Maximal triangle trades with foundation 5 (mod 6) — the final case

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Abstract

The maximum possible volume of a simple, non-Steiner $(v, 3, 2)$ trade was determined for all v by Khosrovshahi and Torabi (Ars Combinatoria **51** (1999), 211–223); except that in the case $v \equiv 5 \pmod 6$, $v \geq 23$, they were only able to provide an upper bound on the volume. In this paper we construct trades with volume equal to that bound for all $v \equiv 5 \pmod 6$, thus completing the problem.

Keywords: Trade; Triangle trade; Maximal trade; Triples

1 Introduction and Definitions

We begin with some necessary definitions. Using design theory terminology, a *block* is defined to be any subset chosen from some given set of size v. We call this v-set the *foundation*. A (k, t) trade of volume m and foundation size v is a pair $\{T_1, T_2\}$ of disjoint sets of blocks, with each of T_1 and T_2 containing m blocks of size k, based on a foundation set of size v, such that every t-set chosen from the foundation occurs equally often in the blocks of T_1 and in the blocks of T_2. Every element in the foundation set must occur in the trade.

Various classes of combinatorial trades (typically the trades with a particular t and k) have been investigated since the concept was first formally applied in design theory in the 1960s (see [2]); the underlying idea was also used earlier than this. The work on combinatorial trades is surveyed in [3] and [1].

The case $t = 2$ has received the greatest attention to date. This case can be easily represented in terms of graph theory; the foundation is a set of v vertices, blocks are complete graphs of size k on the foundation set, and the t-sets with $t = 2$ are edges.

The union of the blocks of T_1 is then a graph H on v vertices, which is also equal to the union of the blocks of T_2. Hence a $(k, 2)$ trade of volume m and foundation size v is equivalent to two disjoint decompositions of some graph H on v vertices into m copies of K_k.

A trade is *simple* if the sets T_1 and T_2 have no repeated blocks (that is, they are true sets rather than multisets). A *Steiner* trade has the further condition that each t-set may occur in at most one block of T_1 and at most one block of T_2. For a graphical trade (with $t = 2$) this is equivalent to saying that the graph H must be simple. If the trade is not Steiner then H will be a multigraph (with no loops).

In this paper we are concerned with simple non-Steiner $(3, 2)$ trades (K_3 or triangle trades in graphical terms), which we shall henceforth refer to as merely trades, denoted $\{T_1, T_2\}$.

A simple non-Steiner $(3, 2)$ trade of foundation size v may be viewed as a partition of the $\binom{v}{3}$ triples on a v-set into three sets T_1, T_2, T^c, such that every pair occurs the same number of times in triples of T_1 as in triples of T_2. We have $|T_1| = |T_2|$; this quantity is the volume of the trade. The set T^c is the leave, containing the 'unused' triples.

A trade of maximal volume on v vertices is a trade with volume greater than or equal to the volume of every other trade on v vertices. The size of T^c is thus minimised.

The volume of a trade of maximal volume on v vertices is denoted by $\text{vol}(T_M(v))$. In [4], this quantity is fully determined for all v except when v is congruent to 5 modulo 6, where only partial results were obtained. The results of that paper are summarised in the following theorem (we take $\mathbb{N}$ to be the set of positive integers, so zero is not included).

Theorem 1.1 *([4], Theorems 1,3,2) The maximal possible volume of a simple, non-Steiner* $(3, 2)$ *trade of foundation size* v*, or in some cases an upper bound for this volume, is given below for all* $v \in \mathbb{N}$*:*

Foundation size (v)	*Maximal trade volume*
$v \le 5$	$vol(T_M(v)) = 0$
$v \equiv 0 \ (mod\ 4)$	$vol(T_M(v)) = v(v+1)(v-4)/12$
$v \equiv 2 \ (mod\ 4)$	$vol(T_M(v)) = v(v-1)(v-2)/12$
$v = 7$	$vol(T_M(v)) = 12$
$v \equiv 1,3 \ (mod\ 6),\ v > 7$	$vol(T_M(v)) = v(v-1)(v-3)/12$
$v \in \{11, 17\}$	$vol(T_M(v)) = (v(v-1)(v-3)-16)/12$
$v \equiv 5 \ (mod\ 6),\ v > 17$	$vol(T_M(v)) \le (v(v-1)(v-3)-16)/12$

For the rest of this paper, we let $v = 6m + 5$, for $m \in \mathbb{N}$.

The upper bound on vol$(T_M(v))$, given in Theorem 1.1, can be written:

$$\text{vol}(T_M(v)) \le 18m^3 + 33m^2 + 19m + 2, \qquad (1)$$

with equality in the cases $m = 1$ and $m = 2$. We give trades, for every other positive integer m, with volume equal to the bound (1), thus showing equality in general. This completes the determination of vol$(T_M(v))$, $v \in \mathbb{N}$.

