EXTREMAL SIZE IN GRAPHS WITH BOUNDED DEGREE

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Abstract

A graph on n vertices having no vertex of degree greater than f, $2 \le f \le n-2$, is called an *f-graph* of order n. For a given f the vertices of degree less than f are called *orexic*. An f-graph to which no edge can be added without violating the f-degree restriction is called an edge maximal f-graph (EM f-graph). An upper bound, as a function of n and f, for the number of m orexic vertices in an EM f-graph and the structure of the subgraph induced by its orexic vertices is given. For any n and f, the maximum size, minimum size, and realizations of extremal size EM f-graphs having m orexic vertices and order n are obtained. This is also done for any given n and f independent of m. The number of size classes of EM f-graphs of order n and fixed m is determined. From this the maximum number of size classes over all m follows. These results are related to the study of $(f+1)$-star-saturated graphs.

1. Introduction

Simple graphs of *order n* and *size t* having no vertex of degree greater than f, $2 \le f \le n-2$, are considered. Such graphs are called *f-graphs* and their vertices of degree less than f are called *orexic vertices* (cf. [14]). An f-graph to which no edge can be added without violating the degree restriction is said to be *edge maximal* (EM). Graphs with bounded degree are of importance and interest because in most applications the graphs involved have specific restrictions on their vertex degrees. One such instance of this is in chemistry, where in graph models for molecules the vertex degrees correspond to chemical bonding restrictions, such as, degree 4 corresponds to carbon, degree 1 corresponds to hydrogen, etc. (see [11, 16]).

Perhaps the most important aspect of EM f-graphs is the fact that these graphs are a generalization of regular graphs in the following sense. The terminal graph G of a process involving the evolution of a graph obtained by sequentially adjoining one edge at a time to a set of n fixed vertices under the restriction that no vertex shall have degree greater than f, is not an f-regular graph, but G is an EM f-graph of order n (see [8]). Such processes play a role in both physical applications [10, 13] and in theoretical studies [2, 3]. It is known that for this process, with nf even and n going to infinity, almost all f-graphs will be regular [1]. On the other hand, for the uniform distribution of EM f-graphs, for nf even and n going to infinity, the proportion of f-regular graphs is 2/5 when $f = 2$ and in general equal to $2/(f^2 + 1)$ (see [4]). However, when n and f are fixed, it is an open problem to determine, the proportion of f-regular graphs relative to the set of EM f-graphs. The problem is for both the above sequential process and the uniform distribution case (see [5, 7]).

In the context of extremal graph theory and in particular relative to Turán type problems (see [12] and references therein) we can state our results as follows. A graph is $(f + 1)$-*star saturated* means it does not contain an $(f + 1)$-star and the addition of any edge will introduce an $(f + 1)$-star. Let $G(n, f, m)$ denote the set of all EM f-graphs of order n having m orexic vertices and $G(n, f)$ the union of $G(n, f, m)$ over all m. Thus, $G(n, f)$ is the set of all $(f + 1)$-star saturated graphs of order n. In [12], included with other results on saturated graphs, there is a theorem on minimum size star-saturated graphs (see Theorem 4 [12] and compare to Theorem 3.3).

We determine the extremal sizes (maximum and minimum) and explicit realizations for graphs in each $G(n, f, m)$ and for $G(n, f)$ (see Sections 2 and 3, respectively). In Section 4, the determination and study of size classes of EM f-graphs is carried out. A pair of open problems is given in Section 5.

Some structural properties of EM f-graphs are given in the following theorems (cf. [14]).

Theorem 1.1. *If G is an EM f-graph of order n with $m \geq 1$ orexic vertices, then the orexic vertices induce a complete subgraph of order m in G. Furhermore, $m \leq f$.* ■

Corollary 1.2. *If G is an EM f-graph of order n, then*
1) *if $1 \leq m \leq f - 1$, then either the m orexic vertices form a K_m component of G or they induce a K_m which is adjacent in various ways to vertices of degree f, and*
2) *if $m = f$, then the orexic vertices form a K_f component of G.* ■

If f is such that $(n - 1)/2 < f$, then there is a bound smaller than f for the number of orexic vertices (see Theorem 1.3, its corollary, and Figure 1.1).

