Decomposing complete graphs into isomorphic subgraphs with six vertices and seven edges*

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Abstract: Let K_v be the complete multigraph with v vertices. Let G be a finite simple graph. A G-design of K_v, denoted by G-$GD(v)$, is a pair of $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called blocks, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly one block of $\mathcal{B}$. In this paper, the discussed graphs are sixteen graphs with six vertices and seven edges. We give a unified method for constructing such G-designs.

Key words: G-design; G-holey design; quasigroup

*Research supported by NSFC Grant 10371031 and NSFHB Grant 103146.
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1 Introduction

A *complete graph* of order v, denoted by K_v, is a graph with v vertices, where any two distinct vertices x and y are joined by one edge $\{x, y\}$. Let G be a finite simple graph. A *G-design* of K_v, denoted by G-$GD(v)$, is a pair of $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly one block of $\mathcal{B}$. The necessary conditions for the existence of a G-$GD(v)$ are

$$v(v-1) \equiv 0 \pmod{2e}, \ (v-1) \equiv 0 \pmod{d} \text{ and } v \geq g,$$

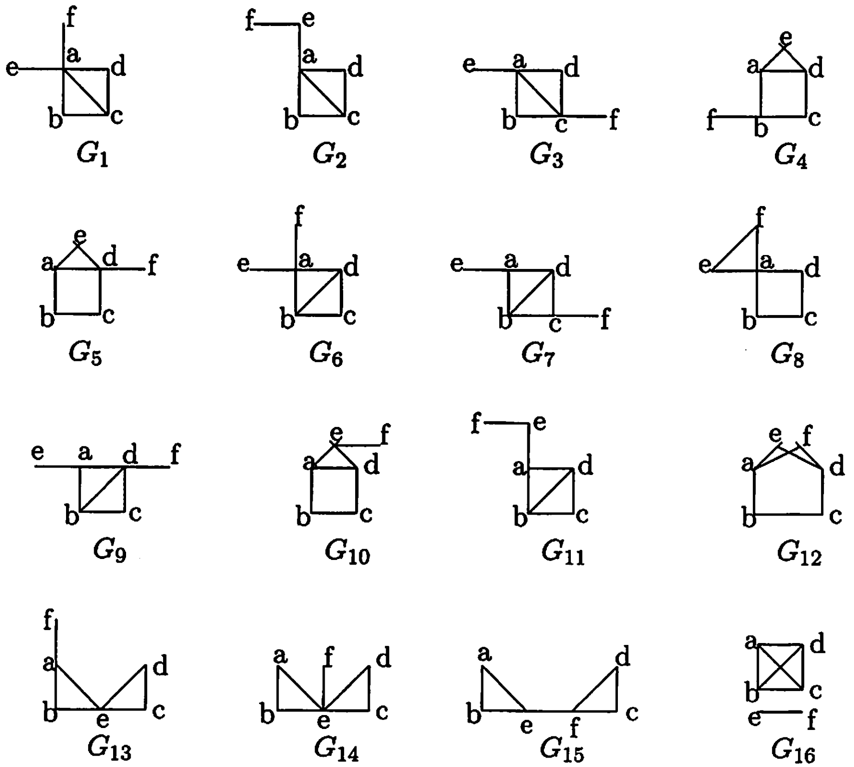
where $V(G)$ and $E(G)$ denotes the set of vertices and edges of G respectively, $e = |E(G)|$, $g = |V(G)|$, and d is the greatest common divisor of the degrees of all vertices in G.

Let $X = \bigcup_{i=1}^{t} X_i$ be the vertex set of $K_{n_1,n_2,\cdots,n_t}$, a complete multipartite graph consisting of t parts with size $n_1, n_2, \cdots, n_t$ respectively, where the sets X_i $(1 \leq i \leq t)$ are disjoint and $|X_i| = n_i$. Let $v = \sum_{i=1}^{t} n_i$ and $\mathcal{G} = \{X_1, X_2, \cdots, X_t\}$. For any given graph G, if the edges of $K_{n_1,n_2,\cdots,n_t}$ can be decomposed into edge-disjoint subgraphs $\mathcal{A}$, each member of which is isomorphic to G and is called a *block*, then the system $(X, \mathcal{G}, \mathcal{A})$ is called a *holey G-design*, denoted by G-$HD(T)$, where $T = n_1^1 n_2^1 \cdots n_t^1$ is the type of the holey G-design. Usually, the type is denoted by exponential form, for example, the type $1^i 2^r 3^k \cdots$ denotes i occurrences of 1, r occurrences of 2, etc. A G-$HD(1^{v-w} w^1)$ is called an *incomplete G-design*, denoted by G-$ID(v, w) = (V, W, \mathcal{A})$, where $|V| = v$, $|W| = w$ and $W \subset V$. Obviously, a G-$GD(v)$ is a G-$HD(1^v)$ or a G-$ID(v, w)$ with $w = 0$ or 1.

For the path P_k, the star $K_{1,k}$ and the cycle C_n, the existence problem of P_k-$GD(v)$, $K_{1,k}$-$GD(v)$ and C_k-$GD(v)$ has been solved, refer [1, 6, 11]. The graph design problem for some other graphs, e.g.,

k-cube[14], cycle with one chord[4,14] and so on[12,13,16], has also been researched. On the other hand, for the graphs with fewer vertices and fewer edges, the existence of their graph design has already been solved[2,3,9].

In this paper, we will discuss the graphs with six vertices and seven edges. There are twenty such graphs without isolated vertices (see the Appendix I in [10], pp 254-255). For two graphs among them, i.e. 6-cycle with a chord, the existence of their graph designs has been given in [14]. For other two graphs among them, i.e. $K_{2,3}$ with a pendent edge, the existence of their graph designs has been given in [8]. The remaining 16 graphs G_k, $1 \le k \le 16$, are listed as follows and are denoted by (a, b, c, d, e, f) according the vertex-labels in each graph. Note that the graph design for graph G_{12}, as a theta-graph, has been already given in [5]. However, for the sake of completeness, we will still show our construction.



In what follows, element (x,i) in $Z_m \times Z_n$ may be denoted by x_i for brevity. Moreover, $x_i+y_j = (x,i)+(y,j) = (x+y,i+j) = (x+y)_{i+j}$, $\infty+x = \infty$, $\infty+x_i = \infty$. For the block $B = (x,y,z,u,v,w)$, B mod m denotes the blocks $(x+t, y+t, z+t, u+t, v+t, w+t)$, $0 \le t \le m-1$. In $Z_m \times Z_n$, a block mod (m,n) denotes that the first (resp. second) coordinate taken modulo m (resp. n), while mod $(m,-)$ denotes that the first coordinate taken modulo m, the second invariant.

In this paper, we shall prove that the necessary conditions for the existence of a G_k-$GD(v)$ are also sufficient for each G_k with the exceptions $(v,k) \in \{(7,6),(7,7),(7,12),(7,15),(7,16), (8,11),(8,14), (8,16)\}$ and possible exceptions $(v,k) \in \{(14t+8,16) : t \ge 1\}$. The main way to get all constructions is the following lemma.

Lemma 1. *For given graph G and positive integers h,w,m, if there exist G-$HD(h^m)$, G-$ID(h+w,w)$ and G-$GD(w)$ (or G-$GD(h+w)$), then G-$GD(mh+w)$ exists, too.*

2 Constructions for holey designs

A *quasigroup* is a set Q with a binary operation "$\cdot$", denoted by $(Q,\cdot)$, such that the equations $a \cdot x = b$ and $y \cdot a = b$ are uniquely solvable for every pair of elements a, $b \in Q$. It is well known that the multiplication table of a quasigroup defines a Latin square. Similarly, a quasigroup can be obtained from a Latin square. A quasigroup is said to be *idempotent* (or *symmetric*) if the identity $x \cdot x = x$ (or $x \cdot y = y \cdot x$) holds for all $x \in Q$ (or $x, y \in Q$). Let S be a finite set and $H = \{S_1, S_2, \cdots, S_n\}$ be a partition of S. A *holey Latin square* with holes H is a $|S| \times |S|$ array L on S such that:

(1) every cell of L either contains an element of S or is empty;

(2) every element of S occurs at most once in any row or column of L;

(3) the subarrays indexed by $S_i \times S_i$ are empty for $1 \le i \le n$;

(4) element $s \in S$ occurs in row (or column) t if and only if $(s,t) \in (S \times S) \setminus \bigcup_{i=1}^{n} (S_i \times S_i)$.

