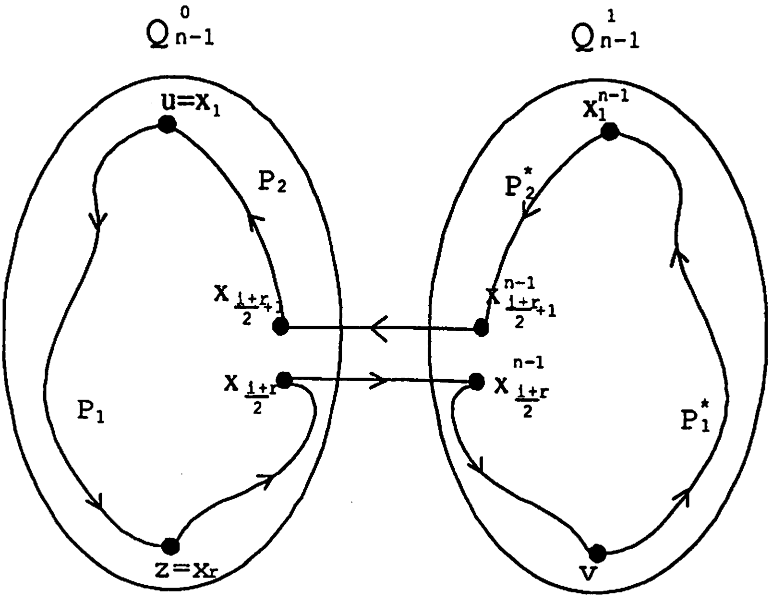

Figure 1: The hamiltonian cycle in Theorem 1.

$P_1 = \langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r, \ldots, \mathbf{x}_{\frac{i+r}{2}} \rangle$ and
$P_2 = \langle \mathbf{x}_{\frac{i+r}{2}+1}, \mathbf{x}_{\frac{i+r}{2}+2}, \ldots, \mathbf{x}_{2^{n-1}}, \mathbf{x}_1 \rangle$. We set
$P_1^* = \langle \mathbf{x}^{n-1}_{\frac{i+r}{2}}, \mathbf{x}^{n-1}_{\frac{i+r}{2}-1}, \ldots, \mathbf{x}^{n-1}_r, \ldots, \mathbf{x}^{n-1}_1 \rangle$ and
$P_2^* = \langle \mathbf{x}^{n-1}_1, \mathbf{x}^{n-1}_{2^{n-1}}, \mathbf{x}^{n-1}_{2^{n-1}-1}, \ldots, \mathbf{x}^{n-1}_{\frac{i+r}{2}+1} \rangle$.
Let $C_i = \langle \mathbf{x}_1, P_1, \mathbf{x}_{\frac{i+r}{2}}, \mathbf{x}^{n-1}_{\frac{i+r}{2}}, P_1^*, \mathbf{x}^{n-1}_1, P_2^*, \mathbf{x}^{n-1}_{\frac{i+r}{2}+1}, \mathbf{x}_{\frac{i+r}{2}+1}, P_2, \mathbf{x}_1 \rangle$.
Obviously, C_i be a hamiltonian cycle of Q_n and $d_C(\mathbf{u}, \mathbf{v}) = i$. See Figure 2 as an illustration. □

Theorem 2 *$G(n; 1, 3)$ is bipanpositionable hamiltonian if and only if n is an even integer and $n \geq 6$.*

Proof. Let $H = G(n; 1, 3)$. Obviously, H is bipartite if and only if n is even. Thus, H is not bipanpositionable hamiltonian if n is odd. Assume that n is an even integer with $n \geq 6$. With the symmetric property of H, it suffices to show that there exists a hamiltonian cycle C such that $d_C(0,u) = k$ for any vertex u of H with $1 \leq u \leq \frac{n}{2}$, and any integer k with $d_H(0,u) \leq k \leq \frac{n}{2}$ and $k - d_H(0,u)$ is even. It is easy to see that $d_H(0,u) = \lceil \frac{u}{3} \rceil$. We set $r = \lceil \frac{u}{3} \rceil$. To describe the required hamiltonian cycles, we define some path patterns:

$$\begin{aligned} p(i,j) &= \langle i, i+1, i+2, \ldots, j-1, j \rangle; \\ q(i,i+3) &= \langle i, i+3 \rangle; \\ q^{-1}(i,i-3) &= \langle i, i-3 \rangle. \end{aligned}$$

