EDGE ZETA FUNCTIONS OF GRAPH COVERINGS

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Abstract

We give a decomposition formula for the edge zeta function of a regular covering $\tilde{G}$ of a graph G. Furthermore, we present a determinant expression for some L-function of an oriented line graph $\vec{L}(G)$ of G. As a corollay, we obtain a factorization formula for the edge zeta function of $\tilde{G}$ by L-functions of $\vec{L}(G)$.

Key words: zeta function, graph covering, L-function

1 Introduction

Graphs and digraphs treated here are finite and simple. Let $G = (V(G), E(G))$ be a connected graph with vertex $V(G)$ and arc set $E(G)$, and D the symmetric digraph corresponding to G. Note that $E(G) = E(D)$. For $e = (u, v) \in E(G)$, let $o(e) = u$ and $t(e) = v$. The inverse arc of e is denoted by $\bar{e}$. A path P of length n in D(or G) is a sequence $P = (v_0, v_1, \cdots, v_{n-1}, v_n)$ of $n + 1$ vertices and n arcs(or edges) such that consecutive vertices share an arc(or edge) (we do not require that all vertices are distinct). Also, P is called a (v_0, v_n)-path. If $e_i = (v_i, v_{i+1})$ for $i = 1, \cdots, n-1$, then we can write $P = (e_1, \cdots, e_{n-1})$. We say that a path has a backtracking if a subsequence of the form $\cdots, x, y, x, \cdots$ appears. A (v, w)-path is called a cycle (or closed path) if $v = w$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (v_1, \cdots, v_m)$ and $C_2 = (w_1, \cdots, w_m)$ are called equivalent if there exists an integer k such that $w_j = v_{j+k}$ for all j. Let $[C]$ be the equivalnce class

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which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a multiple of B. A cycle C is said to be reduced if both C and C^2 have no backtracking. A cycle C is prime if $C \neq B^r$ for any other cycle B and $r \geq 2$.

The (Ihara) zeta function of a graph G is defined to be a formal power series of a variable u, by

$$\mathbf{Z}(G,u) = \mathbf{Z}_G(u) = \prod_{[C]} (1 - u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G, and $|C|$ is the length of C.

Zeta functions of graphs started from zeta functions of regular graphs by Ihara [7]. In [7], he showed that their reciprocals are explicit polynomials. A zeta function of a regular graph G associated to a unitary representation of the fundamental group of G was developed by Sunada [14,15]. Hashimoto [6] treated multivariable zeta functions of bipartite graphs. Bass [2] generalized Ihara's result on the zeta function of a regular graph to an irregular graph, and showed that the reciprocal of the zeta function of a graph G is a polynomial:

$$\mathbf{Z}(G,u)^{-1} = (1-u^2)^{r-1} \det(\mathbf{I} - u\mathbf{A}(G) + u^2(\mathbf{D} - \mathbf{I})),$$

where r and $\mathbf{A}(G)$ is the Betti number and the adjacency matrix of G, respectively, and $\mathbf{D} = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg v_i (V(G) = \{v_1, \cdots, v_n\})$.

Stark and Terras [12] gave an elementary proof of Bass' Theorem, and discussed three different zeta functions of any graph. Furthermore, various proofs of Bass' Theorem were given by Foata and Zeilberger [4], Kotani and Sunada [9]. Mizuno and Sato [10] obtained a decomposition formula for the zeta function of a regular covering of a graph.

Stark and Terras [12] introduced two zeta functions of graphs based on edge and path, and presented determinant expressions of them.

Let G be a connected graph and $V(G) = \{v_1, \ldots, v_n\}$. We associate with each of its arc $e = (v_i, v_j)$ a complex variable $w_e = w(e) = w(v_i, v_j)$. For each path $P = (v_{i_1}, \ldots, v_{i_r})$ of G, the weight $w(P)$ of P is defined as follows: $w(P) = w(v_{i_1}, v_{i_2}) w(v_{i_2}, v_{i_3}) \ldots w(v_{i_{r-1}}, v_{i_r})$. The edge zeta function of G is defined by

$$\zeta_G(w) = \prod_{[C]} (1 - w(C))^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

Let $G = (V, E)$ be a connected graph. The oriented line graph $\vec{L}(G) = (V_L, E_L)$ of G is defined as follows:

$$V_L = E; \; E_L = \{(e_1, e_2) \in E \times E \mid \bar{e}_1 \neq e_2, t(e_1) = o(e_2)\}.$$