The type of L is the multiset $T = \{|S_i| : 1 \le i \le n\}$ and will be denoted by exponential notation. A holey symmetric quasigroup corresponding a holey symmetric Latin square with type T is denoted by $HSQ(T) = (S, \mathcal{H}, \cdot)$. Two Latin squares L_1 and L_2 on a set S said to be *orthogonal* if their superposition yields every ordered pair in $S \times S$. A Latin square is called *self-orthogonal* if it is orthogonal to its transpose. A self-orthogonal quasigroup corresponding to a self-orthogonal Latin Square of order v is denoted by $SOQ(v)$. An idempotent SOQ is denoted by $ISOQ$.

Lemma 2.[7,9]

(1) *There exists an idempotent quasigroup of order v if and only if $v \neq 2$;*

(2) *There exists an idempotent symmetric quasigroup of order v if and only if v is odd;*

(3) *There exists an $HSQ(2^n)$ for all $n \ge 3$;*

(4) *There exists an $ISOQ(v)$ for $v \neq 2, 3, 6$.*

Let $(I_n, \cdot)$ be an idempotent quasigroup and $(I_{2n}, \mathcal{H}, \cdot)$ be an $HSQ(2^n)$ with holes $\mathcal{H} = \{\{2r-1, 2r\} : 1 \le r \le n\}$, where $I_n = \{1, 2, \cdots, n\}$ and $I_{2n} = \{1, 2, \cdots, 2n\}$. Define six subsets of $I_n \times I_n$ and three subsets of $I_{2n} \times I_{2n}$ as follows:

$$P = \{(i, i \cdot j) : 1 \le i < j \le n\},\ Q = \{(j, j \cdot i) : 1 \le i < j \le n\},$$
$$R = \{(i, j \cdot i) : 1 \le i < j \le n\},\ S = \{(j, i \cdot j) : 1 \le i < j \le n\},$$
$$M = \{(i, j) : 1 \le i < j \le n\},\ N = \{(i \cdot j, j \cdot i) : 1 \le i < j \le n\}.$$
$$P' = \{(i, i \cdot j) : 1 \le i < j \le 2n, \{i, j\} \notin \mathcal{H}\},$$
$$Q' = \{(j, i \cdot j) : 1 \le i < j \le 2n, \{i, j\} \notin \mathcal{H}\},$$
$$M' = \{(i, j) : 1 \le i < j \le 2n, \{i, j\} \notin \mathcal{H}\}.$$

Lemma 3. *Under the definitions above-mentioned,*

(1) $P \cup Q$ *(or* $R \cup S$*, or* $M \cup M^{-1}$*) forms all ordered 2-subsets in* I_n*.*

(2) $N \cup N^{-1}$ *forms all ordered 2-subsets in* I_n*, if* $(I_n, \cdot)$ *is self-orthogonal.*

(3) $P' \cup Q'$ *(or* $R' \cup S'$*, or* $M' \cup M'^{-1}$*) forms all ordered 2-subsets in* I_{2n} *without* $\mathcal{H}$*.*

Proof.

(1) From $|P| = |Q| = |R| = |S| = |M| = |N| = \binom{n}{2}$, we have

$|P| + |Q| = |R| + |S| = |M| + |M^{-1}| = |N| + |N^{-1}| = n(n-1)$,

which is just the number of all ordered 2-subsets in I_n. And, it is easy to see that the ordered 2-subsets in each one of P, Q, R, S, M and M^{-1} are distinct. Furthermore, we have

$$P \cap Q = \emptyset,\ R \cap S = \emptyset \text{ and } M \cap M^{-1} = \emptyset.$$

In fact, if $(u, v) \in P \cap Q$, let $(u, v) = (i, i \cdot j) = (j', j' \cdot i')$ then $i = j'$ and $i \cdot j = j' \cdot i'$. Therefore $j = i'$ and $i = j'$, the conditions $i < j$ and $i' < j'$ can't be simultaneously satisfied. Similarly, we can prove that $R \cap S = \emptyset$ and $M \cap M^{-1} = \emptyset$.

(2) The cell in the ith row and the jth column of the superposition of the corresponding Latin squares L and L^T is $(i \cdot j, j \cdot i)$. These cells are distinct if $(I_n, \cdot)$ is self-orthogonal, which implies that the ordered 2-subsets in N or N^{-1} are distinct. Suppose $N \cap N^{-1} \neq \emptyset$, then there is $(u, v) = (i \cdot j, j \cdot i) = (j' \cdot i', i' \cdot j')$. Thereby $(i, j) = (j', i')$ by the self-orthogonality, which is still a contradiction with $i < j$ and $i' < j'$.

(3) From $|P'| = |Q'| = |M'| = |M'^{-1}| = \binom{2n}{2} - n = 2n(n-1)$, we have

$$|P'| + |Q'| = |M'| + |M'^{-1}| = 4n(n-1),$$

which is just the number of all ordered 2-subsets in I_{2n} without $\mathcal{H}$. And, similar to (1), we can show that the ordered 2-subsets in each one of P', Q', M' and M'^{-1} are distinct and

$$P' \cap Q' = \emptyset \text{ and } M' \cap M'^{-1} = \emptyset$$

both hold also. □

Let G be a given simple graph and let $e = |E(G)|$. In order to construct a holey graph design $G\text{-}HD(e^n)$, we may take $Z_e \times I_n$ as the vertex set and Z_e as the automorphism group of the block set, where $(I_n, \cdot)$ is an idempotent quasigroup on the set $I_n = \{1, 2, \cdots, n\}$. A $G\text{-}HD(e^n)$ consists of $\frac{\binom{n}{2}e^2}{e} = \frac{n(n-1)e}{2}$ blocks. For our methods, the range of the subscripts of $\frac{n(n-1)}{2}$ base blocks $A_{i,j}$ is taken as $1 \le i < j \le n$.

On the other hand, in order to construct a holey graph design $G\text{-}HD((2e)^n)$, we may take $Z_e \times I_{2n}$ as the vertex set, Z_e as the automorphism group of the block set, where $I_{2n} = \{1, 2, \cdots, 2n\}$ and $(I_{2n}, \mathcal{H}, \cdot)$ forms an $HSQ(2^n)$ with holes $\mathcal{H} = \{\{2r-1, 2r\} : 1 \le r \le n\}$, which exists for $n \ge 3$ by Lemma 2(3). In fact, for the original $G\text{-}HD((2e)^n)$, the vertex set $Z_{2e} \times I_n$ contains n holes with size $2e$: $H_i = Z_{2e} \times \{i\}$, $1 \le i \le n$. Now, halve each hole H_i into $\overline{H}_{2i-1}$ and $\overline{H}_{2i}$, where each $\overline{H}_j = Z_e \times \{j\}$ has size e, $1 \le j \le 2n$. Then, equivalently, the holes of the $G\text{-}HD((2e)^n)$ can be regarded as $\overline{H}_1, \overline{H}_2, \cdots, \overline{H}_{2n}$ with such restriction that there is no edge between $\overline{H}_{2i-1}$ and $\overline{H}_{2i}$, $1 \le i \le n$. A $G\text{-}HD((2e)^n)$ consists of $\frac{\binom{n}{2}(2e)^2}{e} = 2n(n-1)e$ blocks. For our methods, the range of the subscripts of $2n(n-1)$ base blocks $A_{i,j}$ is taken as $1 \le i < j \le 2n$ and $\{i, j\} \notin \mathcal{H}$. Below, it suffices to construct only one base block $A_{i,j}$ for constructing $G\text{-}HD(e^n)$ and $G\text{-}HD((2e)^n)$, where i, j are variable in the given range.