Theorem 1.3. ([6]) *Let G be an EM f-graph of order n with m orexic vertices, $2 \leq f \leq n - 2$. Then*

$$0 \leq m \leq \min\left\{\frac{n(n-f-1)}{2(n-f)-1}, f\right\}. \quad (1.1)$$

Corollary 1.4. ([6]) *Let G be an EM f-graph of order n.*

$$\textit{If } 2 \le f \le (n-1)/2 \textit{, then } 0 \le m \le f$$

and

$$\textit{if } (n-1)/2 < f \le n-2 \textit{, then } 0 \le m \le \frac{n(n-f-1)}{2(n-f)-1}. \blacksquare$$

In Figure 1.1 (cf. [9]) the upper bound for the number of orexic vertices is shown as a function of f with n fixed. Note that if $f = n-2$, then $m \le n/3$.

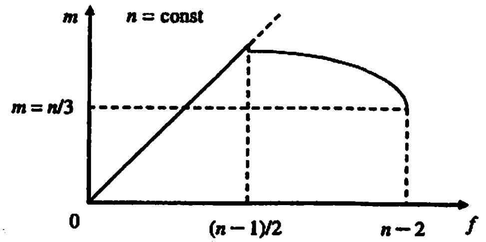


Figure 1.1. The upper bound for the number of orexic vertices as a function of f with n fixed

The bounds on the number of edges of an EM f-graph are given in the following theorem.

Theorem 1.5. ([14]) *Let G be an EM f-graph of order n and size t. If G has m orexic vertices, then*

$$(n-m)f + m(m-1) \le 2t \le nf - m. \blacksquare \tag{1.2}$$

2. The maximum size and its realizations

By the *global* maximum size of EM f-graphs of order n we mean the maximum number of edges that such a graph can have. A realization of this maximum size depends on the parity of nf. It is known (cf. [15], p.249) that a regular graph of order n with degree of regularity f exists if nf is even and $f \le n-1$.

Let $R_{n,f}$ be a *regular f*-graph and $A_{n,f}$ denote an *almost regular f*-graph of order n, the latter being a graph having $n-1$ vertices of degree f and one vertex of degree $f-1$. We define a method for constructing a graph $A_{n,f}$ when nf is odd, i.e., both n and f are odd (cf. Figure 2.1).

Construction A. Given n and f with nf odd construct an almost regular f-graph $A_{n,f}$ of order n, as follows.
1. Start with an initial graph, $G = R \cup K_1$ where $R = R_{n-1,f}$ is a regular f-graph of order $n-1$ (note that $n-1$ and $f-1$ are even). If $f = n-2$, then the initial graph is $G = K_{n-1} \cup K_1$.
2. Select $k = (f-1)/2$ independent edges in R and delete them.
3. Add $l = 2k$ new edges between the endvertices of the deleted edges and the K_1.•

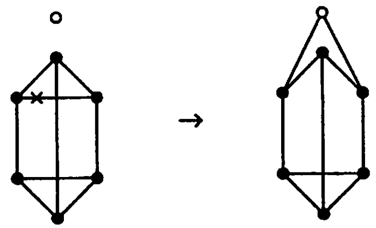


Figure 2.1. An example of Construction A for $n = 7, f = 3$; A: $R_{6,3} \cup K_1 \rightarrow A_{7,3}$

