Graceful labellings for an infinite class of generalised Petersen graphs

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Abstract

Graceful labellings have both a mathematical beauty in their own right and considerable connections with pure and applied combinatorics (edge-decomposition of graphs, coding systems, communication networks, etc.). In the present paper we exhibit a graceful labelling for each generalised Petersen graph $P_{8t,3}$ with $t \geq 1$. As a consequence we obtain, for any fixed t, a cyclic edge-decomposition of the complete graph K_{48t+1} into copies of $P_{8t,3}$. Due to its extreme versatility, the technique employed looks promising for finding new graceful labellings, not necessarily involving generalised Petersen graphs.

Keywords: Edge decomposition, generalised Petersen graph, graceful graph, graph labelling.
AMS Subject Classification: 05C78.

1 Introduction

Let $G = (V, E)$ be a connected graph with no loops and $\lambda : V \rightarrow \{0, 1, 2, ..., |E|\}$ be an injective vertex labelling. Let us also denote by λ' the induced labelling, on the edges, assigning the value $|\lambda(u) - \lambda(v)|$ to the edge $\{u, v\}$. In keeping with the standards, we provide the following notion.

Definition 1.1. In the above setting, the labelling λ is termed *graceful* if λ' is injective (equivalently, if it is a bijection on $\{1, 2, ..., |E|\}$). A graph which admits a graceful labelling is termed *graceful* as well.

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Starting from the work of Rosa in [9], many types of graphs have so far been proved to be graceful, whereas a number of necessary conditions for gracefulness have been recorded which, because of their absence, have ruled out some other types of graphs. Nonetheless, a considerable amount of graphs still keep their secret as to being graceful or not. For example the "graceful tree conjecture", stating that every tree is graceful, has not been proved or disproved yet (see [7], a thorough survey on graph labellings).

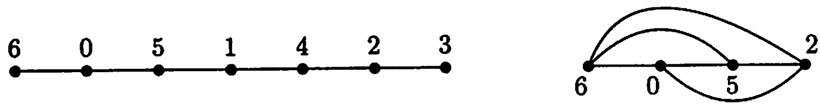

Figure 1: Graceful labellings for the 6-path and the complete graph K_4

The gracefulness problem has also been tackled from a more general viewpoint, namely, that of *relaxed graceful labellings* (see [10]). For a particular kind of these labellings, λ and λ' are allowed to take larger values than $|E|$, injectivity still being required for both maps. Relaxed graceful labellings have been recently investigated also from the asymptotic point of view (see [3]), while connections with the so-called *Golomb rulers* can be found, for example, in [1].

Among the classes of graphs which have unveiled only partial information on gracefulness, we find that of *generalised Petersen graphs.*

Definition 1.2. Let n, k be positive integers such that $n \geq 3$ and $1 \leq k \leq \lfloor (n-1)/2 \rfloor$. The *generalised Petersen graph* $P_{n,k}$ is the graph whose vertex set is $\{a_i, b_i : 1 \leq i \leq n\}$ and whose edge set is $\{\{a_i, b_i\}, \{a_i, a_{i+1}\}, \{b_i, b_{i+k}\} : 1 \leq i \leq n\}$, where $a_{n+c} = a_c$ and $b_{n+c} = b_c$ for every $c \geq 1$.

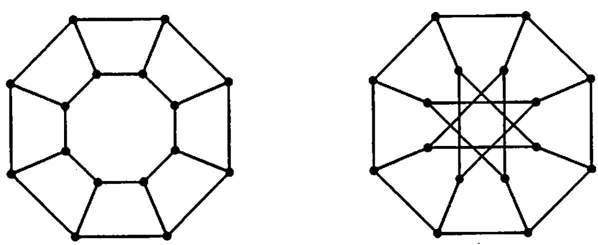

Figure 2: The generalised Petersen graphs $P_{8,1}$ and $P_{8,3}$

In [6] gracefulness has been established for all graphs $P_{n,1}$. Such graphs are known as *prisms*. Other proofs that prisms are graceful – each proof not covering the whole spectrum – were provided earlier (see [7]). Furthermore, by a comparatively recent computer search gracefulness was certified for finitely many $P_{n,k}$'s with $k > 1$ (see [8]).

In the present paper we exhibit graceful labellings for an infinite class of generalised Petersen graphs having $k > 1$. More precisely, in the second section we prove the following.