Then we define the path pattern q^t by executing the path pattern q for t times. Similarly for $(q^{-1})^t$. More precisely,

$$\begin{aligned} q^t(i,i+3t) &= \langle i, q(i,i+3), i+3, q(i+3,i+6), \ldots, \\ &\quad i+3(t-1), q(i+3(t-1), i+3t), i+3t \rangle; \\ (q^{-1})^t(i,i-3t) &= \langle i, q^{-1}(i,i-3), i-3, q^{-1}(i-3,i-6), \ldots, \\ &\quad i-3(t-1), q^{-1}(i-3(t-1), i-3t), i-3t \rangle. \end{aligned}$$

There are three cases:

Case 1. $u \equiv 0 \pmod 3$.

(1.1) $r \leq k \leq u$. Let $l = \frac{k-r}{2}$. The hamiltonian cycle is

$$\begin{aligned} C = \; & \langle 0, p(0,3l), 3l, q^{\frac{u-3l}{3}}(3l,u), u, u+1, \\ & (q^{-1})^{\frac{u-3l}{3}}(u+1, 3l+1), 3l+1, 3l+2, \\ & q^{\frac{u-3l}{3}}(3l+2, u+2), u+2, p(u+2, n-1), \\ & n-1, 0 \rangle. \end{aligned}$$

(1.2) $u < k \leq \frac{n}{2}$. Let $l = \frac{k-u}{2}$. The hamiltonian cycle is

$$\begin{aligned} C = \; & \langle 0, p(0,u-1), u-1, q^l(u-1, u+3l-1), u+3l-1, \\ & u+3l-2, (q^{-1})^{l-1}(u+3l-2, u+1), u+1, u, \\ & q^l(u, u+3l), u+3l, p(u+3l, n-1), n-1, 0 \rangle. \end{aligned}$$

Case 2. $u \equiv 1 \pmod 3$.

(2.1) $r \le k \le u$. Let $l = \frac{k-r}{2}$. The hamiltonian cycle is

$$\begin{aligned} C \;=\; & \langle 0, 1, p(1, 3l+1), 3l+1, q^{\frac{u-3l-1}{3}}(3l+1, u), u, u+1, \\ & (q^{-1})^{\frac{u-3l-1}{3}}(u+1, 3l+2), \\ & 3l+2, 3l+3, q^{\frac{u-3l-1}{3}}(3l+3, u+2), \\ & u+2, p(u+2, n-1), n-1, 0\rangle. \end{aligned}$$

(2.2) $u < k \le \frac{n}{2}$. Let $l = \frac{k-u}{2}$. The hamiltonian cycle is

$$\begin{aligned} C \;=\; & \langle 0, p(0, u-1), u-1, q^{l}(u-1, u+3l-1), u+3l-1, \\ & u+3l, (q^{-1})^{l}(u+3l, u), u, u+1, q^{l}(u+1, u+3l+1), \\ & u+3l+1, p(u+3l+1, n-1), n-1, 0\rangle. \end{aligned}$$

Case 3. $u \equiv 2 \pmod 3$.

(3.1) $r \le k < u$. Let $l = \frac{k-r}{2}$. The hamiltonian cycle is

$$\begin{aligned} C \;=\; & \langle 0, p(0, 3l+2), 3l+2, q^{\frac{u-3l-2}{3}}(3l+2, u), u, u-1, \\ & (q^{-1})^{\frac{u-3l-5}{3}}(u-1, 3l+4), 3l+4, 3l+3, \\ & q^{\frac{u-3l-2}{3}}(3l+3, u+1), u+1, p(u+1, n-1), n-1, 0\rangle. \end{aligned}$$

(3.2) $k = u$. The hamiltonian cycle is

(1) $$C = \langle 0, p(0, n-1), n-1, 0\rangle.$$

(3.3) $u < k \le \frac{n}{2}$. Let $l = \frac{k-u}{2}$. The hamiltonian cycle is

$$\begin{aligned} C \;=\; & \langle 0, p(0, u-2), u-2, q^{l}(u-2, u+3l-2), u+3l-2, \\ & u+3l-1, (q^{-1})^{l}(u+3l-1, u-1), u-1, u, \\ & q^{l}(u, u+3l), u+3l, p(u+3l, n-1), n-1, 0\rangle. \end{aligned}$$

The theorem is proved. □

3 Some panpositionable hamiltonian graphs

Theorem 3 $G(n; 1, 2)$ *is panpositionable hamiltonian if and only if* $n \in \{5, 6, 7, 8, 9, 11\}$.