The construction will require the following additional definitions:

A *Steiner Triple System* (STS) of order m consists of a set of triples on a given m-set such that each pair of elements in the m-set occurs together in exactly one triple.

Let m be an integer and K be a set of integers. A *pairwise balanced design* PBD$(m; K; 1)$ consists of a family of blocks (subsets) on a given m-set such that the size of each block is contained in K, with the property that each pair of elements from the m-set is contained in exactly one block. An integer in the set K may be marked with an asterix to indicate that there is exactly one block of that size in the design.

A *group divisible design* (GDD) consists of a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$, where $\mathcal{V}$ is a set, $\mathcal{G}$ is a partition of $\mathcal{V}$ into *groups*, and $\mathcal{B}$ is a family of blocks (subsets) of $\mathcal{V}$ with the following property: every pair of elements from $\mathcal{V}$ occurs in exactly one block or one group, but not both. A k-GDD of type $s_1^{t_1} s_2^{t_2} ... s_r^{t_r}$ is a group divisible design with exactly t_i groups of size s_i, $1 \le i \le r$, and with all blocks of size k. Necessarily $|\mathcal{V}| = \sum_{i=1}^{r} s_i t_i$.

A *1-factor* is a graph consisting of v isolated edges on $2v$ vertices, for some $v \in \mathbb{N}$. It is well known that for all $v \in \mathbb{N}$, K_{2v} may be decomposed into $2v - 1$ 1-factors (this decomposition is known as a 1-factorisation). A 1-factor can also be considered as a set of v disjoint pairs on a $2v$-set, with each element of the $2v$-set occurring in exactly one of the pairs. A 1-factorisation is then a partition of the $\binom{2v}{2}$ pairs on the $2v$-set into $2v - 1$ of these 1-factors.

1.1 Notation

We partition the v-set, where $v = 6m + 5$, into one subset of size 5, labelled F, and m subsets of size 6, labelled G_x, $1 \le x \le m$.

We partition the $\binom{v}{3}$ triples on the v-set according to the subsets to which the three points of the triple belong. For example, $F^1G_3^2$ is the set of triples with one point in F and two points in G_3.

We partition the $\binom{v}{2}$ pairs on the v-set in a similar way, so for example G_1^2 is the set of pairs containing two points in G_1. In terms of this notation, we assume that point sets written separately are distinct; for instance in the triple set $F^1G_x^1G_y^1$, it is assumed that $x \ne y$. We will call these *basic* edge and triple sets.

All pairs are contained in the basic sets

$$F^2,\ F^1G_x^1,\ G_x^2,\ G_x^1G_y^1,$$

for $1 \le x \le m$ or $1 \le x < y \le m$; while all triples are contained in the sets

$$F^3,\ F^2G_x^1,\ F^1G_x^2,\ F^1G_x^1G_y^1,\ G_x^3,\ G_x^2G_y^1,\ G_x^1G_y^1G_z^1,$$

for $1 \le x \le m$, or $1 \le x < y \le m$, or $1 \le x < y < z \le m$.

1.2 Outline of construction

Our constructions are based on partitioning the $\binom{v}{3}$ triples on the v-set into subsets of standard sizes and structures. These subsets are equal to either one of the basic triple sets (for example $F^1G_2^2$), or to the union of several of them (for example $F^3 \cup F^2G_1^1 \cup F^1G_1^2 \cup G_1^3$).

We give subtrades of maximal volume based on each of these triple sets; these subtrades are trades which may use only the specified set of triples, rather than all triples on a given foundation set. It is not necessary to prove that these subtrades are of maximal volume, only that their combined volume is equal to the bound (1). This is achieved by choosing the subsets into which the triples are partitioned so that we can construct 'efficient' subtrades on them, with the fewest possible triples discarded in total (so we are minimising the size of T^c).

The constructions rely on the following well-known result:

Lemma 1.1 *Let $m \in \mathbb{N}$. Depending on the congruency class of m modulo 6, the following designs can be constructed on an m-set:*

(i) If $m \equiv 1,3 \pmod 6$ then there exists a Steiner triple system (STS) of order m.

(ii) If $m \equiv 5 \pmod 6$ then there exists a PBD$(m; \{3, 5^*\}; 1)$.

(iii) If $m \equiv 0,2 \pmod 6$ then there exists a 3-GDD of type $2^{m/2}$.

(iv) If $m \equiv 4 \pmod 6$ then there exists a 3-GDD of type $2^{m/2-2}4^1$.

Proof.

(i) See for example the Skolem and Bose constructions, Sections 1.3 and 1.2 of [6]. This was originally proved in [5].

(ii) See for example Section 1.4 of [6]. This was first proved in [8].

(iii) We can construct a 3-GDD of type $2^{m/2}$ by removing one element from a STS of order $m+1$ (which exists by (i)). Each block which contained the removed element is converted into a group.

(iv) We can construct a 3-GDD of type $2^{m/2-2}4^1$ by removing one of the elements which occurs in the block of size 5 in a PBD$(m+1; \{3, 5^*\}; 1)$ (which exists by (ii)). Each block which contained the removed element is converted into a group.