Theorem 1.3. *$P_{8t,3}$ is graceful for every $t \geq 1$.*

The basic idea in the proof is that of splitting any given graph into a cycle and a residual family of 3-ray stars, whose endvertices are connected to the cycle, and whose edges are altogether as many as those forming the cycle. After endowing the cycle with a labelling that yields all even differences, and whose labels are all even integers, it is not an impossible task to provide the star centres with suitable odd numbers that complete the labelling.

Besides its unquestionable aesthetic appeal, gracefulness has many connections with real-life models. A relevant starting point in the literature appears to be [2], by Bloom and Golomb, in which graceful labellings and other kinds of labellings were shown to be quite helpful when dealing with e.g. coding systems, communication networks, X-ray analysis.

Without going far from pure mathematics, we can find an immediate and precious application of graceful graphs in the realm of graph theory itself. It is indeed not difficult to see that any gracefully labelled graph G on v vertices yields a *cyclic decomposition* of the complete graph K_{2v+1} into copies of G (this property was first highlighted by Rosa in [9]). Accordingly, in the third section of this paper, after recalling some basic notions on graph decompositions, we exploit the main theorem to obtain decompositions of complete graphs by means of graceful generalised Petersen graphs. Subsequent considerations will eventually flow into a conjecture involving a stronger notion than gracefulness, namely the *α-labelling* property.

2 The main theorem

The whole section is devoted to the proof of the main theorem.

Proof of Theorem 1.3. The initial spark of the proof is the splitting of some given $P_{8t,3}$ into a cycle of length $12t$ and a family of $4t$ 3-ray stars, with the three endvertices of each star lying on the $(12t)$-cycle. The high symmetry of the initial graph reflects in the easily understandable, and harmonious, law that governs the linkages between the stars and the cycle.

The length of the cycle allows for a graceful labelling of this subgraph – in fact, a well-known result (see [9]) states that the graceful cycles are precisely those of length congruent to 0 or 3 (mod 4). After gracefully

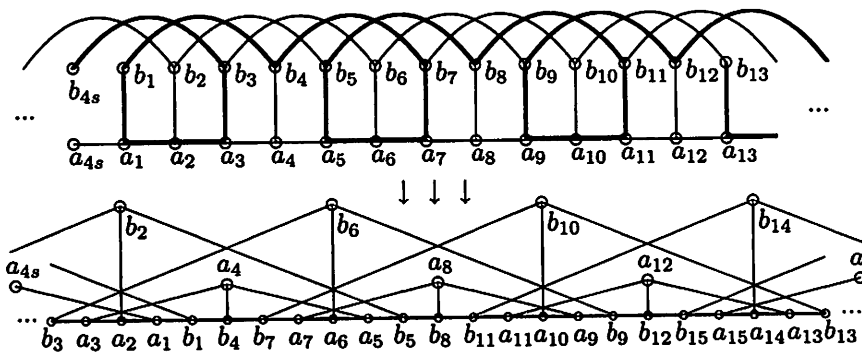

Figure 3: Looking at $P_{4s,3}$ in a different way

labelling this $12t$-cycle, we shall multiply labels by 2, thus obtaining all the required even differences. Subsequently, labelling the star centres by suitable odd numbers will result in a complete graceful labelling of the initial graph.

As the first step let us formally introduce the above-mentioned $(12t)$-cycle. Bearing in mind Definition 1.2, we define the required cycle as

$$(b_3 \ a_3 \ a_2 \ a_1 \ b_1 \ b_4 \ b_7 \ a_7 \ a_6 \ a_5 \ b_5 \ b_8 \ b_{11} \ a_{11} \ \dots$$

$$\dots \ a_{8t-5} \ a_{8t-6} \ a_{8t-7} \ b_{8t-7} \ b_{8t-4} \ b_{8t-1} \ a_{8t-1} \ a_{8t-2} \ a_{8t-3} \ b_{8t-3} \ b_{8t} \ )$$

(see Figure 3). After a little thought, the reader will realise that the above cycle is indeed contained in $P_{8t,3}$ and that the stars completing the graph can be grouped in two subfamilies: stars, say, *of class* 1, having the centre b_{4i-2} connected to b_{4i-5}, a_{4i-2}, b_{4i+1} for $1 \le i \le 2t$, and stars *of class* 2, having a_{4i} connected to a_{4i-1}, b_{4i}, a_{4i+1} for $1 \le i \le 2t$.[1]

On the above cycle, and in the same order, we now define and amplify by 2 a graceful labelling, as follows.