Let $x, d \in Z_e$ and i, j be in the given range for $A_{i,j}$ in abovementioned constructions $G\text{-}HD(e^n)$ and $G\text{-}HD((2e)^n)$. Each vertex in the base block may be labelled as one among four forms: $(x, i), (x, j), (x, i \cdot j)$ and $(x, j \cdot i)$, where $(x, i \cdot j)$ and $(x, j \cdot i)$ are the same for the symmetric quasigroup. Each unordered edge in the base block may be one among six forms:

$\{(x, i), (x + d, j)\}$, $\{(x, i), (x + d, i \cdot j)\}$, $\{(x, j \cdot i), (x + d, j)\}$,

$\{(x,i),(x+d,j\cdot i)\}$, $\{(x,i\cdot j),(x+d,j)\}$, $\{(x,i\cdot j),(x+d,j\cdot i)\}$. For given $d\in Z_e$, $u,v\in\{i,j,i\cdot j,j\cdot i\}$ and $u\neq v$, the edge joining vertices (x,u) and $(x+d,v)$ in base block $A_{i,j}$ is denoted by $d(u,v)$, which represents a mixed difference orbit $\{\{(x,u),(x+d,v)\}:x\in Z_e\}$. And, denote $D(u,v)=\{d:d(u,v)\in A_{i,j}\}$.

Lemma 4A. *Let $(I_n,\cdot)$ be an idempotent quasigroup on the set $I_n=\{1,2,\cdots,n\}$ and G be a graph with e edges, then $\mathcal{A}=\{A_{i,j}:1\le i<j\le n\}$ can be taken as a base of a G-$HD(e^n)$ under the action of automorphism group Z_e if the following conditions hold.*
(1) $D(i,i\cdot j)=D(j,j\cdot i)$, $D(i,j\cdot i)=D(j,i\cdot j)$, $D(i,j)=D(j,i)$;
(2) $D(i\cdot j,j\cdot i)=D(j\cdot i,i\cdot j)$ when $(I_n,\cdot)$ is self-orthogonal;
(3) $D(i,j)\cup D(i,i\cdot j)\cup D(j\cdot i,j)\cup D(i,j\cdot i)\cup D(i\cdot j,j)\cup D(i\cdot j,j\cdot i)=Z_e$.
Proof. By the conditions and the conclusion (1), (2) of Lemma 3, each ordered (mixed) difference between any two of n holes appears in the base $\mathcal{A}$. Note that, for symmetric $(I_n,\cdot)$, the conditions become $D(i,i\cdot j)=D(j,i\cdot j)$ and $D(i,j)\cup D(i,i\cdot j)\cup D(i\cdot j,j)=Z_e$. □

Lemma 4B. *Let $(I_{2n},\mathcal{H},\cdot)$ be an $HSQ(2^n)$ with holes $\mathcal{H}=\{\{2r-1,2r\}:1\le r\le n\}$ and G be a graph with e edges. Then $\{A_{i,j}:1\le i<j\le 2n,\{i,j\}\notin\mathcal{H}\}$ can be taken as a base of a G-$HD((2e)^n)$ under the action of automorphism group Z_e if*

$$D(i,i\cdot j)=D(j,i\cdot j)\ \text{ and }\ D(i,j)\cup D(i,i\cdot j)\cup D(i\cdot j,j)=Z_e.$$

Proof. By the conditions and the conclusion (3) of Lemma 3, each ordered (mixed) difference between any two holes of n holes appears in the base $\mathcal{A}$. □

Lemma 5. *There exist G_k-$HD(7^{2t+1})$ and G_k-$HD(14^{t+2})$ for $1\le k\le 14$ and $t\ge 1$.*

Proof. By Lemma 2(2), there exists an idempotent symmetric quasigroup $(I_{2t+1},\cdot)$ on the set $I_{2t+1}=\{1,2,\dots 2t+1\}$. For each $G_k, 1\le k\le 14$, define a system $\mathcal{A}_k=\{A_k(i,j)\bmod(7,-):1\le i<j\le 2t+1\}$ on the set $X=Z_7\times I_{2t+1}$, where each base $A_k(i,j)$ is as

follows.

$$A_1(i,j) = ((0,i),(2,i\cdot j),(0,j),(1,i\cdot j),(4,j),(3,j)),$$
$$A_2(i,j) = ((0,i),(2,i\cdot j),(0,j),(1,i\cdot j),(3,j),(6,i)),$$
$$A_3(i,j) = ((0,i),(2,i\cdot j),(0,j),(1,i\cdot j),(4,j),(4,i)),$$
$$A_4(i,j) = ((0,j),(3,i),(5,j),(0,i),(1,i\cdot j),(6,j)),$$
$$A_5(i,j) = ((0,i),(5,j),(3,i),(0,j),(1,i\cdot j),(4,i)),$$
$$A_6(i,j) = ((2,i\cdot j),(0,i),(1,i\cdot j),(0,j),(5,i),(5,j)),$$
$$A_7(i,j) = ((1,i\cdot j),(0,i),(2,i\cdot j),(0,j),(4,i),(5,j)),$$
$$A_8(i,j) = ((0,i),(6,j),(2,i),(3,j),(0,j),(2,i\cdot j)),$$
$$A_9(i,j) = ((0,i),(1,i\cdot),(3,i),(0,j),(3,j),(5,i\cdot j)),$$
$$A_{10}(i,j) = ((0,i\cdot j),(1,j),(3,i),(3,j),(1,i),(5,i\cdot j)),$$
$$A_{11}(i,j) = ((0,i),(1,i\cdot j),(3,i),(3,j),(4,j),(5,i\cdot j)),$$
$$A_{12}(i,j) = ((0,i),(3,j),(3,i),(0,j),(1,i\cdot j),(2,i\cdot j)),$$
$$A_{13}(i,j) = ((5,j),(1,i\cdot j),(3,i\cdot j),(2,j),(0,i),(5,i)),$$
$$A_{14}(i,j) = ((5,j),(1,i\cdot j),(3,i\cdot j),(2,j),(0,i),(0,j)).$$

It is not difficult to verify that each $A_k(i,j)$ satisfies the conditions in Lemma 4A, so each $\mathcal{A}_k$ forms a G_k-$HD(7^{2t+1})$. Furthermore, let's consider the $HSQ(2^{t+2}) = (I_{2t+4}, \mathcal{H}, \cdot)$ with holes $\mathcal{H} = \{\{2r-1,2r\} : 1 \le r \le t+2\}$, which exists for $t \ge 1$ by Lemma 2(3). So, by Lemma 4B, $\mathcal{A}_k = \{A_k(i,j) \bmod (7,-) : 1 \le i < j \le 2t+4, \{i,j\} \notin \mathcal{H}\}$ will form a G_k-$HD(14^{t+2})$ for $1 \le k \le 14$. □

Lemma 6. *There exist a G_{15}-$HD(7^t)$ for $t \neq 2,3,6$, a G_{15}-$HD(14^t)$ for $t \ge 3$ and a G_{16}-$HD(14^t)$ for $t \ge 4$.*

Proof. By Lemma 2(4), there exists an $ISOQ(t) = (I_t, \cdot)$ on the set $I_t = \{1,2,\cdots t\}$ for $t \neq 2,3,6$. It is easy to verify that the base blocks in following each construction satisfy the conditions in Lemma 4A.