□

For any $m \in \mathbb{N}$, one of the four cases of Lemma 1.1 holds.

The constructions, corresponding to the four cases of Lemma 1.1, are given in the next section. Section 3 lists (or references) the subtrades used in these constructions, numbered from (1) to (10).

2 Constructions

Theorem 2.1 *For any $m \in \mathbb{N}$ with $m \equiv 1,3 \pmod 6$, there is a trade of volume $18m^3 + 33m^2 + 19m + 2$ and foundation size v, where $v = 6m + 5$.*

Proof. Take a STS of order m on the set $\{g_i \mid 1 \le i \le m\}$. Now 'blow up' each element by six, so each point g_i is replaced by the set $G_i = \{i_1, i_2, i_3, i_4, i_5, i_6\}$, $1 \le i \le m$. We add an extra set $F = \{f_1, f_2, f_3, f_4, f_5\}$, to give a foundation set of size $v = 6m + 5$.

We place a subtrade of type (1) (see Section 3; this has volume 72) on $F^3 \cup F^2G_1^1 \cup F^1G_1^2 \cup G_1^3$.

We place subtrades of types (2), (3), and (4) (volumes 30, 10, and 30) on $F^2G_i^1$, G_i^3, and $F^1G_i^2$ respectively, for $2 \le i \le m$.

For each triple $g_ig_jg_k$ in the STS, we place a subtrade of type (5) (volume 630) on $\left(\bigcup_{\{x,y\}\in\{i,j,k\},x<y} F^1G_x^1G_y^1 \cup G_x^2G_y^1 \cup G_x^1G_y^2\right) \cup G_i^1G_j^1G_k^1$ (there are $m(m-1)/6$ of these blocks). By definition each pair g_ig_j occurs in exactly one block of the STS, and hence each triple set $F^1G_i^1G_j^1$, $G_i^2G_j^1$, and $G_i^1G_j^2$ will be used in exactly one subtrade.

For each triple $g_ig_jg_k$ which is not a block of the STS, we place a subtrade of type (7) (volume 108) on $G_i^1G_j^1G_k^1$. There are $m(m-1)(m-2)/6 - m(m-1)/6 = m(m-1)(m-3)/6$ such triples.

This gives a trade of volume $72 + (30+10+30)(m-1) + 630(m(m-1)/6) + 108(m(m-1)(m-3)/6) = 18m^3 + 33m^2 + 19m + 2$, as required. □

Theorem 2.2 *For any $m \in \mathbb{N}$ with $m \equiv 5 \pmod 6$, there is a trade of volume $18m^3 + 33m^2 + 19m + 2$ and foundation size v, where $v = 6m + 5$.*

Proof. Take a PBD$(m; \{3, 5^*\}; 1)$ on the set $\{g_i \mid 1 \le i \le m\}$. Assume without loss of generality that the block of size five is $g_1g_2g_3g_4g_5$. Now 'blow up' each element by six, so each point g_i is replaced by the set $G_i = \{i_1, i_2, i_3, i_4, i_5, i_6\}$, $1 \le i \le m$.

We add an extra set $F = \{f_1, f_2, f_3, f_4, f_5\}$, to give a foundation set of size $v = 6m + 5$.

We place a subtrade of type (1) (volume 72) on $F^3 \cup F^2G_1^1 \cup F^1G_1^2 \cup G_1^3$.

We place subtrades of types (2), (3), and (4) (volumes 30, 10, and 30) on $F^2G_i^1$, G_i^3, and $F^1G_i^2$ respectively, for $2 \le i \le m$.

For each block $g_ig_jg_k$ of size three in the PBD$(m; \{3, 5^*\}; 1)$, we place a subtrade of type (5) (volume 630) on

$$\left(\bigcup_{\{x,y\}\in\{i,j,k\},x<y} F^1G_x^1G_y^1 \cup G_x^2G_y^1 \cup G_x^1G_y^2\right) \cup G_i^1G_j^1G_k^1$$

(there are $(m(m-1)/2 - 10)/3 = (m^2 - m - 20)/6$ of these blocks). For the block of size five, $g_1g_2g_3g_4g_5$, we place a subtrade of type (6) (volume 2820) on

$$\left(\bigcup_{1\le x<y\le 5} F^1G_x^1G_y^1 \cup G_x^2G_y^1 \cup G_x^1G_y^2\right) \cup \left(\bigcup_{1\le x<y<z\le 5} G_x^1G_y^1G_z^1\right).$$

For each triple $g_ig_jg_k$ which is not a block of the PBD$(m;\{3,5^*\};1)$, and is not a subset of the block of size five (so $\{i,j,k\} \not\subseteq \{1,2,3,4,5\}$), we place a subtrade of type (7) (volume 108) on $G_i^1G_j^1G_k^1$. There are $m(m-1)(m-2)/6-(m^2-m-20)/6-10=(m^3-4m^2+3m-40)/6$ such triples.