$$(\overset{b_3}{24t}\ ,\ \overset{a_3}{0}\ ,\ \overset{a_2}{24t-2}\ ,\ \overset{a_1}{2}\ ,\ \dots\ ,\ \overset{b_{4t-3}}{18t+2}\ ,\ \overset{b_{4t}}{6t-2}\ ,\ \overset{b_{4t+3}}{18t}\ ,$$

$$\overset{a_{4t+3}}{6t+2}\ ,\ \overset{a_{4t+2}}{18t-2}\ ,\ \overset{a_{4t+1}}{6t+4}\ ,\ \dots\ ,\ \overset{b_{8t-3}}{12t+2}\ ,\ \overset{b_{8t}}{12t}\,).$$

In formal terms, every a_i with $i \equiv 1, 2, 3 \pmod 4$ is labelled respectively by $2 + 3(i-1)/2 + \varepsilon_i$, $24t - 2 - 3(i-2)/2$, $3(i-3)/2 + \varepsilon_i$, where $\varepsilon_i = 2$

[1]This splitting is applicable to the more general class $\{P_{4s,3} : s \ge 1\}$. However, if s is odd the resulting cycle is not graceful, because $6s \equiv 2 \pmod 4$. It would be nice to adapt the present construction to the odd s case.

if $i \geq 4t+1$ and $\varepsilon_i = 0$ otherwise; also, every b_i with $i \equiv 0, 1, 3 \pmod 4$ is labelled respectively by $4+3(i-4)/2+\varepsilon_i$, $24t-4-3(i-1)/2$, $24t-3(i-3)/2$, where $\varepsilon_i = 2$ if $i \geq 4t+4$ and $\varepsilon_i = 0$ otherwise.

Leaving aside four exceptions, two for each class, it can be easily seen that the labels of the three endvertices of each class-1 star turn out to be of the form $X, X-8, X-16$, while those of each class-2 star are of the form $X, X+4, X+8$ (we have used different orderings merely to take into account the cycle orientation). More precisely, denoting respectively by $[X]$ and $[[X]]$ the above triples, with little effort the reader can prove that, if $t \geq 2$, the arising triples are

$$\{[6x]: 2t+3 \leq x \leq 4t\} \cup \{[[6x]]: 0 \leq x \leq t-2\} \cup \{[[6x+2]]: t \leq x \leq 2t-2\}$$

together with the triples $\{\{12t+12, 12t+4, 24t-4\}, \{12t+6, 24t-2, 24t-10\}, \{6t-6, 6t-2, 6t+4\}, \{12t-4, 12t, 2\}\}$, whereas if $t=1$ the only triples are the last four. In the sequel, these exceptional triples will be shortly denoted by $\langle\alpha\rangle$, where α is the first entry of each of them.

The assignment procedure for the star centres splits in two infinite families of constructions, depending on the parity of $t \geq 4$, plus three constructions "ad hoc" for the smallest values of t. With a slight abuse of notation, we shall often identify triples and corresponding centres.

Case 1*: t even and greater than* 2.

A short computation shows that the labelling

$$[6x] \mapsto 24t-5-6x \ (x \text{ odd}), \quad [6x] \mapsto 24t+3-6x \ (x \text{ even}),$$

where $3t+3 \leq x \leq 4t$, generates the differences $\{12t+25+4z: 0 \leq z \leq 3t-7\}$. The labels employed are $\{1+12u, 3+12u: 0 \leq u \leq t/2-2\}$. Similarly, the labelling

$$[6x] \mapsto 24t+11-6x \ (x \text{ odd}), \quad [6x] \mapsto 24t+19-6x \ (x \text{ even}),$$

where $2t+3 \leq x \leq 3t$, generates the differences $\{9+4z: 0 \leq z \leq 3t-7\}$. This time the labels employed are $\{6t+17+12u, 6t+19+12u: 0 \leq u \leq t/2-2\}$.

The centres of the remaining two stars of class 1 are labelled by $[18t+12] \mapsto 6t-11$ and $[18t+6] \mapsto 6t-13$, thus obtaining the differences $\{12t+3+4z: 0 \leq z \leq 5\}$.

Now we deal with the stars of class 2. Using the labelling

$$[[6x]] \mapsto 24t-1-6x \ : \ 0 \leq x \leq t-3$$

we obtain the differences $\{12t+27+4z: 0 \leq z \leq 3t-7\}$, while the labelling

$$[[6x+2]] \mapsto 24t-3-6x \ : \ t \leq x \leq 2t-2$$

generates the differences $\{11+4z\colon 0\le z\le 3t-4\}$. The labels employed in the above cases are, respectively, $\{18t+17+6u\colon 0\le u\le t-3\}$ and $\{12t+9+6u\colon 0\le u\le t-2\}$.