Theorem 2.1. *The global maximum size for an EM f-graph of order n is*

$$t^*(n,f) = \left\lfloor \frac{nf}{2} \right\rfloor \tag{2.1}$$

and its realization is $R_{n,f}$ (a regular f-graph of order n with $m = 0$) when nf is even, and $A_{n,f}$ (an almost regular f-graph of order n with $m = 1$) when nf is odd.
Proof. By Theorem 1.5 the size of EM f-graph of order n is bounded from above: $t \le \frac{nf-m}{2}$. This is maximum when $m = 0$ and a realization is $R_{n,f}$, a regular f-graph of order n. However if nf is odd, such a regular graph does not exist and the upper bound is maximum when $m = 1$. Here a realization is $A_{n,f}$, an almost regular f-graph of order n. Therefore, $t^*(n,f) = \lfloor nf/2 \rfloor$ in all cases. ■

We next obtain the maximum size for an EM f-graph of order n having m orexic vertices where m satisfies (1.1). To do this we define Construction **B** which will provide us with realizations of the extremal graphs.

Construction B. Given n, f, and m ($3 \le f \le n-2$, and m as in (1.1)) construct $B_{n,f,m}$, an EM f-graph of order n with m orexic vertices and size $t^*(n,f,m) = \left\lfloor \frac{nf-m}{2} \right\rfloor$ as follows.

Case 1. If $n - m > f + 1$, then let $k = \lfloor m(f-m)/2 \rfloor$.

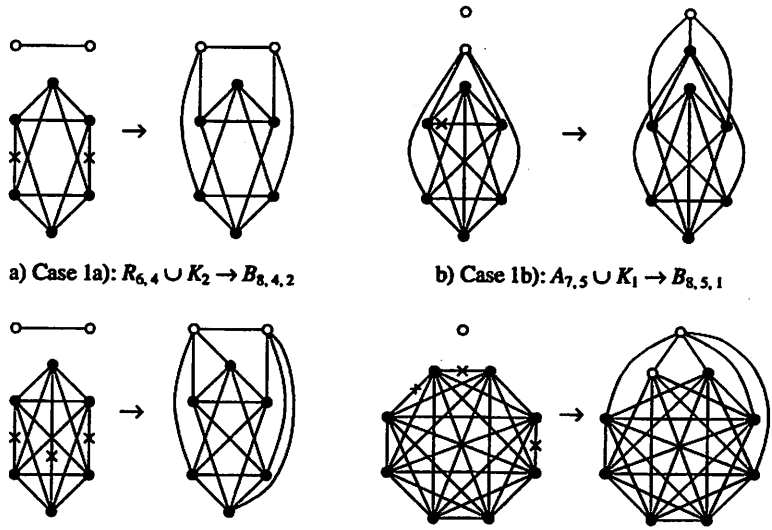
a) If $(n-m)f$ is even, then choose an initial graph $G = R \cup K_m$ with $R = R_{n-m,f}$, delete k edges from R and add $l = 2k$ edges between the endvertices of the deleted edges and the K_m.
b) If $(n-m)f$ is odd, then choose an initial graph $G = A \cup K_m$, $A = A_{n-m,f}$, delete $k-1$ edges from A and add new edges: one edge between the orexic ver-

tex of A and any vertex of the K_m, and $2(k-1)$ edges between the endvertices of the deleted edges and the K_m.

Case 2. If $n-m=f+1$, then the initial graph is $G=K_{f+1}\cup K_m$. Delete $k=\lfloor m(f-m)/2\rfloor$ edges from K_{f+1} and add $l=2k$ edges (one edge between of a deleted edge and the K_m).

Case 3. If $n-m<f+1$, then the initial graph is $G=K_{f+1}\cup K_{m-x}$ with $x=(f+1)-(n-m)$, the number of vertices of K_{f+1} that are of degree f before the construction and are orexic (of degree $f-1$) after the construction (we call these vertices "shared").
Delete from K_{f+1}:
a) $k'=x(m-x+1)$ edges with exactly one end in the shared orexic vertices and
b) $k''=\left\lfloor\dfrac{m(f-m)-x(f+1-x)}{2}\right\rfloor$ edges between vertices not used in a).
Obtain $B=B_{n,f,m}$ by adding:
a) $l'=m-x$ edges between each shared vertex and the K_{m-x} and
b) $l''=k'+2k''$ edges, i.e., one edge between each other endvertex of a deleted edge and the K_{m-x}. •