This gives a trade of volume $72+(30+10+30)(m-1)+630((m^2-m-20)/6)+2820+108((m^3-4m^2+3m-40)/6)=18m^3+33m^2+19m+2$, as required. □

Theorem 2.3 *For any $m \in \mathbb{N}$ with $m \equiv 0,2 \pmod 6$, there is a trade of volume $18m^3+33m^2+19m+2$ and foundation size v, where $v=6m+5$.*

Proof. Take a 3-GDD of type $2^{m/2}$ on the set $\{g_i \mid 1 \le i \le m\}$, with groups $\{g_1,g_2\}$, $\{g_3,g_4\}$,...,$\{g_{m-1},g_m\}$.

Now 'blow up' each element by six, so each point g_i is replaced by the set $G_i=\{i_1,i_2,i_3,i_4,i_5,i_6\}$, $1 \le i \le m$.

We add an extra set $F=\{f_1,f_2,f_3,f_4,f_5\}$, to give a foundation set of size $v=6m+5$.

We place a subtrade of type (8) (volume 316) on $(F\cup G_1\cup G_2)^3 = F^3\cup F^2G_1^1\cup F^2G_2^1\cup F^1G_1^2\cup F^1G_2^2\cup F^1G_1^1G_2^1\cup G_1^2G_2^1\cup G_1^1G_2^2\cup G_1^3\cup G_2^3$.

We place a subtrade of type (2) (volume 30) on $F^2G_i^1$, for $3 \le i \le m$.

We place a subtrade of type (9) (volume 254) on $F^1(G_{2i-1}\cup G_{2i})^2\cup(G_{2i-1}\cup G_{2i})^3 = F^1G_{2i-1}^2\cup F^1G_{2i}^2\cup F^1G_{2i-1}^1G_{2i}^1\cup G_{2i-1}^2G_{2i}^1\cup G_{2i-1}^1G_{2i}^2\cup G_{2i-1}i^3\cup G_{2i}^3$, for $2\le i\le m/2$.

The construction is now completed in the same way as the construction for the case m congruent to 1 or 3 (mod 6).

For each block $g_ig_jg_k$ in the GDD, we place a subtrade of type (5) (volume 630) on $\left(\bigcup_{\{x,y\}\in\{i,j,k\},x<y} F^1G_x^1G_y^1\cup G_x^2G_y^1\cup G_x^1G_y^2\right)\cup G_i^1G_j^1G_k^1$ (there are $m(m-2)/6$ of these blocks). By definition each pair g_ig_j occurs in either one group or one block of the GDD, hence each triple set $F^1G_i^1G_j^1$, $G_i^2G_j^1$, and $G_i^1G_j^2$ will be used in exactly one subtrade.

For each triple $g_ig_jg_k$ which is not a block of the GDD, we place a subtrade of type (7) (volume 108) on $G_i^1G_j^1G_k^1$. There are $m(m-1)(m-2)/6-m(m-2)/6=m(m-2)^2/6$ such triples.

This gives a trade of volume $316+30(m-2)+254(m-2)/2+630(m(m-2)/6)+108(m(m-2)^2/6)=18m^3+33m^2+19m+2$, as required. □

Theorem 2.4 *For any $m \in \mathbb{N}$ with $m \equiv 4 \pmod 6$, there is a trade of volume $18m^3 + 33m^2 + 19m + 2$ and foundation size v, where $v = 6m + 5$.*

Proof. Take a 3-GDD of type $2^{m/2-2}4^1$ on the set $\{g_i \mid 1 \leq i \leq m\}$, with groups $\{g_1, g_2, g_3, g_4\}$, $\{g_5, g_6\}$, $\{g_7, g_8\}, \ldots, \{g_{m-1}, g_m\}$. Now 'blow up' each element by six, so each point g_i is replaced by the set $G_i = \{i_1, i_2, i_3, i_4, i_5, i_6\}$, $1 \leq i \leq m$.

We add an extra set $F = \{f_1, f_2, f_3, f_4, f_5\}$, to give a foundation set of size $v = 6m + 5$.

We place a subtrade of type (10) (volume 1758) on $(F \cup G_1 \cup G_2 \cup G_3 \cup G_4)^3 = F^3 \cup \bigcup_{1 \leq x \leq 4} (F^2 G_x^1 \cup F^1 G_x^2 \cup G_x^3) \cup \bigcup_{1 \leq x < y \leq 4} (F^1 G_x^1 G_y^1 \cup G_x^2 G_y^1 \cup G_x^1 G_y^2) \cup \bigcup_{1 \leq x < y < z \leq 4} G_x^1 G_y^1 G_z^1$.

We place a subtrade of type (2) (volume 30) on $F^2 G_i^1$, for $5 \leq i \leq m$.