The remaining triple, namely $[[6t-12]]$, is mapped to $18t+9$ so as to obtain the differences $\{12t+13+4z\colon 0\le z\le 2\}$.

Finally, we deal with the exceptional stars, as follows. $\langle 12t+12\rangle \mapsto 24t-3, \langle 12t+6\rangle \mapsto 12t+1, \langle 6t-6\rangle \mapsto 18t+3, \langle 12t-4\rangle \mapsto 12t+3$. These assignments produce altogether the differences $\{1,3,5,7\}\cup\{12t-15+4z\colon 0\le z\le 6\}\cup\{12t-1\}$.

It can be easily checked that, in the whole procedure, no label appears more than once and that all the required differences are obtained, with no repetition.

Case 2: t odd and greater than 3.

The labelling

$$[6x]\mapsto 24t-5-6x\ (x \text{ odd})\ ,\ [6x]\mapsto 24t+3-6x\ (x \text{ even}),$$

where $3t+4\le x\le 4t$, generates the differences $\{12t+37+4z\colon 0\le z\le 3t-10\}$. The labels employed are $\{1+12u, 3+12u\colon 0\le u\le (t-5)/2\}$. Furthermore, if $t\ne 5$, the labelling

$$[6x]\mapsto 24t+9-6x\ (x \text{ odd})\ ,\ [6x]\mapsto 24t+17-6x\ (x \text{ even}),$$

where $2t+3\le x\le 3t-3$, generates the differences $\{11+4z\colon 0\le z\le 3t-16\}$ (if $t=5$ such a labelling is not required, and we skip to the next assignment). The labels employed are $\{6t+33+12u, 6t+35+12u\colon 0\le u\le (t-7)/2\}$.

The remaining six labellings for stars of class 1 are $[18t+18]\mapsto 6t-17$, $[18t+12]\mapsto 6t-19$, and $[18t-12+6x]\mapsto 6t+11+4x$ with $0\le x\le 3$, the first two yielding the differences $\{12t+15+4z\colon 0\le z\le 5\}$, the other four yielding $\{12t-39+2z\colon 0\le z\le 11\}$.

We now deal with the stars of class 2. First we consider the labelling

$$[[6x]]\mapsto 24t-1-6x\ :\ 0\le x\le t-4\ ,$$

which produces the differences $\{12t+39+4z\colon 0\le z\le 3t-10\}$ and, if $t\ne 5$, the labelling

$$[[6x+2]]\mapsto 24t-5-6x\ :\ t+4\le x\le 2t-2$$

(if $t=5$ we skip this labelling, as above) generating the differences $\{9+4z\colon 0\le z\le 3t-16\}$. The labels employed in these cases are, respectively, $\{18t+23+6u\colon 0\le u\le t-4\}$ and $\{12t+7+6u\colon 0\le u\le t-6\}$.

It remains to label the centres of six stars, which is done as follows. (I) $[[6t-18]] \mapsto 18t+15$ and $[[6t-12]] \mapsto 18t+9$. (II) $[[6t+2]] \mapsto 18t-3$ and $[[6t+8]] \mapsto 18t+19$. (III) $[[6t+14]] \mapsto 18t-27$ and $[[6t+20]] \mapsto 18t-23$. These assignments yield, respectively, the differences $\{12t+13+4z\colon 0 \le z \le 5\}$, $\{12t-13+4z\colon z \in \{0,1,2,4,5,6\}\}$, $\{12t-51+2z\colon 0 \le z \le 5\}$. The six corresponding labels make up the set $\{18t+u\colon u \in \{-27,-23,-3,9,15,19\}\}$.

Finally, the four exceptional centres are labelled as in the even case. Again, the not so troublesome routine checks are left to the reader.

Case 3: ad hoc constructions for $t = 1,2,3$.

If $t=1$, we use the following labelling. $\langle 24\rangle \mapsto 1, \langle 18\rangle \mapsto 13, \langle 0\rangle \mapsto 21, \langle 8\rangle \mapsto 15$. If $t=2$, we make the assignments $[48] \mapsto 1, [42] \mapsto 25, [[0]] \mapsto 45, [[14]] \mapsto 33, \langle 36\rangle \mapsto 23, \langle 30\rangle \mapsto 3, \langle 6\rangle \mapsto 39, \langle 20\rangle \mapsto 27$. Finally, the case $t=3$ is managed as follows. $[72] \mapsto 3, [66] \mapsto 1, [60] \mapsto 33, [54] \mapsto 23, [[0]] \mapsto 71, [[6]] \mapsto 65, [[20]] \mapsto 67, [[26]] \mapsto 43, \langle 48\rangle \mapsto 69, \langle 42\rangle \mapsto 37, \langle 12\rangle \mapsto 57, \langle 32\rangle \mapsto 39$. Routine calculations are, as usual, left to the reader.