Examples of Construction **B** are given in Figure 2.2 (cf. [9]).



a) Case 1a): $R_{6,4}\cup K_2\to B_{8,4,2}$ b) Case 1b): $A_{7,5}\cup K_1\to B_{8,5,1}$

c) Case 2a): $K_6\cup K_2\to B_{8,5,2}$ d) Case 2b): $K_8\cup K_1\to B_{9,7,2}$

Figure 2.2. Examples of Construction **B**. a) $n=8, f=4, m=2$, b) $n=8, f=5, m=1$, c) $n=8, f=5, m=2$, and d) $n=9, f=7, m=2$

Theorem 2.2. *The maximum size for an EM f-graph of order n with m (m as in* (1.1)) *orexic vertices is*

$$t^*(n, f, m) = \left\lfloor \frac{nf - m}{2} \right\rfloor \tag{2.2}$$

and a realization of this size is $B = B_{n,f,m}$ *given by Construction* **B**.

Proof. By using the inequality of Theorem 1.5 and Construction **B**, we show there exist an EM f-graph $B_{n,f,m}$ of order n with m orexic vertices and size $t^*(n, f, m)$ for all cases that parameters n, f, and m fulfill. A construction of such a graph starts with an initial graph $G = F \cup K_m$ with $F = F_{n-m,f}$, an EM f-graph of order $n - m$ and size $t = t^*(n - m, f)$. The size of F being the maximum size, over all m, for an EM f-graph of order $n - m$, provided $n - m$ is not less than $f + 1$ (the order of a complete graph with vertices of degree f).

In the case when $n - m$ is less than $f + 1$ the initial graph and a procedure of obtaining the result is modified. The basic idea of this construction is to increase the number of edges of G by deleting some (independent) edges from the main component (in most cases with vertices of degree f) of G and then adding new edges between the two components (two new edges for each deleted edge). In all cases a resulting graph B is of order n (with m orexic vertices) and size $t = t^*(n, f, m)$. ∎

3. Minimum size and its realizations

We first obtain the minimum size for an EM f-graph of order n with m orexic vertices and realizations of graphs with this size. At the end of this section, the global minimum size of such graphs and their realizations are determined. Specifically, the *global minimum size* of an EM f-graph of order n is the minimum size independent of its number of orexic vertices.

Theorem 3.1. *The minimum size for an EM f-graph of order n with m orexic vertices,*

$$0 \le m \le \min\left\{\frac{n(n-f-1)}{2(n-f)-1}, f\right\},$$

is

$$t_*(n, f, m) = \left\lceil \frac{(n-m)f}{2} \right\rceil + \binom{m}{2}. \tag{3.1}$$

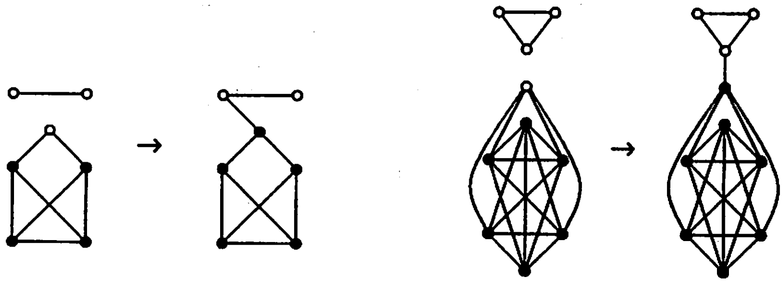
Proof. By using the inequality of Theorem 1.5. ∎

We define two constructions, **H** and **J**, for EM f-graphs of order n and minimum size as given in Theorem 3.1.

Construction H. Given n, f, and m (with m as in (1.1), $(n-m)f$ odd, and $n-m \geq f+1$) construct $H_{n,f,m}$, an EM f-graph of order n with m orexic vertices and size $t_*(n, f, m)$ as follows.