We place a subtrade of type (9) (volume 254) on $F^1(G_{2i-1} \cup G_{2i})^2 \cup (G_{2i-1} \cup G_{2i})^3 = F^1 G_{2i-1}^2 \cup F^1 G_{2i}^2 \cup F^1 G_{2i-1}^1 G_{2i}^1 \cup G_{2i-1}^2 G_{2i}^1 \cup G_{2i-1}^1 G_{2i}^2 \cup G_{2i-1}^3 \cup G_{2i}^3$, for $3 \leq i \leq m/2$.

The construction is now completed in the same way as the construction for the case m congruent to 1 or 3 (mod 6).

For each block $g_i g_j g_k$ in the GDD, we place a subtrade of type (5) (volume 630) on $\left(\bigcup_{\{x,y\} \in \{i,j,k\}, x<y} F^1 G_x^1 G_y^1 \cup G_x^2 G_y^1 \cup G_x^1 G_y^2\right) \cup G_i^1 G_j^1 G_k^1$ (there are $(m(m-1)/2 - 6 - (m-4)/2)/3 = (m^2 - 2m - 8)/6$ of these blocks). By definition each pair $g_i g_j$ occurs in either one group or one block of the GDD, and hence each triple set $F^1 G_i^1 G_j^1$, $G_i^2 G_j^1$, and $G_i^1 G_j^2$ will be used in exactly one subtrade.

For each triple $g_i g_j g_k$ which is not a block of the GDD, nor contained in the group of size four ($\{g_1, g_2, g_3, g_4\}$), we place a subtrade of type (7) (volume 108) on $G_i^1 G_j^1 G_k^1$. There are $m(m-1)(m-2)/6 - (m^2 - 2m - 8)/6 - 4 = (m^3 - 4m^2 + 4m - 16)/6$ such triples.

This gives a trade of volume $1758 + 30(m-4) + 254(m-4)/2 + 630(m^2 - 2m - 8)/6 + 108(m^3 - 4m^2 + 4m - 16)/6 = 18m^3 + 33m^2 + 19m + 2$, as required. □

3 Subtrades

For $m, n \in \mathbb{N}$, let $[m, n]$ be the set $\{k \mid k \in \mathbb{N},\ m \leq k \leq n\}$.

3.1 Subtrade type (1): volume 72, with 21 triples in T^c

A (maximal volume) trade of foundation 11 and volume 72 is given in [4].

Given a set F of size five and a set G of size six, we use this known trade as a subtrade of volume 72 on the triples $(F \cup G)^3 = F^3 \cup F^2G^1 \cup F^1G^2 \cup G^3$.

3.2 Subtrade type (2): volume 30, with no triples in T^c

Given a set $F = \{f_1, f_2, f_3, f_4, f_5\}$ of five points and a set $G = \{1, 2, 3, 4, 5, 6\}$ of six points, the following is a trade on the triples F^2G^1:

$$\{f_i f_j x \mid i - j \equiv 1 \pmod 5,\ x \in [1,3]\} \subseteq T_1$$
$$\{f_i f_j x \mid i - j \equiv 2 \pmod 5,\ x \in [4,6] \subseteq T_1$$

$$\{f_i f_j x \mid i - j \equiv 2 \pmod 5,\ x \in [1,3] \subseteq T_2$$
$$\{f_i f_j x \mid i - j \equiv 1 \pmod 5,\ x \in [4,6] \subseteq T_2$$

3.3 Subtrade type (3): volume 10, with no triples in T^c

Let $G = \{1, 2, 3, 4, 5, 6\}$ be a set of six points.

The following is a trade on the triples G^3:

$$\{123, 124, 345, 346, 561, 562, 135, 146, 236, 245\} \subseteq T_1,$$

$$\{125, 126, 341, 342, 563, 564, 136, 145, 235, 246\} \subseteq T_2.$$

This trade of volume ten is the union of two familiar trades of volumes six and four (the Pasch trade), on blocks of size three.

3.4 Subtrade type (4): volume 30, with 15 triples in T^c

Let $F = \{f_1, f_2, f_3, f_4, f_5\}$ be a set of five points and $G = \{1, 2, 3, 4, 5, 6\}$ be a set of six points. Partition the $\binom{6}{2}$ pairs in G^2 into five one-factors O_1, O_2, O_3, O_4, O_5, each of which contains three pairs.

The following is a trade on the triples F^1G^2:

$$\{f_ixy \mid xy \in O_j,\ i-j \equiv \pm 1 \pmod 5,\ i,j \in [1,5]\} \subseteq T_1,$$

$$\{f_ixy \mid xy \in O_j,\ i-j \equiv \pm 2 \pmod 5,\ i,j \in [1,5]\} \subseteq T_2,$$

$$\{f_ixy \mid xy \in O_i,\ i \in [1,5]\} \subseteq T^c.$$

3.5 Subtrade type (5): volume 630, with 36 triples in T^c

Let $F = \{f_1, f_2, f_3, f_4, f_5\}$ be a set of five points and G_1, G_2, and G_3, where $G_x = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, be three sets of six points.