<u>G_{15}-$HD(7^t)$</u> $X = Z_7 \times I_t$,

$(3_j, 0_i, 3_i, 0_j, 1_{i\cdot j}, 1_{j\cdot i}) \bmod (7,-),\ \ 1 \le i < j \le t.$

<u>G_{15}-$HD(14^t)$</u> $X = Z_{14} \times I_t,\ 1 \le i < j \le t,$

$(6_{i\cdot j}, 1_j, 1_i, 4_{i\cdot j}, 0_i, 0_j),\ (5_{j\cdot i}, 2_j, 9_i, 13_{j\cdot i}, 0_i, 7_j) \bmod (14,-).$

$\underline{G_{15}\text{-}HD(14^3)}$ $\quad X = Z_{14} \times Z_3$,

$(0_1, 6_2, 11_2, 1_0, 0_0, 2_1),\ (3_1, 2_2, 12_2, 9_0, 0_0, 5_1) \mod (14, 3).$

$\underline{G_{15}\text{-}HD(14^6)}$ $\quad X = Z_{14} \times Z_6$,

$(10_1, 7_2, 13_0, 11_1, 0_0, 10_2),\ (11_3, 7_1, 5_0, 7_3, 0_0, 13_2),$
$(2_1, 5_2, 1_1, 9_0, 0_0, 4_3),\ (4_1, 9_2, 0_1, 5_0, 0_0, 6_3), \mod (14, 6);$
$(0_1, 1_2, 1_5, 0_4, 0_0, 0_3) + i_j,\ (0_2, 2_4, 9_1, 7_5, 0_0, 7_3) + i_j,$
$0 \le i \le 13,\quad 0 \le j \le 2.$

$\underline{G_{16}\text{-}HD(14^t)}$ $\quad X = Z_{14} \times I_t,\ 1 \le i < j \le t$,

$(1_{i \cdot j}, 0_i, 6_j, 4_{j \cdot i}, 1_i, 1_j),\ (4_{i \cdot j}, 6_i, 0_j, 1_{j \cdot i}, 0_i, 7_j) \mod (14, -).$

$\underline{G_{16}\text{-}HD(14^6)}$ $\quad X = Z_{14} \times Z_6$,

$(4_3, 0_0, 3_1, 11_2, 5_0, 0_1), (2_1, 0_0, 12_2, 4_4, 5_0, 0_2),$
$(6_3, 0_0, 1_1, 0_2, 1_0, 5_1), (11_1, 0_0, 2_2, 6_4, 1_0, 8_2) \mod (14, 6);$
$(0_1, 0_0, 3_3, 1_4, 1_0, 1_3) + i_j,\ (0_4, 0_3, 3_0, 1_1, 1_0, 8_3) + i_j,$
$0 \le i \le 13,\quad 0 \le j \le 2.$ □

3 Constructions for GD

In this section, we will give the existence for G_k-$GD(v)$, $1 \le k \le 16$.

Lemma 7. *There exist no G_k-$GD(7)$ for $k = 6, 7, 12, 15, 16$ and no G_k-$GD(8)$ for $k = 11, 14, 16$.*

Proof. Let graph G have m_i vertices with degree d_i, where $1 \le i \le r$ and $\sum m_i = 6$. Consider the existence of G-$GD(v)$ with b blocks by the following steps:

1° Solving the equations

$$\sum_{i=1}^{r} d_i x_i = v - 1 \text{ with conditions } \sum_{i=1}^{r} x_i \le b \text{ and } x_i \ge 0, \qquad (*)$$

obtain s integer solutions $(x_1, x_2, \cdots, x_r) = (a_{1j}, a_{2j}, \cdots, a_{rj})$, $1 \le j \le s$. The jth solution means that some element α of v-set may appear in a_{ij} blocks as degree d_i vertex, $1 \le i \le r$. We will say the element α has the degree-type $d_1^{a_{1j}} d_2^{a_{2j}} \cdots d_r^{a_{rj}}$.

2° Solve the further equations

$$\sum_{j=1}^{s} y_j = v \text{ and } \sum_{j=1}^{s} a_{ij}y_j = m_i b \quad 1 \le i \le r. \qquad (**)$$

Each solution $(y_1, y_2, \cdots, y_s)$ means that a possible structure of G-$GD(v)$: y_j elements in the v-set have degree-type $d_1^{a_{1j}} d_2^{a_{2j}} \cdots d_r^{a_{rj}}, 1 \le j \le s$.

3° For each solution obtained above, discuss the existence of such structure.

Now, let us prove this Lemma by these steps above-mentioned.

(1) G_6-$GD(7)$, $v = 7, b = 3$, $(d_i, m_i) = (1,2),(2,1),(3,2),(4,1)$. There are five solutions for $(*)$, where $r = 4$ and $s = 5$.

$$(a_{ij})_1^{r,s} = \begin{pmatrix} 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

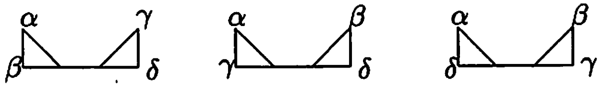
And, the equations $(**)$ have only two solutions: $(y_1, y_2, y_3, y_4, y_5) = (1,2,2,2,0)$, $(0,3,3,0,1)$, neither is viable. In fact, if $y_2 \ge 2$ then there are two elements α, β having degree-type $1^2 4^1$. But, for graph G_6, it will imply that the edge $\alpha\beta$ will appear twice.

(2) G_7-$GD(7)$ and G_{16}-$GD(7)$, $v = 7, b = 3$, $(d_i, m_i) = (1,2),(3,4)$. There is only one solution for $(*)$: $(a_{11}, a_{21}) = (0,2)$. Obviously, it is impossible.

(3) G_{12}-$GD(7)$ and G_{15}-$GD(7)$, $v = 7, b = 3$, $(d_i, m_i) = (2,4),(3,2)$. The solutions for $(*)$ are

$$(a_{ij})_1^{2,2} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

Furthermore, $(**)$ has only one solution $(y_1, y_2) = (4,3)$. But, for graph G_{12}, if $y_2 \ge 2$ then there are two elements α, β having degree-type 3^2. It is not available since the edge $\alpha\beta$ will not appear. As for graph G_{15}, let the four elements having degree-type 2^3 be $\alpha, \beta, \gamma, \delta$, then the six edges in the three blocks will form a one-factorization of the K_4 on the set $\{\alpha, \beta, \gamma, \delta\}$ as follows.



But, the other three elements having degree-type 3^2 can not be arranged well.

(4) G_{11}-$GD(8)$, $v = 8, b = 4$, $(d_i, m_i) = (1,1),(2,2),(3,3)$. The solutions for $(*)$ are

$$(a_{ij})_1^{3,4} = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \end{pmatrix}.$$

And, the unique solution for $(**)$ is $(y_1, y_2, y_3, y_4) = (4,4,0,0)$. Let the four elements having degree-type $1^1 3^2$ form a 4-set F. The six edges in K_4 over F must appear in the triangle consisting of three degree 3 vertices since the degree 1 and degree 3 vertices are disjoint. The unique possibility to form six edges by 4×2 vertices is a partition of K_4 into two triangles. Of course, it is impossible.

(5) G_{14}-$GD(8)$, $v = 8, b = 4$, $(d_i, m_i) = (1,1),(2,4),(5,1)$. The solutions for $(*)$ are

$$(a_{ij})_1^{3,3} = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 0 \end{pmatrix}.$$

And, the unique solution for $(**)$ is $(y_1, y_2, y_3) = (4,0,4)$. Let the four elements having degree-type $2^1 5^1$ form a 4-set F. There are six edges in K_4 over F. But, in the G_{14}-blocks, four degree 2 vertices and four degree 5 vertices form only four edges.

(6) G_{16}-$GD(8)$, $v = 8, b = 4$, $(d_i, m_i) = (1,2),(3,4)$. The unique solution for $(*)$ is $(x_1, x_2) = (1,2)$. So, the unique possibility is that each element has the same degree-type $1^1 3^2$. Suppose the K_2 in a G_{16}-block ($G_{16} = K_4 \cup K_2$) is edge $\alpha\beta$. The element α (and β) will

appear in two of other three G_{16}-blocks, so α and β must simultaneously appear in some block. Thus, edge $\alpha\beta$ will be repeated. □

Lemma 8. *There exists no G_{16}-$ID(14+8,8)$.*

Proof. Let $X = I_{14} \cup \{\infty_1, \infty_2, \cdots, \infty_8\}$. Since $G_{16} = K_4 \cup K_2$, any infinity element may be arranged only in one degree 3 vertex or in one degree 1 vertex of G_{16}. The number of G_{16}-blocks is 29, and the total degree of all infinity elements is $8 \times 14 = 112$. There are only two cases for arranging infinity elements:

(1) in 29 degree 3 vertices and 25 degree 1 vertices;

(2) in 28 degree 3 vertices and 28 degree 1 vertices.