Theorem 1 **(Stark and Terras)** *Let G be a connected graph. Then we have*

$$\zeta_G(w)^{-1} = \det(\mathbf{I} - \mathbf{U}\mathbf{A}(\vec{L}(G))) = \det(\mathbf{I} - \mathbf{A}(\vec{L}(G))\mathbf{U}),$$

where $\mathbf{U}$ *is the diagonal matrix*

$$\mathbf{U} = \mathrm{diag}(w_{e_1}, \ldots, w_{e_{2l}}), E(G) = \{e_1, \ldots, e_l, e_{l+1}, \ldots, e_{2l}\}.$$

Let $E(G) = \{e_1, \ldots, e_l, e_{l+1}, \ldots, e_{2l}\}$, where $e_{l+i} = \bar{e}_i (1 \leq i \leq l)$. Furthermore, let $\vec{\mathbf{W}} = \mathbf{W}(\vec{L}(G))$ be a $2l \times 2l$ matrix with ij entry the variable w_{e_i} if $t(e_i) = o(e_j)$, $e_j \neq \bar{e}_i$, and 0 otherwise. The matrix $\vec{\mathbf{W}} = \mathbf{W}(\vec{L}(G))$ is called the weighted matrix of G.

Mizuno and Sato [11] gave another determinant expression for the edge zeta function of a graph.

Theorem 2 **(Mizuno and Sato)** *Let G be a connected graph. Then the reciprocal of the edge zeta function of G is*

$$\zeta_G(w)^{-1} = \det(\mathbf{I} - \vec{\mathbf{W}}).$$

Foata and Zeilberger [4] gave a new proof of Bass' Theorem by using the algebra of Lyndon words. Let X be a finite nonempty set, $<$ a total order in X, and X^* the free monoid generated by X. Then the total order $<$ on X derives the lexicographic order $<$ on X^*. A Lyndon word in X is defined to a nonempty word in X^* which is prime, i.e., not the power l^r of any other word l for any $r \geq 2$, and which is also minimal in the class of its cyclic rearrangements under $<$(see [8]). Let L denote the set of all Lyndon words in X.

Foata and Zeilberger[4] gave a short proof of Amitsur's identity [1].

Theorem 3 **(Amitsur)** *For square matrices* $\mathbf{A}_1, \cdots, \mathbf{A}_k$,

$$\det(\mathbf{I} - (\mathbf{A}_1 + \cdots \mathbf{A}_k)) = \prod_{l \in L} \det(\mathbf{I} - \mathbf{A}_l),$$

where the product runs over all Lyndon words in $\{1, \cdots, k\}$, *and* $\mathbf{A}_l = \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_p}$ *for* $l = i_1 \cdots i_p$.

In Section 2, we give a decomposition formula for the edge zeta function of a regular covering $\tilde{G}$ of a graph G. In Section 3, we present a determinant expression for some L-function of an oriented line graph $\vec{L}(G)$ of G. As a corollay, we obtain a factorization formula for the edge zeta function of $\tilde{G}$ by L-functions of $\vec{L}(G)$.

For a general theory of the representation of groups, the reader is referred to [3].

2 Edge zeta functions of regular coverings

Let G be a connected graph, and let $N(v) = \{w \in V(G) \mid (v,w) \in E(G)\}$ for any vertex v in G. A graph H is called a covering of G with projection $\pi : H \longrightarrow G$ if there is a surjection $\pi : V(H) \longrightarrow V(G)$ such that $\pi|_{N(v')} : N(v') \longrightarrow N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. When a finite group Π acts on a graph G, the quotient graph G/Π is a simple graph whose vertices are the Π-orbits on $V(G)$, with two vertices adjacent in G/Π if and only if some two of their representatives are adjacent in G. A covering $\pi : H \longrightarrow G$ is said to be regular if there is a subgroup B of the automorphism group $AutH$ of H acting freely on H such that the quotient graph H/B is isomorphic to G.