1. Start with an initial graph $G = A \cup K_m$, with $A = A_{n-m,f}$ (an almost regular f-graph of order $n-m$).
2. Add an edge between the orexic vertex of A and a vertex of K_m. •

In Figure 3.1 two examples of Construction **H** are shown (cf. [9]).



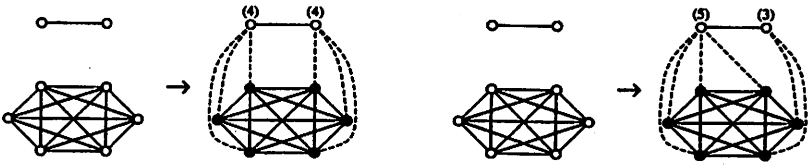
a) **H**: $A_{5,3} \cup K_2 \rightarrow H_{7,3}$ b) **H**: $A_{7,5} \cup K_3 \rightarrow H_{10,5}$

Figure 3.1. Two examples of Construction **H**: a) $n = 7, f = 3$, b) $n = 10, f = 5$

Construction J. Given $n, f \geq n/2$, m as in (1.1), and $f \geq n-m$ construct $J_{n,f,m}$, an EM f-graph of order n with m orexic vertices and size $t_*(n, f, m)$.

1. Start with the initial graph $K = K_{n-m} \cup K_m$.
2. Let $d = f - (n - m - 1)$, where $n - m - 1$ is the degree of the vertices of K_{n-m}.
3. Add $l = d(n-m)$ edges, i.e., d edges from each vertex of the K_{n-m} to vertices of the K_m in such a way that the vertices of the K_m remain orexic. Note that in some cases not all l edges will be used.•

Two realizations of Construction **J** for $n = 8, f = 6$ and $m = 2$ are shown in Figure 3.2 (cf. [9]).



a) b)

Figure 3.2. Two realizations of Construction **J** for $n = 8, f = 6$ and $m = 2$ ($d = 1$) with distinct degree sequences of orexic vertices a) deg = 4, 4 and b) deg = 5, 3

Theorem 3.2. *A realization of the minimum size $t_*(n, f, m)$ for EM f-graphs of order n with m orexic vertices (m as in (1.1)) is as follows.*

Case 1. *If $n - m > f + 1$, then there are two cases:*

a) *if $(n - m)f$ is even, then the realization is $G = R \cup K_m$, where $R = R_{n-m,f}$ is a regular f-graph of order $n - m$, and*

b) *if $(n - m)f$ is odd, then the realization is $H = H_{n,f,m}$ (cf. Construction* **H**).

Case 2. *If $n - m = f + 1$, then the realization is $K = K_{n-m} \cup K_m$.*

Case 3. *If $n - m < f + 1$, then the realization is $J = J_{n,f,m}$ (cf. Construction* **J**).

Proof. Cases 1 and 2 are obvious. In Case 3, for a given n and f the value of m is too big and vertices of a K_{n-m} are of degree less than f. Degrees of these vertices are increased (up to the value of f) as defined in Construction **J**. ■

Remark. In cases when for a given n, f, and m the maximum and the minimum sizes are equal we obtain the same structure of EM f-graph using constructions for the corresponding size (e.g., **B** for maximum size and **J** for minimum size with $n = 6, f = 4, m = 2$, cf. [9]).

Theorem 3.3. *The global minimum size of EM f-graphs of order n is*

$$t_*(n, f) = \left\lceil \frac{(n-m)f}{2} \right\rceil + \binom{m}{2} \tag{3.2}$$

with the value of m determined as follows.