Therefore, the K_{14} over the set I_{14} needs to be partitioned into twenty nine K_3 and four K_2 for case (1), or into twenty eight K_3, one K_4 and one K_2. It is impossible, since the maximum packing number $P(14,3,1) = 28$ (see [7]) and $K_3 \subset K_4$. □

Theorem. *There exists a G_k-$GD(v)$ if and only if $v \equiv 0,1$ (mod 7) for $k \neq 8$ or $v \equiv 1,7$ (mod 14) for $k = 8$ with the exceptions $(v,k) = (7,6),(7,7),(7,12),(7,15),(7,16),(8,11), (8,14),(8,16)$ and possible exceptions $(v,k) \in \{(14t+8,16) : t \geq 1\}$.*

Proof. Obviously, the necessary conditions for the existence of G_k-$GD(v)$ are $v \equiv 0,1 \pmod 7$ for $k \neq 8$, and $v \equiv 1,7 \pmod{14}$ for $k = 8$. By Lemmas 1, 5, 6, 7 and 8, we list the following table, where $t \geq 1$, $s \geq 4$ and $s \neq 6$. The desired HD-designs in the table have been already given in §2. The other desired designs, i.e. ID and GD, will be constructed in the Appendix. □

The authors should like to thank the referees for their useful suggestions.

$v \equiv (\text{mod } 14)$		0	1	7	8
$G_1 - G_5$,	HD	14^{t+2}	14^{t+2}	7^{2t+1}	7^{2t+1}
G_9, G_{10}, G_{13}	$GD(v)$	14,28	15,29	7	8
	HD	14^{t+2}	14^{t+2}	14^{t+2}	7^{2t+1}
G_6, G_7, G_{12}	$ID(v,w)$			(21,7)	
	$GD(v)$	14,28	15,29	21,35	8
G_8	HD		14^{t+2}	7^{2t+1}	
	$GD(v)$		15,29	7	
	HD	14^{t+2}	14^{t+2}	7^{2t+1}	14^{t+2}
G_{11}, G_{14}	$ID(v,w)$				(22,8)
	$GD(v)$	14,28	15,29	7	22,36
	HD	14^{t+2}	7^s	14^{t+2}	7^s
G_{15}	$ID(v,w)$			(21,7)	
	$GD(v)$	14,28	8,15,43	21,35	8,22
	HD	14^{t+3}	14^{t+3}	14^{t+3}	
G_{16}	$ID(v,w)$			(21,7)	?
	$GD(v)$	14,28,42	15,29,43	21,35,49	

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Appendix

In the following constructions, the block set of each G_k-$GD(v)$ is denoted by $\mathcal{B}_k(v)$, and its vertex set is taken as $(Z_3 \times Z_2)\bigcup\{\infty\}$ for $v = 7$, or Z_v for $v = 8, 15, 29$, or $Z_{v-1}\bigcup\{\infty\}$ for $v = 14, 28$ besides other set, which will be specially denoted.

1. $\mathcal{B}_1(7)$: $(0_0, \infty, 1_1, 0_1, 1_0, 2_1)$ mod $(3, -)$;
$\mathcal{B}_1(8)$: $(0, 2, 1, 3, 5, 6), (6, 2, 3, 7, 1, 5),$
$(7, 4, 1, 5, 0, 2), (4, 2, 5, 3, 0, 6)$;
$\mathcal{B}_1(14)$: $(0, 3, 1, 5, 6, \infty)$ mod 13;
$\mathcal{B}_1(15)$: $(0, 3, 1, 5, 6, 7)$ mod 15;
$\mathcal{B}_1(28)$: $(0, 5, 1, 7, 2, 12), (0, 11, 3, 13, 9, \infty)$ mod 27;
$\mathcal{B}_1(29)$: $(0, 5, 1, 7, 2, 9), (0, 11, 3, 13, 12, 14)$ mod 29.

2. $\mathcal{B}_2(7)$: $(0_0, \infty, 2_1, 1_0, 0_1, 1_1)$ mod $(3, -)$;
$\mathcal{B}_2(8)$: $(0, 2, 4, 3, 1, 5), (7, 3, 2, 5, 0, 6),$
$(6, 2, 1, 3, 5, 0), (4, 1, 7, 6, 5, 3)$;
$\mathcal{B}_2(14)$: $(0, 3, 1, 5, 6, \infty)$ mod 13;
$\mathcal{B}_2(15)$: $(0, 3, 1, 5, 6, 13)$ mod 15;
$\mathcal{B}_2(28)$: $(0, 5, 1, 7, 2, 14), (0, 11, 3, 13, 9, \infty)$ mod 27;
$\mathcal{B}_2(29)$: $(0, 5, 1, 7, 2, 14), (0, 11, 3, 13, 9, 23)$ mod 29.

3. $\mathcal{B}_3(7)$: $(0_0, \infty, 2_1, 1_0, 0_1, 1_1)$ mod $(3, -)$;
$\mathcal{B}_3(8)$: $(0, 2, 1, 3, 6, 7), (2, 6, 3, 7, 4, 5),$
$(4, 0, 5, 1, 3, 2), (6, 4, 7, 5, 1, 0)$;
$\mathcal{B}_3(14)$: $(0, 5, 1, 3, 6, \infty)$ mod 13;
$\mathcal{B}_3(15)$: $(0, 5, 1, 3, 6, 8)$ mod 15;
$\mathcal{B}_3(28)$: $(0, 5, 1, 7, 2, 10), (0, 11, 3, 13, 12, \infty)$ mod 27;
$\mathcal{B}_3(29)$: $(0, 5, 1, 7, 2, 10), (0, 11, 3, 13, 12, 17)$ mod 29.

4. $\mathcal{B}_4(7)$: $(1_0, 1_1, \infty, 0_0, 2_1, 0_1)$ mod $(3, -)$;

$\mathcal{B}_4(8)$: $(2,4,3,0,1,5), (5,0,4,7,2,6),$
$(3,7,1,6,2,0), (5,6,4,1,3,7)$;

$\mathcal{B}_4(14)$: $(3,9,4,0,1,\infty)$ mod 13;

$\mathcal{B}_4(15)$: $(3,9,4,0,1,2)$ mod 15;

$\mathcal{B}_4(28)$: $(25,9,4,0,12,6), (9,26,6,0,8,\infty)$ mod 27;

$\mathcal{B}_4(29)$: $(26,10,4,0,12,20), (9,16,5,0,8,14)$ mod 29.

5. $\mathcal{B}_5(7)$: $(1_0, \infty, 1_1, 2_1, 0_0, 2_0)$ mod $(3, -)$;

$\mathcal{B}_5(8)$: $(2,3,4,0,1,5), (0,7,5,6,3,4),$
$(6,2,5,1,7,3), (7,3,5,4,2,1)$;

$\mathcal{B}_5(14)$: $(0,4,9,3,1,\infty)$ mod 13;

$\mathcal{B}_5(15)$: $(0,4,9,3,1,10)$ mod 15;

$\mathcal{B}_5(28)$: $(0,4,9,25,12,\infty), (0,6,26,9,8,12)$ mod 27;

$\mathcal{B}_5(29)$: $(0,4,10,26,12,16), (0,5,16,9,8,7)$ mod 29.