Let G be a graph and Γ a finite group. Then a mapping $\alpha : E(G) \longrightarrow \Gamma$ is called an ordinary voltage assignment if $\alpha(v,u) = \alpha(u,v)^{-1}$ for each $(u,v) \in E(G)$. The pair (G,α) is called an ordinary voltage graph. The derived graph G^α of the ordinary voltage graph (G,α) is defined as follows:

$$V(G^\alpha) = V(G) \times \Gamma \text{ and } ((u,h),(v,k)) \in E(G^\alpha) \text{ if and only if } (u,v) \in E(G) \text{ and } k = h\alpha(u,v).$$

The natural projection $\pi : G^\alpha \longrightarrow G$ is defined by $\pi(u,h) = u, (u,h) \in V(G^\alpha)$. The graph G^α is called a derived graph covering of G with voltages in Γ or a Γ-covering of G. The natural projection π commutes with the right multiplication action of the $\alpha(e), e \in E(G)$ and the left action of $g \in \Gamma$ on the fibers: $g \circ (u,h) = (u,gh), g \in \Gamma$, which is free and transitive. Thus, the Γ-covering G^α is a $\mid \Gamma \mid$-fold regular covering of G with covering transformation group Γ. Futhermore, every regular covering of a graph G is a Γ-covering of G for some group Γ(see [5]).

Let G be a connected graph, Γ a finite group and $\alpha : E(G) \longrightarrow \Gamma$ an ordinary voltage assignment. In the Γ-covering G^α, set $v_g = (v,g)$ and $e_g = (e,g)$, where $v \in V(G), e \in E(G), g \in \Gamma$. For $e = (u,v) \in E(G)$, the arc e_g emanates from u_g and terminates at $v_{g\alpha(e)}$. Note that $\bar{e_g} = (\bar{e})_{g\alpha(e)}$.

Let $w : E(G) \longrightarrow \mathbf{C}$ be a weight of G. Then we define the weight $\tilde{w}$ of G^α derived from w as follows:

$$\tilde{w}(u_g, v_h) := \begin{cases} w(u,v) & \text{if } (u,v) \in E(G) \text{ and } h = g\alpha(u,v), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the weighted matrix $\tilde{\mathbf{W}} = \mathbf{W}(\vec{L}(G^\alpha)) = (\tilde{w}(e_g, f_h))$ of G^α is given by

$$\tilde{w}(e_g, f_h) := \begin{cases} w(e) & \text{if } (e,f) \in E(\vec{L}(G)) \text{ and } h = g\alpha(e), \\ 0 & \text{otherwise.} \end{cases}$$

For $g \in \Gamma$, let the matrix $\vec{\mathbf{W}}_g = (w_{ef}^{(g)})$ be defined by

$$w_{ef}^{(g)} := \begin{cases} w(e) & \text{if } \alpha(e) = g \text{ and } (e,f) \in E(\vec{L}(G)), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$ be the block diagonal sum of square matrices $\mathbf{M}_1, \cdots, \mathbf{M}_s$. If $\mathbf{M}_1 = \mathbf{M}_2 = \cdots = \mathbf{M}_s = \mathbf{M}$, then we write $s \circ \mathbf{M} = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$. The Kronecker product $\mathbf{A} \bigotimes \mathbf{B}$ of matrices $\mathbf{A}$ and $\mathbf{B}$ is considered as the matrix $\mathbf{A}$ having the element a_{ij} replaced by the matrix $a_{ij}\mathbf{B}$.

Theorem 4 *Let G be a connected graph with l unoriented edges, Γ a finite group and $\alpha : E(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Furthermore, let $\rho_1 = 1, \rho_2, \cdots, \rho_t$ be all inequivalent irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$. Suppose that the Γ-covering G^α of G is connected. Then the reciprocal of the edge zeta function of G^α is*

$$\zeta_{G^\alpha}(\tilde{w})^{-1} = \zeta_G(w)^{-1} \cdot \prod_{i=2}^{t} \det(\mathbf{I}_{2lf_i} - \sum_{h\in\Gamma} \rho_i(h) \bigotimes \vec{\mathbf{W}}_h)^{f_i}.$$