Case A. *If $3 \le f \le \lfloor n/2 \rfloor$, then*

1) *if f is even, then can use either $m = \lfloor (f + 1)/2 \rfloor$ or $m = \lceil (f + 1)/2 \rceil$; a realization is $G = R_{n-m,f} \cup K_m$,*

2) *if f is odd, then $m = (f + 1)/2$ is an integer; thus, if $(n - m)$ is even, then a realization is $G = R_{n-m,f} \cup K_m$; otherwise a realization is $H = H_{n,f,m}$.*

Case B. *If $\lfloor n/2 \rfloor < f \le n - 2$, then $m = n - (f + 1)$ and the realization is $K = K_{f+1} \cup K_{n-(f+1)}$.*

Proof. From Theorem 1.5 we have $(n - m)f + m(m - 1) \le 2t$. The left hand side is quadratic with absolute minimum value attained at $m = (f + 1)/2$. In the event $(f + 1)/2$ is not an integer and since m must be an integer, the minimum value of $(n - m)f + m(m - 1)$ at integer values of m is attained at $m = \lfloor (f + 1)/2 \rfloor$ and, by symmetry, also at $m = \lceil (f + 1)/2 \rceil$. If $(n - m)f$ is odd, then $2t$ can be no smaller than $(n - m)f + m(m - 1) + 1$. Thus, the lower bound for t is in all cases equal to $\left\lceil \frac{(n-m)f}{2} \right\rceil + \binom{m}{2}$.

If $3 \le f \le \lfloor n/2 \rfloor$, there are the following types of EM f-graphs that realize the above bound. Namely, if f is odd, then $m = (f + 1)/2$ is an integer. When $n - m$

is odd, then the bound is realized by $H = H_{n,f,m}$ obtained by Construction H. For all other cases in this range of f, the bound is realized by a regular f-graph of order $n - m$ union K_m.

If $\lfloor n/2 \rfloor < f \leq n - 2$, we have $f > n/2$. Thus, $m \leq \frac{n(n-f-1)}{2(n-f)-1}$ (see Corollary 1.4). For a regular f-graph of order $n - m$ to exist we need $f \leq n - m - 1$, which yields

$$m \leq n - f - 1 < \frac{n(n-f-1)}{2(n-f)-1}.$$

To obtain the minimum size we want the smallest order regular f-graph R such that $R \cup K_m$ has minimum size. This is achieved when $m = n - f - 1$ and $R = K_{f+1}$. ■

4. Size classes

The set of all EM f-graphs of order n is subdivided into size classes. When the number m of orexic vertices is fixed, the number of size classes is

$$S(n, f, m) = t^*(n, f, m) - t_*(n, f, m) + 1.$$

Theorem 4.1. *The number of size classes of EM f-graphs of order n with m orexic vertices* (m as in (1.1)) *is*

$$S(n, f, m) = \left\lfloor \frac{nf - m}{2} \right\rfloor - \left\lceil \frac{(n-m)f}{2} \right\rceil - \binom{m}{2} + 1. \tag{4.1}$$

Proof. Follows from Theorems 2.2 and 3.1. ■

Let $S(n, f)$ denote the number of *size classes* of EM f-graphs of order n. Then

$$S(n, f) = t^*(n, f) - t_*(n, f) + 1. \tag{4.2}$$

Theorem 4.2. *The number of size classes of EM f-graphs of order n is as follows.*

<u>*Case A*</u>. *If* $2 \leq f \leq \lfloor n/2 \rfloor$ *then* $S(n, f) = \begin{cases} a, & f \equiv 3 \pmod 4,\ n \text{ odd} \\ a+1, & \text{otherwise} \end{cases}$,

where $a = \left\lfloor \frac{mf}{2} \right\rfloor + \binom{m}{2}$ *with* $m = \lfloor (f+1)/2 \rfloor$.

<u>*Case B*</u>. *If* $\lfloor n/2 \rfloor < f \leq n - 2$, *then* $S(n, f) = \left\lfloor \frac{nf}{2} \right\rfloor - \binom{f+1}{2} - \binom{m}{2} + 1$,

with $m = n - (f + 1)$.