6. $\mathcal{B}_6(8)$: $(0,2,5,4,3,6), (1,5,7,0,2,6),$
$(6,2,3,7,4,5), (3,1,7,4,5,6)$;

$\mathcal{B}_6(14)$: $(0,1,7,3,5,\infty)$ mod 13;

$\mathcal{B}_6(15)$: $(0,1,7,3,5,8)$ mod 15;

$\mathcal{B}_6(28)$: $(0,1,10,3,4,12), (0,5,19,11,10,\infty)$ mod 27;

$\mathcal{B}_6(29)$: $(0,1,10,3,12,13), (0,5,19,11,10,4)$ mod 29;

$\mathcal{B}_6(21)$: $X = Z_3 \times Z_7$,
$(1_0, 0_0, 1_4, 1_3, 0_3, 2_5), (1_1, 0_1, 2_2, 1_6, 1_4, 2_4),$
$(1_3, 0_3, 2_6, 1_5, 2_1, 0_1), (1_4, 0_4, 2_5, 1_6, 0_2, 2_2),$
$(1_2, 0_2, 1_5, 1_6, 0_3, 2_3), (1_5, 0_5, 1_3, 0_4, 2_1, 2_0),$
$(1_6, 0_6, 1_4, 0_3, 2_0, 1_0), (0_0, 0_1, 0_3, 0_2, 0_4, 1_2),$
$(0_1, 2_0, 1_4, 1_2, 0_5, 2_4), (0_5, 2_1, 1_6, 0_0, 1_2, 0_2)$ mod $(3, -)$;

$\mathcal{B}_6(35)$: $X = (Z_{17} \times Z_2) \bigcup \{\infty\}$,
$(0_0, 3_0, 5_1, 4_0, 6_0, 7_1), (0_1, 3_1, 5_0, 4_1, 6_0, 4_0),$
$(0_1, 7_1, 10_0, 5_1, 7_0, 8_1), (0_0, 7_0, 10_1, 5_0, 4_1, 8_0),$
$(6_1, 0_0, \infty, 0_1, 14_0, 15_0)$ mod $(17, -)$.

7. $\mathcal{B}_7(8)$: $(0,2,4,3,1,5), (5,0,7,6,3,2),$

$(4,1,3,6,0,7),(7,1,2,5,4,6);$

$\mathcal{B}_7(14)$: $(0,1,7,3,5,\infty)$ mod 13;

$\mathcal{B}_7(15)$: $(0,1,7,3,8,2)$ mod 15;

$\mathcal{B}_7(28)$: $(0,1,10,3,12,20),(0,5,19,11,4,\infty)$ mod 27;

$\mathcal{B}_7(29)$: $(0,1,10,3,12,23),(0,5,19,11,4,9)$ mod 29;

$\mathcal{B}_7(21)$: $X = Z_3 \times Z_7$,

$(1_0,0_0,1_2,1_3,1_6,0_4),(1_1,0_1,2_2,1_6,1_5,1_3),$
$(1_2,0_2,1_5,1_6,0_5,2_0),(1_3,0_3,2_6,1_5,2_0,0_0),$
$(1_4,0_4,2_5,1_6,0_0,2_2),(1_5,0_5,1_3,0_4,0_0,1_1),$
$(1_6,0_6,1_4,0_3,2_1,0_2),(0_0,0_1,1_3,0_2,1_6,1_4),$
$(0_1,2_0,1_4,1_2,2_3,2_1),(0_5,2_1,0_4,0_0,1_1,0_1)$ mod $(3,-)$;

$\mathcal{B}_7(35)$: $X = (Z_{17} \times Z_2)\bigcup\{\infty\}$,

$(0_0,3_0,5_1,4_0,7_1,1_0)$, $(0_1,3_1,5_0,4_1,9_0,\infty)$,
$(0_0,7_0,10_1,5_0,13_1,2_1),(0_1,7_1,10_0,5_1,7_0,2_0)$,
$(6_1,0_0,6_0,0_1,\infty,15_1)$ mod $(17,-)$.

8. $\mathcal{B}_8(7)$: $(0_0,\infty,1_1,0_1,1_0,2_1)$ mod $(3,-)$;

$\mathcal{B}_8(15)$: $(0,4,11,5,1,3)$ mod 15;

$\mathcal{B}_8(29)$: $(0,28,9,25,2,7),(0,8,25,11,3,9)$ mod 29.

9. $\mathcal{B}_9(7)$: $(0_0,1_0,\infty,0_1,1_1,2_1)$ mod $(3,-)$;

$\mathcal{B}_9(8)$: $(0,1,4,5,2,3),(1,2,5,6,7,4),$
$(2,3,6,7,4,5),(3,4,7,0,1,6);$

$\mathcal{B}_9(14)$: $(0,1,7,3,\infty,8)$ mod 13;

$\mathcal{B}_9(15)$: $(0,1,7,3,5,10)$ mod 15;

$\mathcal{B}_9(28)$: $(0,12,2,13,\infty,8),(0,7,1,9,3,5)$ mod 27;

$\mathcal{B}_9(29)$: $(0,13,2,14,3,8),(0,8,1,10,4,5)$ mod 29.

10. $\mathcal{B}_{10}(7)$: $(0_1,2_1,\infty,1_0,0_0,1_1)$ mod $(3,-)$;

$\mathcal{B}_{10}(8)$: $(0,6,7,1,4,3),(1,6,4,2,5,0),$
$(2,7,5,6,3,1),(3,5,4,7,0,2);$

$\mathcal{B}_{10}(14)$: $(3,9,4,0,1,\infty)$ mod 13;

$\mathcal{B}_{10}(15)$: $(3,9,4,0,1,8)$ mod 15;

$\mathcal{B}_{10}(28)$: $(0,12,2,13,6,\infty),(0,4,13,5,2,1)$ mod 27;

$\mathcal{B}_{10}(29)$: $(0,13,2,14,4,3),(0,9,2,8,3,1)$ mod 29.

11. $\mathcal{B}_{11}(7)$: $(0_0,1_0,\infty,0_1,1_1,2_1)$ mod $(3,-)$;

$\mathcal{B}_{11}(14)$: $(0,1,7,3,5,\infty)$ mod 13;

$\mathcal{B}_{11}(15)$: $(0,1,7,3,5,12)$ mod 15;

$\mathcal{B}_{11}(28)$: $(0,12,2,13,4,\infty),(0,7,1,9,3,8)$ mod 27;

$\mathcal{B}_{11}(29)$: $(0,13,2,14,3,7),(0,8,1,10,5,11)$ mod 29;

$\mathcal{B}_{11}(22)$: $X = Z_{11}\times Z_2$,
$(0_1,0_0,1_1,4_0,2_1,3_1),(9_1,0_0,10_1,5_0,4_1,1_1)$,
$(3_0,0_0,3_1,1_0,9_1,5_1)$ mod $(11,-)$;

$\mathcal{B}_{11}(36)$: $X = Z_9\times Z_4$,
$(4_1,0_0,3_2,4_0,0_3,3_0),(3_0,0_0,6_1,1_0,1_1,8_0)$ mod $(9,4)$,
$(0_2,0_0,2_2,1_1,1_3,3_1),(0_3,0_1,2_3,1_0,1_2,3_0)$ mod $(9,-)$.