Proof. Let $E(G) = \{e_1, \cdots, e_l, e_{l+1}, \cdots, e_{2l}\}$ and $\Gamma = \{1 = g_1, g_2, \cdots, g_m\}$. Arrange arcs of G^α in m blocks: $(e_1, 1), \cdots, (e_{2l}, 1); (e_1, g_2), \cdots, (e_{2l}, g_2); \cdots; (e_1, g_m), \cdots, (e_{2l}, g_m)$. We consider the weighted matrix $\mathbf{W}(\vec{L}(G^\alpha))$ under this order. For $h \in \Gamma$, the matrix $\mathbf{P}_h = (p_{ij}^{(h)})$ is defined as follows:

$$p_{ij}^{(h)} = \begin{cases} 1 & \text{if } g_i h = g_j, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $p_{ij}^{(h)} = 1$, i.e., $g_j = g_i h$. Then $((e, g_i), (f, g_j)) \in E(\vec{L}(G^\alpha))$ if and only if $(e,f) \in E(\vec{L}(G))$ and $(o(f), g_j) = o(f, g_j) = t(e, g_i) = (t(e), g_i\alpha(e))$, i.e., $\alpha(e) = g_i^{-1}g_j = g_i^{-1}g_i h = h$. Thus we have

$$\tilde{\mathbf{W}} = \mathbf{W}(\vec{L}(G^\alpha)) = \sum_{h\in\Gamma} \mathbf{P}_h \bigotimes \vec{\mathbf{W}}_h.$$

Let ρ be the right regular representation of Γ. Furthermore, let $\rho_1 = 1, \rho_2, \cdots, \rho_t$ be all inequivalent irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$. Then we have $\rho(h) = \mathbf{P}_h$ for $h \in \Gamma$. Furthermore, there exists a regular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\rho(h)\mathbf{P} = (1) \oplus f_2 \circ \rho_2(h) \oplus \cdots \oplus f_t \circ \rho_t(h)$ for each $h \in \Gamma$(see [3]). Putting $\mathbf{B} = (\mathbf{P}^{-1} \bigotimes \mathbf{I}_{2l})\mathbf{W}(\vec{L}(G^\alpha))(\mathbf{P} \bigotimes \mathbf{I}_{2l})$, we have

$$\mathbf{B} = \sum_{h\in\Gamma} \{(1) \oplus f_2 \circ \rho_2(h) \oplus \cdots \oplus f_t \circ \rho_t(h)\} \bigotimes \vec{\mathbf{W}}_h.$$

Note that $\mathbf{W}(\vec{L}(G)) = \sum_{h\in\Gamma} \vec{\mathbf{W}}_h$ and $1 + f_2^2 + \cdots + f_t^2 = m$. Theorem 2 implies that

$$\begin{aligned}\zeta_{G^\alpha}(\tilde{w})^{-1} &= \det(\mathbf{I}_{2lm} - \tilde{\mathbf{W}}) \\ &= \det(\mathbf{I}_{2l} - \vec{\mathbf{W}}) \prod_{i=2}^{t} \det(\mathbf{I}_{2lf_i} - \textstyle\sum_h \rho_i(h) \bigotimes \vec{\mathbf{W}}_h)^{f_i}\end{aligned}$$

□

3 Weighted L-function of oriented line graphs

Let G be a connected graph, Γ a finite group and $\alpha : E(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Furthermore, let ρ be a representation of Γ and d its degree. Then we define the function $\alpha_{\vec{L}} : E(\vec{L}(G)) \longrightarrow \Gamma$ as follows: $\alpha_{\vec{L}}(e, f) = \alpha(e), (e, f) \in E(\vec{L}(G))$. For each path $P = (e_1, \cdots, e_r)$ of $\vec{L}(G)$, let $\alpha_{\vec{L}}(P) = \alpha(e_1) \cdots \alpha(e_r)$. The weighted L-function of $\vec{L}(G)$ associated to ρ and α is defined by

$$\zeta_{\vec{L}(G)}(w, \rho, \alpha) = \prod_{[C]} \det(\mathbf{I}_d - \rho(\alpha_{\vec{L}}(C))w(C))^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of $\vec{L}(G)$.

Let $1 \leq i, j \leq n$. Then, the (i, j)-block $\mathbf{B}_{i,j}$ of an $dn \times dn$ matrix $\mathbf{B}$ is the submatrix of $\mathbf{B}$ consisting of $d(i-1)+1, \cdots, di$ rows and $d(j-1)+1, \cdots, dj$ columns.