□

3 A corollary and some remarks

We are confident that some more familiarity with the present method will allow to construct graceful labellings for other classes of $P_{n,k}$'s. Actually, our hopes are not confined to generalised Petersen graphs, for we believe that even the graceful tree conjecture might become a little more affordable, using the present method. In the case of trees, however, some inductive argument should in our opinion back the above technique, thus keeping the pace of the "rocketing" of trees when the number of vertices increases.

Leaving aside the hoped for consequences in the next future, let us come back to the present theorem and exploit it in a classical fashion. We first recall some terminology. Let $G=(V,E)$ be a finite graph and $L=(V',E')$ be a subgraph of G such that $|E| = \omega|E'|$ for some integer ω. If G contains ω copies of L, say $L = L_1, L_2, \ldots, L_\omega$, whose overall edge set is equal to E, then we are in the presence of a *decomposition of G into copies of L*, also known as a *(G,L)-design* (see for example [5]). When interpreting the vertices of G as elements of $(\mathbf{Z}_{|V|},+)$, and letting this group act on V itself through the sum operation (the action naturally extends to the incidence structure, by defining the new edges as the images of the old edges), the above decomposition is termed *cyclic* if, for any $z \in \mathbf{Z}_{|V|}$ and any copy L_i, we have that $z + L_i = L_j$ for some j.

We are now in a position of proving the following.

Corollary 3.1. *There exists a cyclic $(K_{48t+1}, P_{8t,3})$-design for every $t \ge 1$.*

Proof. Let us think of the labels of a gracefully labelled graph $P_{8t,3}$ as integers (mod $48t+1$). By doing so, this graph becomes a subgraph of the complete graph K_{48t+1} whose vertex set is $\mathbb{Z}_{48t+1}$. Now it can be easily seen that the family $\{z + P_{8t,3} : 0 \leq z \leq 48t\}$ provides an edge-decomposition of K_{48t+1}. Such decomposition is clearly cyclic. □

It is well known that the above argument is applicable to whatever gracefully labelled graph G, thus yielding a cyclic $(K_{2|E(G)|+1}, G)$-design. Such a general scheme dates back to Rosa ([9]). In the same quoted paper, a stronger property than gracefulness was defined which, if satisfied, enables a graph to edge-decompose *infinitely* many complete graphs. The next definition deals with this property.

Definition 3.2. Using the terminology of Definition 1.1, the labelling λ is termed an *α-labelling* if it is graceful and there exists an integer I such that, for any edge $\{u, v\}$ with $\lambda(u) < \lambda(v)$, the inequality $\lambda(u) \leq I < \lambda(v)$ holds.

We remark that – as it could be proved with few difficulties – a graph admits an α-labelling only if it is bipartite. The following result points out the mentioned connection between α-labellings and graph decompositions.

Proposition 3.3. ([9]) *If a graph G having e edges admits an α-labelling, then there exists a cyclic (K_{2ce+1}, G)-design for every $c \geq 1$.*

Our interest in α-labellings stems from the fact that the graceful labellings devised in the above main proof are very close to be α-labellings. More precisely, in the two general constructions the only obstacle is given by the labels of the exceptional stars, $\langle 12t + 12 \rangle$, mapped to $24t - 3$. This assignment prevents the choice of $I = 12t + 1$ for obtaining an α-labelling. Instead, $I = 13$ and $I = 25$ are suitable values for the cases $t = 1, t = 2$ respectively, whereas in the case $t = 3$ the assignment $\langle 48 \rangle \mapsto 69$ prejudices the choice of $I = 37$. It seems then natural to propose the following conjecture.

Conjecture 3.4. *There exists an α-labelling for every graph $P_{8t,3}$ with $t \geq 1$.*

Needless to say, this conjecture would not make sense if some $P_{8t,3}$ were not bipartite. However, using the above cycle-stars splitting, the reader can immediately obtain a vertex bipartition for every t. More generally, and with no resort to the splitting, it would be extremely easy to show that $P_{n,k}$ admits a vertex bipartition if and only if n is even and k is odd.

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