12. $\mathcal{B}_{12}(8)$: $(0,5,4,1,7,6),(2,7,6,3,0,1)$,
$(4,0,1,5,2,3),(6,2,3,7,5,4)$;

$\mathcal{B}_{12}(15)$: $(0,1,4,2,6,7)$ mod 15;

$\mathcal{B}_{12}(29)$: $(0,21,12,2,13,14),(0,3,1,2,6,7)$ mod 29;

$\mathcal{B}_{12}(14)$: $X = Z_{14}$,
$(0,7,10,2,5,6),(0,11,6,1,8,9),(9,13,8,10,6,11)$,
$(1,7,9,2,0,13),(3,12,13,4,0,1),(2,12,4,5,1,11)$,
$(3,9,5,7,2,6),(4,3,7,8,2,9),(3,10,9,12,8,11)$,
$(5,12,10,13,3,6),(4,7,13,5,8,10),(6,12,7,11,4,8)$,
$(0,13,11,1,10,12)$;

$\mathcal{B}_{12}(21)$: $X = (Z_5\times Z_4)\bigcup\{\infty\}$,
$(0_0,0_2,0_3,0_1,\infty,3_3),(1_2,\infty,3_3,0_2,2_2,2_3)$,
$(0_0,4_1,1_3,1_0,3_1,2_0),(0_0,2_3,0_3,1_0,1_1,2_2)$,
$(0_0,1_3,2_2,0_1,3_2,4_2),(0_1,1_3,0_3,1_1,2_1,1_2)$ mod $(5,-)$;

$\mathcal{B}_{12}(28)$: $X = (Z_9\times Z_3)\bigcup\{\infty\}$,
$(0_0,1_0,3_0,6_0,1_1,2_1)$ mod $(9,3)$,
$(0_0,2_2,6_2,\infty,0_1,4_0),(0_1,3_0,3_2,6_0,4_1,0_2)$,
$(0_0,1_2,4_1,2_2,3_1,8_1)$ mod $(9,-)$;

$\mathcal{B}_{12}(35)$: $X = (Z_{17}\times Z_2)\bigcup\{\infty\}$,

$(4_0, 2_0, 2_1, 1_1, \infty, 0_0), (0_0, 3_0, 8_0, 2_0, 15_1, 16_1),$
$(0_0, 8_0, 9_0, 2_0, 11_1, 12_1), (7_1, 1_0, 6_1, 8_1, 1_1, 0_0),$
$(7_1, 11_1, 3_1, 0_0, 2_1, 4_1) \quad \text{mod } (17, -).$

13. $\mathcal{B}_{13}(7)$: $(2_1, 1_0, 0_1, \infty, 0_0, 1_1) \quad \text{mod } (3, -);$

$\mathcal{B}_{13}(8)$: $(1, 3, 7, 4, 0, 6), (2, 4, 7, 5, 1, 0),$
$(5, 3, 7, 6, 2, 4), (3, 4, 0, 5, 6, 7);$

$\mathcal{B}_{13}(14)$: $(1, 4, 11, 5, 0, \infty) \quad \text{mod } 13;$

$\mathcal{B}_{13}(15)$: $(2, 5, 7, 1, 0, 6) \quad \text{mod } 15;$

$\mathcal{B}_{13}(28)$: $(3, 8, 10, 4, 0, \infty), (2, 11, 13, 1, 0, 9) \quad \text{mod } 27;$

$\mathcal{B}_{13}(29)$: $(4, 9, 11, 3, 0, 10), (2, 12, 14, 1, 0, 9) \quad \text{mod } 29.$

14. $\mathcal{B}_{14}(7)$: $(\infty, 0_1, 1_1, 2_1, 0_0, 1_0) \quad \text{mod } (3, -);$

$\mathcal{B}_{14}(14)$: $(1, 4, 11, 5, 0, \infty) \quad \text{mod } 13;$

$\mathcal{B}_{14}(15)$: $(2, 5, 7, 1, 0, 4) \quad \text{mod } 15;$

$\mathcal{B}_{14}(28)$: $(3, 8, 10, 4, 0, \infty), (2, 11, 13, 1, 0, 7) \quad \text{mod } 27;$

$\mathcal{B}_{14}(29)$: $(4, 9, 11, 3, 0, 6), (2, 12, 14, 1, 0, 7) \quad \text{mod } 29;$

$\mathcal{B}_{14}(22)$: $X = Z_{11} \times Z_2,$
$(1_1, 2_1, 9_1, 0_1, 0_0, 4_0), (3_1, 6_1, 8_1, 4_1, 0_0, 5_0),$
$(5_1, 10_1, 3_0, 1_0, 0_0, 7_1) \quad \text{mod } (11, -);$

$\mathcal{B}_{14}(36)$: $X = Z_9 \times Z_4,$
$(3_2, 1_2, 3_1, 7_1, 0_0, 2_1), (0_1, 6_1, 1_1, 2_1, 6_2, 2_0) \quad \text{mod } (9, 4),$
$(8_3, 0_2, 1_3, 1_1, 0_0, 2_2), (7_2, 0_0, 1_3, 0_2, 8_1, 6_3) \quad \text{mod } (9, -).$

15. $\mathcal{B}_{15}(8)$: $(2, 1, 5, 6, 0, 4), (7, 4, 0, 3, 1, 5), (7, 5, 1, 3, 2, 6), (4, 2, 0, 6, 3, 7);$

$\mathcal{B}_{15}(15)$: $(7, 1, 9, 6, 0, 4) \quad \text{mod } 15;$

$\mathcal{B}_{15}(14)$: $X = Z_7 \times Z_2, \quad (2_0, 1_0, 2_1, 3_1, 5_1, 3_0) + i_0 \quad 0 \le i \le 4,$
$(0_0, 4_0, 3_0, 1_0, 2_0, 5_0), (2_1, 1_1, 1_0, 4_0, 2_0, 6_0), (3_0, 1_1, 6_1, 3_1, 0_0, 5_0),$
$(2_1, 4_1, 3_1, 1_1, 0_1, 5_1), (1_0, 4_1, 4_0, 2_1, 0_0, 5_1), (1_1, 6_1, 0_1, 3_1, 4_1, 2_0),$
$(3_0, 4_1, 1_0, 1_1, 6_0, 0_1), (0_0, 6_0, 6_1, 2_1, 3_1, 1_0);$

$\mathcal{B}_{15}(21)$: $X = (Z_5 \times Z_4) \bigcup \{\infty\},$
$(1_0, 3_0, 4_0, 0_2, 3_1, 1_2) \quad \text{mod } (5, 4),$
$(3_0, \infty, 0_0, 1_3, 4_1, 0_2), (3_3, \infty, 0_3, 1_0, 4_2, 0_1) \quad \text{mod } (5, -);$

$\mathcal{B}_{15}(22)$: $X = Z_{11} \times Z_2,$

$(3_0,1_0,1_1,3_1,0_0,0_1),(4_0,0_0,0_1,6_1,10_1,3_0),$
$(5_0,0_0,1_1,8_1,9_1,7_0)$ mod $(11,-)$;

$\mathcal{B}_{15}(28)$: $X=(Z_9\times Z_3)\bigcup\{\infty\}$, $(5_0,2_0,8_0,4_2,0_0,7_1)$ mod $(9,3)$,
$(1_0,0_0,4_1,2_0,1_1,6_2),(1_1,0_1,0_2,6_1,1_2,1_0),$
$(3_1,0_0,3_0,1_2,\infty,0_2)$ mod $(9,-)$;

$\mathcal{B}_{15}(35)$: $X=(Z_{17}\times Z_2)\bigcup\{\infty\}$,
$(0_1,\infty,3_1,2_1,1_0,2_0)$ mod $(17,-)$,
$(8_0,5_0,3_0,14_0,0_0,5_1),(2_0,0_0,6_0,10_0,6_1,13_1)$ mod $(17,2)$;

$\mathcal{B}_{15}(43)$: $X=Z_{43}$,
$(26,19,13,4,14,0),(27,17,18,3,11,0),$
$(29,10,21,1,8,0)$ mod 43.