Proof. Let $H = G(n; 1, 2)$. We first show that H is panpositionable if $n \in \{5, 6, 7, 8, 9, 11\}$. With the symmetric property of H, it suffices to show that for any vertex u with $1 \leq u \leq \frac{n}{2}$ and for any integer k with $d_H(0, u) \leq k \leq \frac{n}{2}$, there exists a hamiltonian cycle C such that $d_C(0, u) = k$. It is easy to see that $d_H(0, u) = \lceil \frac{u}{2} \rceil$. We set $r = \lceil \frac{u}{2} \rceil$. To describe the required hamiltonian cycles, we define some path patterns:

$$\begin{aligned} p(i, j) &= \langle i, i+1, i+2, \ldots, j-1, j \rangle; \\ q(i, j) &= \langle i, i+2, i+4, \ldots, j-2, j \rangle; \\ q^{-1}(j, i) &= \langle j, j-2, j-4 \ldots, i+2, i \rangle. \end{aligned}$$

Case 1. $n \in \{5, 7, 9, 11\}$.

$\{0, u\}$	$d_C(0, u)$	Hamiltonian cycle C
$\{0, 1\}$	1	$\langle 0, p(0, n-1), n-1, 0 \rangle$.
$n \in \{5, 7, 9, 11\}$	2	$\langle 0, 2, 1, 3, p(3, n-1), n-1, 0 \rangle$.
	$3, n \in \{7, 9, 11\}$	$\langle 0, 2, 3, 1, n-1, q^{-1}(n-1, 4), 4, 5, q(5, n-2), n-2, 0 \rangle$.
	$4, n \in \{9, 11\}$	$\langle 0, 2, 4, 3, 1, n-1, q^{-1}(n-1, 6), 6, 5, q(5, n-2), n-2, 0 \rangle$.
	$5, n = 11$	$\langle 0, 9, q^{-1}(9, 1), 1, 2, q(2, 10), 10, 0 \rangle$.

$\{0,u\}$	$d_C(0,u)$	Hamiltonian cycle C
$\{0,2\}$	1	$\langle 0,2,1,3,p(3,n-1),$ $n-1,0\rangle$.
$n \in \{5,7,9,11\}$	2	$\langle 0,p(0,n-1),n-1,0\rangle$.
	3, $n \in \{7,9,11\}$	$\langle 0,n-1,1,p(1,n-2),$ $n-2,0\rangle$.
	4, $n \in \{9,11\}$	$\langle 0,n-1,1,3,2,4,$ $p(4,n-2),n-2,0\rangle$.
	5, $n=11$	$\langle 0,9,q^{-1}(9,3),3,2,$ $q(2,10),10,1,0\rangle$.
$\{0,3\}$	2	$\langle 0,2,p(2,n-1),n-1,$ $1,0\rangle$.
$n \in \{7,9,11\}$	3	$\langle 0,p(0,n-1),n-1,0\rangle$.
	4, $n \in \{9,11\}$	$\langle 0,1,2,4,3,5,$ $p(5,n-1),n-1,0\rangle$.
	5, $n=11$	$\langle 0,10,1,2,4,3,5,$ $p(5,9),9,0\rangle$.
$\{0,4\}$	2	$\langle 0,q(0,n-1),n-1,1,$ $q(1,n-2),n-2,0\rangle$.
$n \in \{9,11\}$	3	$\langle 0,2,p(2,n-1),n-1,$ $1,0\rangle$.
	4	$\langle 0,p(0,n-1),n-1,0\rangle$.
	5, $n=11$	$\langle 0,p(0,3),3,5,4,6,$ $p(6,10),10,0\rangle$.
$\{0,5\}$	3	$\langle 0,1,q(1,9),9,10,$ $q^{-1}(10,0),0\rangle$.
$n=11$	4	$\langle 0,10,1,q(1,9),9,8,$ $q^{-1}(8,0),0\rangle$.
	5	$\langle 0,p(0,10),10,0\rangle$.