The following is a trade on the triples

$$\left(\bigcup_{1\le x<y\le 3} F^1G_x^1G_y^1 \cup G_x^2G_y^1 \cup G_x^1G_y^2\right) \cup G_1^1G_2^1G_3^1 :$$

Taking subscripts modulo 6, let

$\{f_i1_j2_k, f_i1_j3_k, f_i2_j3_k \mid j,k \in [1,6],$ $j+k$ odd, $i \in \{1,2,3\}\} \subseteq T_1,$	(162 triples)
$\{f_i1_j2_k, f_i1_j3_k, f_i2_j3_k \mid j,k \in [1,6],$ $j+k$ even, $i \in \{4,5\}\} \subseteq T_1,$	(108 triples)
$\{1_i2_j3_{i+j+1} \mid i,j \in [1,6]\} \subseteq T_1,$	(36 triples)
$\{1_i2_j3_{i+j+2} \mid i,j \in [1,6]\} \subseteq T_1,$	(36 triples)
$\{1_i2_j3_{i+j+3} \mid i,j \in [1,6],$ i and j odd$\} \subseteq T_1,$	(9 triples)
$\{1_i2_j3_{i+j+4} \mid i,j \in [1,6],$ i and j even$\} \subseteq T_1,$	(9 triples)
$G_1^2G_2^1, G_2^2G_3^1, G_3^2G_1^1 \subseteq T_1,$	(270 triples)
$\{f_i1_j2_k, f_i1_j3_k, f_i2_j3_k \mid j,k \in [1,6],$ $j+k$ odd, $i \in \{4,5\}\} \subseteq T_2,$	(108 triples)
$\{f_i1_j2_k, f_i1_j3_k, f_i2_j3_k \mid j,k \in [1,6],$ $j+k$ even, $i \in \{1,2,3\}\} \subseteq T_2,$	(162 triples)
$\{1_i2_j3_{i+j+3} \mid i,j \in [1,6],$ i or j even$\} \subseteq T_2,$	(27 triples)
$\{1_i2_j3_{i+j+4} \mid i,j \in [1,6],$ i or j odd$\} \subseteq T_2,$	(27 triples)
$\{1_i2_j3_{i+j+5} \mid i,j \in [1,6]\} \subseteq T_2,$	(36 triples)
$G_1^2G_3^1, G_2^2G_1^1, G_3^2G_2^1 \subseteq T_2,$	(270 triples)
$\{1_i2_j3_{i+j} \mid i,j \in [1,6]\} \subseteq T^c.$	(36 triples)

3.6 Subtrade type (6): volume 2820, with 120 triples in T^c

Let $F = \{f_1, f_2, f_3, f_4, f_5\}$ be a set of five points and G_1, G_2, G_3, G_4, and G_5, where $G_x = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, be five sets of six points.

We give a trade on the triples

$$\left(\bigcup_{1\le x<y\le 5} F^1G_x^1G_y^1 \cup G_x^2G_y^1 \cup G_x^1G_y^2\right) \cup \left(\bigcup_{1\le x<y<z\le 5} G_x^1G_y^1G_z^1\right).$$

To begin with we give three disjoint subsets of the triple set $\bigcup_{1\le x<y<z\le 5} G_x^1G_y^1G_z^1$, each of which forms a decomposition of the pair set $\bigcup_{1\le x<y\le 5} G_x^1G_y^1$:

Taking sums modulo 5,

$$\begin{aligned} D_1 &= \{a_j(a+1)_i(a+2)_k, a_i(a+2)_j(a+3)_k \mid a \in [1,5], \\ &\quad i,j,k \in [1,6],\ j \equiv k \equiv i+1 \pmod 3),\ i+j+k \text{ odd}\}, \end{aligned}$$

$$\begin{aligned} D_2 &= \{a_j(a+1)_i(a+2)_k, a_i(a+2)_j(a+3)_k \mid a \in [1,5], \\ &\quad i,j,k \in [1,6],\ j \equiv k \equiv i+1 \pmod 3),\ i+j+k \text{ even}\}, \end{aligned}$$

$$\begin{aligned} D_3 &= \{a_j(a+1)_i(a+2)_k, a_i(a+2)_j(a+3)_k \mid a \in [1,5], \\ &\quad i,j,k \in [1,6],\ j \equiv k \equiv i-1 \pmod 3),\ i+j+k \text{ odd}\}. \end{aligned}$$