16. $\mathcal{B}_{16}(14)$: $(1,0,4,6,\infty,2)$ mod 13;

$\mathcal{B}_{16}(15)$: $(1,0,4,6,2,9)$ mod 15;

$\mathcal{B}_{16}(29)$: $(13,0,7,2,1,11),(12,0,8,9,1,15)$ mod 29;

$\mathcal{B}_{16}(21)$: $X=(Z_5\times Z_4)\bigcup\{\infty\}$,
$(0_2,\infty,1_3,0_0,2_2,2_3),(4_1,0_0,1_0,2_1,\infty,0_1),$
$(4_2,0_0,2_0,3_2,0_2,2_3),(4_2,0_1,1_1,2_2,0_3,1_0),$
$(2_1,0_3,1_3,2_2,3_3,0_0),(0_0,0_3,2_3,0_1,2_1,3_3)$ mod $(5,-)$;

$\mathcal{B}_{16}(28)$: $X=(Z_9\times Z_3)\bigcup\{\infty\}$,
$(6_1,0_0,4_0,0_2,\infty,1_0),(1_0,0_0,3_0,8_1,2_0,3_1)$ mod $(9,3)$;

$\mathcal{B}_{16}(42)$: $X=Z_{41}\bigcup\{\infty\}$,
$(20,0,3,19,\infty,1),(14,0,2,10,3,12),$
$(18,0,5,11,1,16)$ mod 41;

$\mathcal{B}_{16}(43)$: $X=Z_{43}$,
$(20,0,3,19,1,22),(4,0,2,10,3,12),$
$(18,0,5,11,1,16)$ mod 43;

$\mathcal{B}_{16}(49)$: $X=(Z_{24}\times Z_2)\bigcup\{\infty\}$, $0\le i\le 11$,
$(23_1,0_0,5_0,17_1,1_0,13_0)+i_0$, $(11_1,12_0,17_0,5_1,0_1,12_1)+i_0$,
$(10_0,0_0,11_0,2_0,1_0,20_1),(10_1,0_1,11_1,2_1,0_0,21_1),$
$(20_1,0_0,6_0,15_1,1_0,9_1),(7_1,0_0,7_0,11_1,1_0,17_1),$
$(6_1,0_0,3_0,13_1,\infty,1_0),(2_1,0_0,4_0,5_1,\infty,0_1)$ mod $(24,-)$.

17. Lastly, construct all G_k-$ID(v,w)$ for $k = 6, 7, 11, 12, 14, 15, 16$.

G_6-$ID(21,7)$: $(Z_7 \times Z_2) \bigcup \{x_0, \cdots x_6\}$,

$(x_i, 4_0, 1_1, 6_0, 0_1, 2_1) + i_0, (x_i, 1_1, 0_0, 1_0, 3_1, 4_1) + i_0,$

$(x_i, 2_0, 1_1, 5_0, 0_0, 3_0) + i_0, \quad 0 \le i \le 5;$

$(1_1, 5_1, x_i, 6_1, 3_0, x_6) + j_0, \quad 0 \le j \le 3;$

$(x_6, 1_0, 0_1, 4_0, 2_0, 5_1), (0_1, x_6, 3_0, 5_0, 4_1, 5_1), (5_1, 2_1, x_4, 3_1, 0_0, 4_1),$

$(0_1, 6_0, x_6, 0_0, 3_0, 2_0), (6_1, 3_1, x_5, 4_1, 1_0, x_6).$

G_7-$ID(21,7)$: $(Z_7 \times Z_2) \bigcup \{x_0, \cdots x_6\}$,

$(x_i, 2_0, 1_1, 5_0, 0_0, x_{i+6}) + i_0, \ (x_i, 1_1, 0_0, 1_0, 3_0, 5_1) + i_0 \quad 0 \le i \le 6;$

$(x_i, 4_0, 1_1, 6_0, 4_1, x_{i+5}) + i_0 \quad 0 \le i \le 2;$

$(x_{i+3}, 0_0, 4_1, 2_0, 3_1, x_{i+1}) + i_0 \quad i = 0, 1.$

G_{11}-$ID(22,8)$: $(Z_7 \times Z_2) \bigcup \{x_1, \cdots, x_8\}$,

$(4_0, 5_1, x_2, 0_0, x_1, 0_1), (2_1, 4_1, x_4, 0_0, x_3, 1_0),$

$(6_1, 3_1, x_6, 0_0, x_5, 1_0) \mod (7, -).$

$(0_0, x_7, 6_1, 0_1, x_8, 5_0), (5_0, x_7, 4_1, 5_1, 6_0, 4_0),$

$(2_0, x_7, 1_1, 2_1, 1_0, 3_0), (5_0, 3_0, x_7, 4_0, 0_0, 6_0),$

$(1_0, x_8, 0_1, 1_1, x_7, 6_0), (6_0, x_8, 5_1, 6_1, 1_0, 0_0),$

$(3_0, x_8, 2_1, 3_1, 2_0, 0_0), (4_1, x_8, 2_0, 4_0, 3_1, x_7).$

G_{12}-$ID(21,7)$: $(Z_7 \times Z_2) \bigcup \{x_1, \cdots, x_7\}$,

$(0_1, x_3, 3_0, 0_0, x_1, x_2), (0_0, 4_1, 3_0, 6_1, x_4, 5_1),$

$(6_1, x_5, 5_0, 0_1, 2_1, 0_0) \quad \mod (7, -);$

$(x_6, 1_0, 2_0, x_7, 0_1, 1_1), (x_6, 2_0, 4_0, x_7, 2_1, 3_1),$

$(x_6, 4_0, 6_0, x_7, 4_1, 5_1), (x_6, 6_0, 1_0, x_7, 6_1, 5_0),$

$(0_0, 5_0, 4_0, 3_0, x_6, x_7), (0_0, 6_0, 5_0, 3_0, 1_0, 2_0).$

G_{14}-$ID(22,8)$: $(Z_7 \times Z_2) \bigcup \{x_1, \cdots, x_8\}$,

$(0_0, x_1, x_2, 1_0, 6_1, 3_1), (3_1, x_3, x_4, 4_1, 0_0, 1_0),$

$(0_0, x_5, x_6, 1_0, 1_1, 3_1) \mod (7, -);$

$(1_1, x_7, x_8, 5_0, 0_1, 6_1), (0_0, x_7, x_8, 1_1, 2_1, 3_1),$

$(3_1, x_7, x_8, 2_0, 4_1, 5_1), (2_0, x_7, x_8, 4_0, 6_0, 1_1),$

$(5_1, x_7, 6_0, 1_0, 3_0, x_8), (6_1, x_7, 0_0, 2_0, 4_0, 1_0),$

$(1_0, x_7, 0_0, 3_0, 5_0, 2_0), (5_1, 6_1, 1_0, 3_1, x_8, 0_0).$

G_{15}-$ID(21, 7)$: $(Z_7 \times Z_2) \bigcup \{x_1, \cdots x_7\}$,

$(6_1, x_1, x_2, 0_1, 0_0, 3_0), (2_0, x_3, x_4, 0_0, 0_1, 3_1)$ mod $(7, -)$;

$(3_1, 4_1, 6_0, 0_1, x_5, 6_1), (0_0, 2_1, 2_0, 1_0, x_5, 3_0), (6_0, 1_1, 4_0, 3_0, x_5, 5_0),$

$(5_1, 0_1, 1_1, 0_0, x_5, 1_0), (4_1, 3_0, x_5, 2_0, x_6, 4_0), (6_0, 1_0, 4_1, 2_0, x_6, 2_1),$

$(3_1, 2_0, 0_0, 6_0, x_6, 5_0), (0_1, 0_0, 6_1, 5_0, x_6, 5_1), (6_1, 4_1, 3_1, 2_1, x_7, 1_0),$

$(5_0, 0_1, 5_1, 3_1, x_7, 3_0), (0_0, 2_0, 1_1, 0_1, x_7, 2_1), (1_1, 3_1, 4_1, 4_0, x_7, 5_1),$

$(1_1, x_6, x_7, 6_0, 6_1, 4_0).$

G_{16}-$ID(21, 7)$: $(Z_7 \times Z_2) \bigcup \{x_1, \cdots, x_7\}$,

$(x_0, 1_0, 5_1, 0_0, 2_0, 3_1), (x_0, 4_0, 4_1, 2_0, x_1, 6_0),$

$(x_0, 6_0, 2_1, 3_0, x_1, 0_1)$ mod $(7, -)$;

$(x_0, 1_1, 3_1, 0_1, x_6, 2_1), (x_1, 2_1, 4_1, 1_1, x_6, , 6_1),$

$(x_2, 3_1, 5_1, 2_1, 0_1, 6_1), (x_3, 4_1, 6_1, 3_1, 0_1, 2_1),$

$(x_4, 5_1, 0_1, 4_1, 2_1, 6_1), (x_5, 6_1, 1_1, 5_1, x_6, 0_1).$