Case 2. $n \in \{6,8\}$.

$\{0,u\}$	$d_C(0,u)$	Hamiltonian cycle C
$\{0,1\}$ $n \in \{6,8\}$	1	$\langle 0, p(0,n-1), n-1, 0\rangle$.
	2	$\langle 0,2,1,3,p(3,n-1),n-1,0\rangle$.
	3	$\langle 0,2,3,1,n-1,q^{-1}(n-1,5),5,4,$ $q(4,n-2),n-2,0\rangle$.
	$4, n=8$	$\langle 0,2,4,3,1,7,5,6,0\rangle$.
$\{0,2\}$ $n \in \{6,8\}$	1	$\langle 0,2,1,3,p(3,n-1),n-1,0\rangle$.
	2	$\langle 0,p(0,n-1),n-1,0\rangle$.
	3	$\langle 0,1,3,2,4,p(4,n-1),n-1,0\rangle$.
	$4, n=8$	$\langle 0,7,1,3,2,4,5,6,0\rangle$.
$\{0,3\}$ $n \in \{6,8\}$	2	$\langle 0,2,p(2,n-1),n-1,1,0\rangle$.
	3	$\langle 0,p(0,n-1),n-1,0\rangle$.
	$4, n=8$	$\langle 0,1,2,4,3,5,6,7,0\rangle$.
$\{0,4\}$ $n=8$	2	$\langle 0,q(0,6),7,q^{-1}(7,1),1,0\rangle$.
	3	$\langle 0,2,p(2,7),7,1,0\rangle$.
	4	$\langle 0,p(0,7),7,0\rangle$.

To show that H is not panpositionable hamiltonian if $n = 10$ or $n \geq 12$, we prove that there exists no hamiltonian cycle in H such that the distance between 0 and 2 is 5. Suppose that C is a hamiltonian cycle of H with $d_C(0,2) = 5$. Obviously, $P_1 = \langle 0, n-2, n-1, 1, 3, 2\rangle$, $P_2 = \langle 0, n-1, 1, 3, 4, 2\rangle$ and $P_3 = \langle 0,1,3,5,4,2\rangle$ are all the possible paths of length 5 joining 0 and 2. Then C contains exactly one of P_1, P_2 and P_3.

If C contains P_1, then $\{(0,1),(0,n-1)\} \not\subseteq C$. Thus, C contains $\langle n-2, 0, 2\rangle$. This means C contains a cycle $\langle 0, P_1, 2, 0\rangle$, which is impossible. If C contains P_2 or P_3, then $\{(2,1),(2,3)\} \not\subseteq C$. Thus, C contains $\langle 0,2,4\rangle$. This means that C contains a cycle $\langle 0, P_2, 2, 0\rangle$ or $\langle 0, P_3, 2, 0\rangle$, respectively, which is impossible. The theorem is proved. □

Theorem 4 *$G(n;1,2,3)$ is panpositionable hamiltonian for $n \geq 5$.*

Proof. Let $H = G(n;1,2,3)$ and u be any vertex of H with $1 \leq u \leq \frac{n}{2}$. Since $G(n;1,2)$ is a spanning subgraph of H, with

Theorem 3, H is panpositionable hamiltonian when $n = 5$. It is easy to see that $d_H(0,u) = \lceil \frac{u}{3} \rceil$. We set $r = \lceil \frac{u}{3} \rceil$. With the symmetric property of H, it suffices to show that there exists a hamiltonian cycle C such that $d_C(0,u) = k$ for any integer k with $r \leq k \leq \frac{n}{2}$. Suppose that $k - r$ is even. Since $G(n; 1, 3)$ is a spanning subgraph of H, we can use the similar argument as in Theorem 2, no matter n is odd or even, to prove that there exists a hamiltonian cycle C of H such that $d_C(0,u) = k$. Therefore, we only consider the cases $k - r$ is odd. To describe the required hamiltonian cycles, we define some path patterns:

$$
\begin{aligned}
p(i,j) &= \langle i, i+1, i+2, \ldots, j-1, j \rangle; \\
q(i,i+3) &= \langle i, i+3 \rangle; \\
q^{-1}(i,i-3) &= \langle i, i-3 \rangle; \\
q^t(i,i+3t) &= \langle i, q(i,i+3), i+3, q(i+3,i+6), \ldots, \\
&\quad i+3(t-1), q(i+3(t-1), i+3t), \\
&\quad i+3t \rangle; \\
(q^{-1})^t(i,i-3t) &= \langle i, q^{-1}(i,i-3), i-3, q^{-1}(i-3,i-6), \ldots, \\
&\quad i-3(t-1), q^{-1}(i-3(t-1), i-3t), \\
&\quad i-3t \rangle; \\
r_1^t(0,3t) &= \langle 0, p(0,3t-3), 3t-3, 3t-2, 3t \rangle; \\
s_1^t(u-1,u+1) &= \langle u-1, q^t(u-1,u+3t-1), u+3t-1, \\
&\quad u+3t+1, (q^{-1})^t(u+3t+1,u+1), \\
&\quad u+1 \rangle; \\
r_2^t(0,3t+1) &= \langle 0, p(0,3t-1), 3t-1, 3t+1 \rangle; \\
s_2^t(u-1,u) &= \langle u-1, q^t(u-1,u+3t-1), u+3t-1, \\
&\quad u+3t-3, (q^{-1})^{t-1}(u+3t-3,u), u \rangle; \\
r_3^t(0,3t+2) &= \langle 0, p(0,3t), 3t, 3t+2 \rangle; \\
s_3^t(u-1,u) &= \langle u-1, q^{t+1}(u-1,u+3t+2), u+3t+2, \\
&\quad u+3t, (q^{-1})^t(u+3t,u), u \rangle.
\end{aligned}
$$

There are three cases:

Case 1. $u \equiv 0 \pmod 3$.

(1.1) $r < k < u$. Let $l = \frac{k-r+1}{2}$. The hamiltonian cycle is

$$\begin{aligned} C &= \langle 0, r_1^l(0,3l), 3l, q^{\frac{u-3l}{3}}(3l,u), u, u+1, \\ &\quad (q^{-1})^{\frac{u-3l}{3}}(u+1,3l+1), 3l+1, 3l-1, \\ &\quad q^{\frac{u-3l+3}{3}}(3l-1,u+2), u+2, p(u+2,n-1), n-1, 0\rangle. \end{aligned}$$

(1.2) $u < k \leq \frac{n}{2}$. Let $l = \frac{k-u-1}{2}$. The hamiltonian cycle is

$$\begin{aligned} C &= \langle 0, p(0,u-1), u-1, s_1^l(u-1,u+1), u+1, u, \\ &\quad q^{l+1}(u,u+3l+3), u+3l+3, u+3l+4, u+3l+2, \\ &\quad u+3l+5, p(u+3l+5,n-1), n-1, 0\rangle. \end{aligned}$$

Case 2. $u \equiv 1 \pmod 3$.

(2.1) $r < k < u$. Let $l = \frac{k-r+1}{2}$. The hamiltonian cycle is

$$\begin{aligned} C &= \langle 0, r_2^l(0,3l+1), 3l+1, q^{\frac{u-3l-1}{3}}(3l+1,u), u, u-1, \\ &\quad (q^{-1})^{\frac{u-3l-1}{3}}(u-1,3l), 3l, 3l+2, q^{\frac{u-3l-1}{3}}(3l+2,u+1), \\ &\quad u+1, p(u+1,n-1), n-1, 0\rangle. \end{aligned}$$

(2.2) $u < k \leq \frac{n}{2}$. Let $l = \frac{k-u+1}{2}$. The hamiltonian cycle is

$$\begin{aligned} C &= \langle 0, p(0,u-1), u-1, s_2^l(u-1,u), u, u+1, \\ &\quad q^{l-1}(u+1,u+3l-2), u+3l-2, u+3l, p(u+3l,n-1), \\ &\quad n-1, 0\rangle. \end{aligned}$$

Case 3. $u \equiv 2 \pmod 3$.