Theorem 5 *Let G be a connected graph with l unoriented edges, Γ a finite group and $\alpha : E(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Furthermore, let ρ be a representation of Γ, and d the degree of ρ. Then the reciprocal of the weighted L-function of $\vec{L}(G)$ associated to ρ and α is*

$$\zeta_{\vec{L}(G)}(w, \rho, \alpha)^{-1} = \det(\mathbf{I} - \sum_{h\in\Gamma} \rho(h) \bigotimes \vec{\mathbf{W}}_h).$$

Proof. At first, let $E(G) = \{e_1, \cdots, e_l, e_{l+1}, \cdots, e_{2l}\}$ and consider the lexicographic order on $E(G) \times E(G)$ derived from a total order of $E(G)$: $e_1 < e_2 < \cdots < e_{2l}$. If (e_i, e_j) is the m-th pair under the above order, then we define the $2ld \times 2ld$ matrix $\mathbf{W}_m = ((\mathbf{W}_m)_{p,q})_{1\leq p,q\leq 2l}$ as follows:

$$(\mathbf{W}_m)_{p,q} = \begin{cases} \rho(\alpha(e_i))w(e_i) & \text{if } p = i, q = j \text{ and } (e_i, e_j) \in E(\vec{L}(G)), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, let $\mathbf{B} = \mathbf{W}_1 + \cdots + \mathbf{W}_k, k = 4l^2$. Then we have

$$\mathbf{B} = \sum_h \vec{\mathbf{W}}_h \bigotimes \rho(h).$$

Let L be the set of all Lyndon words in $E(G) \times E(G)$. Then we can also consider L as the set of all Lyndon words in $\{1, \cdots, k\}$: $(e_{i_1}, e_{j_1}) \cdots (e_{i_q}, e_{j_q})$ corresponds to $m_1 m_2 \cdots m_q$, where $(e_{i_r}, e_{j_r})(1 \leq r \leq q)$ is the m_r-th pair. Theorem 3 implies that

$$\det(\mathbf{I}_{2ld} - \mathbf{B}) = \prod_{t \in L} \det(\mathbf{I}_{2ld} - \mathbf{W}_t),$$

where $\mathbf{W}_t = \mathbf{W}_{i_1} \cdots \mathbf{W}_{i_p}$ for $t = i_1 \cdots i_p$. Note that $\det(\mathbf{I}_{2ld} - \mathbf{W}_t)$ is the alternating sum of the diagonal minors of $\mathbf{W}_t$. Thus, we have

$$\det(\mathbf{I} - \mathbf{W}_t) = \begin{cases} \det(\mathbf{I} - \rho(\alpha_{\vec{L}}(C))w(C)) & \text{if } t \text{ is a prime cycle } C, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, it follows that

$$\zeta_{\vec{L}(G)}(w, \rho, \alpha)^{-1} = \det(\mathbf{I}_{2ld} - \sum_{h \in \Gamma} \vec{\mathbf{W}}_h \bigotimes \rho(h)) = \det(\mathbf{I}_{2ld} - \sum_{h \in \Gamma} \rho(h) \bigotimes \vec{\mathbf{W}}_h).$$

□

By Theorems 4,5, the following result holds.

Corollary 1 *Let G be a connected graph, Γ a finite group and $\alpha : E(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Then we have*

$$\zeta_{G^\alpha}(\tilde{w}) = \prod_{\sigma} \zeta_{\vec{L}(G)}(w, \sigma, \alpha)^{\deg \sigma},$$

where σ runs over all inequivalent irreducible representations of Γ.

Let G be a connected graph, Γ a finite group and $\alpha : E(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Furthermore, let ρ be any representation of Γ and $d = \deg \rho$. The <u>L-function</u> of G associated to ρ and α is defined by

$$\mathbf{Z}_G(u, \rho, \alpha) = \prod_{[C]} \det(\mathbf{I}_d - \rho(\alpha(C))u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.
Let $w_{ij} = u$ unless $w_{ij} = 0$. Then we obtain Corollary 2 in [10].

Corollary 2 (Mizuno and Sato) *Let G be a connected graph, Γ a finite group and $\alpha : E(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Suppose that the Γ-covering G^α of G is connected. Then we have*

$$\mathbf{Z}(G^\alpha, u) = \prod_{\rho} \mathbf{Z}_G(u, \rho, \alpha)^d,$$

where ρ runs over all inequivalent irreducible representations of Γ and $d = \deg \rho$.

Proof. At first, we have

$$\zeta_{G^\alpha}(\tilde{w}) = \zeta_{G^\alpha}(u) = \mathbf{Z}(G^\alpha, u)$$

and

$$\zeta_{\vec{L}(G)}(w, \rho, \alpha) = \zeta_{\vec{L}(G)}(u, \rho, \alpha) = \mathbf{Z}_G(u, \rho, \alpha).$$

By Corollary 1, the result follows. □

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