Taking subscripts modulo 6, let

$\{f_i x_j y_k \mid 1 \le x < y \le 5,\ j,k \in [1,6],$
$j+k$ odd, $i \in \{1,2,3\}\} \subseteq T_1,$ (540 triples)
$\{f_i x_j y_k \mid 1 \le x < y \le 5,\ j,k \in [1,6],$
$j+k$ even, $i \in \{4,5\}\} \subseteq T_1,$ (360 triples)
$\{x_i y_j z_k \in D_1 \mid i,j,k \in \{1,3,5\}\} \subseteq T_1,$ (30 triples)
$\{x_i y_j z_k \in D_2 \mid i,j,k \in \{2,4,6\}\} \subseteq T_1,$ (30 triples)
$\{x_i y_j z_k \mid 1 \le x < y < z \le 5,\ i,j,k \in [1,6],$
$i+j+k$ even$\} \setminus D_2 \subseteq T_1,$ (960 triples)
$\{x_i x_j y_k \mid\ x,y \in [1,5],\ y-x \equiv 1,2 \pmod 5),$
$i,j,k \in [1,6],\ i<j\} \subseteq T_1,$ (900 triples)

$\{f_i x_j y_k \mid 1 \le x < y \le 5,\ j,k \in [1,6],$
$j+k$ odd, $i \in \{4,5\}\} \subseteq T_2,$ (360 triples)
$\{f_i x_j y_k \mid 1 \le x < y \le 5,\ j,k \in [1,6],$
$j+k$ even, $i \in \{1,2,3\}\} \subseteq T_2,$ (540 triples)
$\{x_i y_j z_k \in D_1 \mid i,j,k$ not all odd$\} \subseteq T_2,$ (90 triples)
$\{x_i y_j z_k \in D_2 \mid i,j,k$ not all even$\} \subseteq T_2,$ (90 triples)
$\{x_i y_j z_k \mid 1 \le x < y < z \le 5,\ i,j,k \in [1,6],$
$i+j+k$ odd$\} \setminus \{D_1 \cup D_3\} \subseteq T_2,$ (840 triples)
$\{x_i x_j y_k \mid\ x,y \in [1,5],\ y-x \equiv 3,4 \pmod 5),$
$i,j,k \in [1,6],\ i<j\} \subseteq T_2,$ (900 triples)

$D_3 \subseteq T^c.$ (120 triples)

3.7 Subtrade type (7): volume 108, with no triples in T^c

Let $G_x = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $1 \le x \le 3$ be three sets of six points.

The following is a subtrade on the triples $G_1^1 G_2^1 G_3^1$:

$$\{1_i 2_j 3_k \mid i,j,k \in [1,6], i+j+k \text{ even } \} \subseteq T_1,$$

$$\{1_i 2_j 3_k \mid i,j,k \in [1,6], i+j+k \text{ odd } \} \subseteq T_2.$$

3.8 Subtrade type (8): volume 316, with 48 triples in T^c

A (maximal volume) (3, 2) trade of volume 316 and foundation size 17 is given on page 223 of [4].

Given a set F of size five and sets G_1 and G_2 each of size six, we can place a copy of this known trade on the foundation set $F \cup G_1 \cup G_2$. Thus this trade can be considered as a subtrade of volume 316 on the triples $(F \cup G_1 \cup G_2)^3$.

3.9 Subtrade type (9): volume 254, with 42 triples in T^c

Let $F = \{f_1, f_2, f_3, f_4, f_5\}$ be a set of five points and G_1 and G_2, where $G_x = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, be two sets of six points.

We give a trade on the triples $F^1(G_1 \cup G_2)^2 \cup (G_1 \cup G_2)^3 = F^1G_1^2 \cup F^1G_2^2 \cup F^1G_1^1G_2^1 \cup G_1^2G_2^1 \cup G_1^1G_2^2 \cup G_1^3 \cup G_2^3$.

Since the triple set used in this subtrade may be written as $F^1(G_1 \cup G_2)^2 \cup (G_1 \cup G_2)^3$, the sets G_1 and G_2 can be considered as a single combined set of twelve points, for the purposes of this subtrade only (we still need to consider these points as two groups of six for the purposes of other subtrades and the overall trade structure).

In fact, we rearrange the twelve points of $G_1 \cup G_2$ into three groups of four points, labelled P_1, P_2, and P_3 — again just for this subtrade construction. We temporarily relabel the points so that $P_x = \{x_1, x_2, x_3, x_4\}$, for $x = 1, 2, 3$. Note that we are reusing the notation used elsewhere for the points of the sets of size six (that is, the sets G_x, $x \in \mathbb{N}$). The new usage applies to this subtrade construction only.

In this new arrangement, our subtrade uses the triples

$$F^1(P_1 \cup P_2 \cup P_3)^2 \cup (P_1 \cup P_2 \cup P_3)^3$$
$$= \left(\bigcup_{1 \le x \le 3} F^1P_x^2 \cup P_x^3\right) \cup \left(\bigcup_{1 \le x < y \le 3} F^1P_x^1P_y^1 \cup P_x^2P_y^1 \cup P_x^1P_y^2\right) \cup P_1^1P_2^1P_3^1.$$