(3.1) $r < k < u$. Let $l = \frac{k-r+1}{2}$. The hamiltonian cycle is

$$\begin{aligned} C &= \langle 0, p(0,3l-1), 3l-1, q^{\frac{u-3l+1}{3}}(3l-1,u), u, u-1, \\ &\quad (q^{-1})^{\frac{u-3l-2}{3}}(u-1,3l+1), 3l+1, 3l, q^{\frac{u-3l+1}{3}}(3l,u+1), \\ &\quad u+1, p(u+1,n-1), n-1, 0\rangle. \end{aligned}$$

(3.2) $k = u$. The hamiltonian cycle is

$$C = \langle 0, p(0, n-1), n-1, 0 \rangle.$$

(3.3) $u < k \leq \frac{n}{2}$. Let $l = \frac{k-u}{2}$. The hamiltonian cycle is

$$\begin{aligned} C = & \langle 0, p(0, u-2), u-2, q^l(u-2, u-2+3l), u-2+3l, \\ & u-1+3l, q^{-l}(u-1+3l, u-1), u-1, u, q^l(u, u+3l), \\ & u+3l, p(u+3, n-1), n-1, 0 \rangle. \end{aligned}$$

The theorem is proved. □

4 Concluding Remark

A *k-container* $C(x, y)$ in a graph G is a set of k internal vertex-disjoint paths between x and y. Based on Menger's Theorem [7], there exists a k-container between any pair of vertices in a k-connected graph. The length of a k-container $C(x, y)$, written as $l(C(x, y))$, is the length of the longest path in $C(x, y)$. Suppose that G is a k-connected graph. The *k-distance* between x and y, denoted by $d_k(x, y)$, is defined as $\min\{l(C(x, y)) \mid C(x, y)$ is a k-container$\}$. The *k-diameter* of G, denoted by $D_k(G)$, is defined as $\max\{d_k(x, y) \mid x \neq y; x, y \in V(G)\}$. The k-diameter, proposed by Hsu [4], measures the performance of multigraph communication.

Now, we introduce another type of containers. A *k^*-container* $C(x, y)$ is a k-container such that every vertex of G is incident with a path in $C(x, y)$. A graph G is *k^*-connected* if there exists a k^*-container between any two vertices x and y with $x \neq y$. Obviously, a graph G is 1^*-connected if and only if it is hamiltonian connected. Moreover, a graph G is 2^*-connected if it is hamiltonian. The concept of k^*-connected graphs is proposed by Lin et. al.[6].

Suppose that G is a k^*-connected graph. Similar to the definitions of k-distance and k-diameter, we can define the k^*-*distance*, $d_k^*(x,y)$, as $\min\{C(x,y) \mid C(x,y) \text{ is a } k^*\text{-container}\}$. The k^*-*diameter*, denoted by $D_k^*(G)$, is defined by $\max\{d_k^*(x,y) \mid x \neq y; x, y \in V(G)\}$.

Assume that G is a panpositionable hamiltonian graph with n vertices. Obviously, $d_2^*(u,v) = \lceil \frac{n}{2} \rceil$ if u and v are two different vertices in G. Hence $D_2^*(G) = \lceil \frac{n}{2} \rceil$. Similarly, let G be a bipanpositionable hamiltonian graph with n vertices. Obviously, $d_2^*(u,v)$ is either $\lceil \frac{n}{2} \rceil + 1$ or $\lceil \frac{n}{2} \rceil$ depending on the parity of $d(u,v)$. (Note that $d_2^*(u,v) = d(u,v)$.) Thus, $D_2^*(G) = \lceil \frac{n}{2} \rceil + 1$. In particular, $D_2^*(Q_n) = 2^{n-1} + 1$ for $n \geq 2$.

Let $f(n)$ denote the minimum number of edges among any panpositionable hamiltonian graph with n vertices. With Theorem 4, we know that $f(n) \leq 3n$ if $n \geq 6$. It is interesting to find the asymptotic value of $f(n)$ as n is large. Similarly, let $f_b(n)$ be the minimum number of edges among any bipanpositionable hamiltonian graph with n vertices. Obviously, $f(n) = 0$ if n is odd. With Theorem 2, $f_b(n) \leq 2n$ if n is an even integer with $n \geq 6$. It is interesting to find the asymptotic value of $f_b(n)$ as n is large and n is even.

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