Proof. Straightforward calculations with $m = \lfloor (f+1)/2 \rfloor$ and the results of Theorems 2.1 and 3.3 yield the statement of the Theorem. ∎

Theorem 4.3. *The maximum number of size classes of EM f-graphs of order n* ($n \geq 4$) *is*

$$S^*(n) = \lfloor n^2/16 + 1 \rfloor. \tag{4.3}$$

There is one value of f such that $S(n, f) = S^*(n)$:

a) $f^*(n) = \lfloor 3n/4 - 1 \rfloor$ *when* $n \equiv 0, 4$ *or* $7 \pmod 8$,

b) $f^*(n) = \lceil 3n/4 - 1 \rceil$ *when* $n \equiv 1 \pmod 8$,

and there are two values of f:

c) $f^*(n) = \lfloor 3n/4 - 1 \rfloor$ *and* $f^*(n) = \lceil 3n/4 - 1 \rceil$ *when* $n \equiv 2, 3, 5,$ *and* $6 \pmod 8$.

Proof. Maximizing $S(n, f) = \left\lfloor \frac{nf}{2} \right\rfloor - \binom{f+1}{2} - \binom{n-f-1}{2} + 1$ as a function of f shows $S(n, f)$ has its maximum value $\lfloor n^2/16 + 1 \rfloor$ at $f = 3n/4 - 1$. However, f must be an integer. This occurs when $n \equiv 0 \pmod 4$ and a partition of the values of n modulo 4 is sufficient to find when the maximum is achieved at $f = \lfloor 3n/4 - 1 \rfloor$ or $\lceil 3n/4 - 1 \rceil$ or in some cases at both of these values. A further partitioning of the values of n modulo 8 distinguishes precisely when $\lfloor 3n/4 - 1 \rfloor \neq \lceil 3n/4 - 1 \rceil$ and both yield the maximum value of $S(n, f)$. ∎

Table 4.1 shows, for $n = 9$ and $n = 10$, the partition of the number of all EM f-graphs ($f = 2, 3, \ldots, n-2$) of order n into classes with m orexic vertices and a given size $t = t^*(n, f) - i$, with $i = 0, 1, \ldots$ (cf. Table A.3 in [9] with $4 \leq n \leq 10$). Note that the maximum number of size classes is $S^*(9) = 6$ with $f^*(9) = 6$ and $S^*(10) = 7$ with $f^*(10) = 6, 7$. However, there is no general formula for the number of EM f-graphs with $f > 2$.

Table 4.1. The number of EM f-graphs of order $n = 9$, and 10, $t = t^*(n, f) - i$, $0 \leq m \leq f$

$n = 9$																					
	$f=2$			$f=3$			$f=4$					$f=5$			$f=6$				$f=7$		
i \ m	0	1	2	1	2	3	0	1	2	3	4	1	2	3	0	1	2	3	1	2	3
5																	1				
4														1			2				
3									2	1			1	2		1	6	3	1		
2								6	16	3	1	3	10	7		2	11	4	1	2	
1		3	2	6	4	2		28	41	0	0	13	31	11		5	10	0	1	1	1
0	4	0	0	20	0	0	16	0	0	0	0	28	0	0	4	0	0	0	1	0	0

$n = 10$																														
	$f=2$			$f=3$			$f=4$					$f=5$					$f=6$					$f=7$					$f=8$			
i \ m	0	1	2	0	1	2	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3
6																				1				1						
5																			1	2				2						
4														3	1	1			5	9				6	4			1		
3									6	2				29	13	1		4	36	35	2		1	13	10			1	3	
2						6		16	63	9	1		28	206	58	2		20	131	66	3		4	24	13	3		1	2	1
1		4	3		20	24		126	200	0	0		178	361	0	0		60	141	0	0		7	16	0	0		1	1	0
0	5	0	0	21	0	0	60	0	0	0	0	60	0	0	0	0	21	0	0	0	0	5	0	0	0	0	1	0	0	0

5. Two problems

The number of size classes of EM f-graphs of order n and structures of extremal size graphs have been obtained. We propose the following *open problems*.

1. What is the number of extremal size EM f-graphs of order n with m orexic vertices?
2. Determine the number of extremal size EM f-graphs of order n.

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