A Note on the Total Relative Displacement

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Let $G = (V, E)$ be a connected graph and let ϕ be a permutation of V. Define the total relative displacement of the permutation ϕ of V to be

$$\delta_\phi(G) = \sum_{\{x \neq y\} \subset V} |d(x, y) - d(\phi(x), \phi(y))|,$$

where $d(x, y)$ means the distance between x and y, i.e., the length of a shortest path between x and y. Chartrand, Gavlas and VanderJagt[2] considered the total relative displacement of permutations of a graph (this parameter is related to the sorting problem in computer science and measures the disorderliness of data) and studied near-automorphisms of graphs, i.e., permutations that attain the minimum value $\pi(G)$ of the nonzero total relative displacement of the graph G. They got a lot of fundamental properties including the property $\pi(G) \geq 2$ which we will cite later, and they also proposed a conjecture and a problem about it. The conjecture said that $\pi(P_n) = 2n - 4$ for $n \geq 3$; and the problem was to determine $\inf\{\frac{\pi(T)}{|V(T)|} : T$ is a tree of order at least $3\}$. Later, Aiken[1] proved this conjecture and classified the near-automorphisms of paths. Reid[3] determined $\pi(K_{n_1, n_2, \cdots, n_t})$ for complete t-partite graphs.

Now, let's solve the problem.

Problem 1. Determine $\inf\{\frac{\pi(T)}{|V(T)|} | T$ is a tree of order at least $3\}$.

Sol. (i). If T is a tree of order 3, then T is a path and $\pi(T) = 2 \cdot 3 - 4 = 2$. So $\frac{\pi(T)}{|V(T)|} = \frac{2}{3}$.

(ii). If T is a tree of order 4, then T is a path P_4 or a star $K_{1,3}$. Thus we get $\pi(P_4) = 2 \cdot 4 - 4 = 4$ and $\pi(K_{1,3}) = 2(1 + 3 - 2) = 4$. Therefore, we have $\frac{\pi(T)}{|V(T)|} = \frac{4}{4} = 1$.

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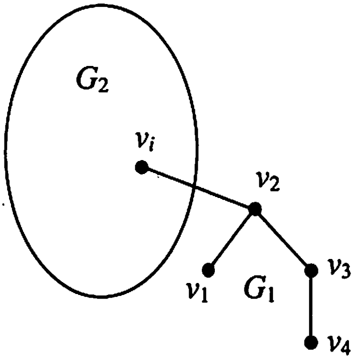

Figure 1: The Tree T

(iii). Construct a tree of order $i \geq 5$ as follows:

Take G_1 a path of order 4 where $V(G_1) = \{v_1, v_2, v_3, v_4\}$ and $E(G_1) = \{v_1v_2, v_2v_3, v_3v_4\}$, and G_2 be any tree with the vertex set $V(G_2) = \{v_5, \cdots, v_i\}$. Let $T = (V, E)$ be a tree where $V = \{v_1, v_2, \cdots, v_i\}$ and $E(T) = E(G_1) \cup E(G_2) \cup \{v_2v_i\}$(see Figure 1).

If $\phi = (v_1, v_3)$ is a transposition of $V(T)$, then $\delta_\phi(T) = 4$. Thus we have $0 < \frac{\pi(T)}{|V(T)|} \leq \frac{\delta_\phi(T)}{|V(T)|} = \frac{4}{i}$ for $i \geq 5$.

By results (i), (ii), and (iii) , we get $\inf\{\frac{\pi(T)}{|V(T)|} | T$ is a tree of order at least $3\} = 0$.

References

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[3] K. B. Reid, Total relative displacement of vertices in complete multipartite graphs, Journal of Graph Theory, 2002.