We use the following five 1-factors on the twelve points $P_1 \cup P_2 \cup P_3$:

$$\begin{aligned}
O_1 &= \{1_11_2, 1_31_4, 2_13_2, 2_23_1, 2_33_4, 2_43_3\},\\
O_2 &= \{2_12_2, 2_32_4, 3_11_2, 3_21_1, 3_31_4, 3_41_3\},\\
O_3 &= \{3_13_2, 3_33_4, 1_12_2, 1_22_1, 1_32_4, 1_42_3\},\\
O_4 &= \{1_11_3, 1_21_4, 2_12_3, 2_22_4, 3_13_3, 3_23_4\},\\
O_5 &= \{1_11_4, 1_21_3, 2_12_4, 2_22_3, 3_13_4, 3_23_3\}.
\end{aligned}$$

The remaining pairs in $(P_1 \cup P_2 \cup P_3)^2$ (those not in one of the above 1-factors) are partitioned into the following two sets:

$$E_1 = \{p_1q_3, p_2q_2, p_3q_1, p_3q_2, p_4q_1, p_4q_4 \mid (p,q) \in \{(1,2),(2,3),(3,1)\}\},$$
$$E_2 = \{p_1q_1, p_1q_4, p_2q_3, p_2q_4, p_3q_3, p_4q_2 \mid (p,q) \in \{(1,2),(2,3),(3,1)\}\}.$$

Every point in $P_1 \cup P_2 \cup P_3$ is in exactly three pairs from E_1 and three pairs from E_2.

The subtrade is as follows, taking $p+1$ and $p-1$ modulo three:

$\{f_ip_jq_k \mid p_jq_k \in O_x, i-x \equiv 1,2 \pmod 5,$
$x,i \in [1,5]\} \subseteq T_1,$ (60 triples)
$\{f_ip_jq_k \mid i \in \{1,2,3\},\ p_jq_k \in E_1\} \subseteq T_1,$ (54 triples)
$\{f_ip_jq_k \mid i \in \{4,5\},\ p_jq_k \in E_2\} \subseteq T_1,$ (36 triples)
$\{1_i2_j3_k \mid i,j,k \in [1,4],\ i+j+k \text{ odd}\}$
$\setminus\{1_12_13_3, 1_12_33_1, 1_32_13_1, 1_42_43_1, 1_42_13_4, 1_12_43_4\} \subseteq T_1,$ (26 triples)
$\{p_ip_j(p+1)_k \mid p \in [1,3],\ i,j,k \in [1,4]\}$
$\setminus\{p_3p_2(p+1)_2 \mid p \in [1,3]\} \subseteq T_1,$ (69 triples)
$\{p_1p_4(p-1)_1 \mid p \in [1,3]\} \subseteq T_1,$ (3 triples)
$\{p_1p_2p_3, p_2p_3p_4 \mid p \in [1,3]\} \subseteq T_1,$ (6 triples)

$\{f_ip_jq_k \mid p_jq_k \in O_x, i-x \equiv 3,4 \bmod 5,$
$x,i \in [1,5]\} \subseteq T_2,$ (60 triples)
$\{f_ip_jq_k \mid i \in \{4,5\},\ p_jq_k \in E_1\} \subseteq T_2,$ (36 triples)
$\{f_ip_jq_k \mid i \in \{1,2,3\},\ p_jq_k \in E_2\} \subseteq T_2,$ (54 triples)
$\{1_i2_j3_k \mid i,j,k \in [1,4],\ i+j+k \text{ even}\}$
$\setminus \{1_22_23_4, 1_22_43_2, 1_42_23_2, 1_32_33_2, 1_32_23_3, 1_22_33_3\} \subseteq T_2,$ (26 triples)
$\{p_ip_j(p-1)_k \mid p \in [1,3],\ i,j,k \in [1,4]\}$
$\setminus\{p_1p_4(p-1)_1 \mid p \in [1,3]\} \subseteq T_2,$ (69 triples)
$\{p_3p_2(p+1)_2 \mid p \in [1,3]\} \subseteq T_2,$ (3 triples)
$\{p_1p_2p_4, p_1p_3p_4 \mid p \in [1,3]\} \subseteq T_2,$ (6 triples)

$\{f_xp_jq_k \mid p_jq_k \in O_x, x \in [1,5]\} \subseteq T^c,$ (30 triples)
$\{1_12_13_3, 1_12_33_1, 1_32_13_1, 1_42_43_1, 1_42_13_4, 1_12_43_4\} \subseteq T^c,$ (6 triples)
$\{1_22_23_4, 1_22_43_2, 1_42_23_2, 1_32_33_2, 1_32_23_3, 1_22_33_3\} \subseteq T^c.$ (6 triples)

3.10 Subtrade type (10): volume 1758, with 138 triples in T^c

Let $F = \{f_1, f_2, f_3, f_4, f_5\}$ be a set of five points and G_1, G_2, G_3, and G_4, where $G_x = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, be four sets of six points.

The following is a subtrade on the triples $(F \cup G_1 \cup G_2 \cup G_3 \cup G_4)^3 = F^3 \cup \bigcup_{1\le x\le 4}(F^2G_x^1 \cup F^1G_x^2 \cup G_x^3) \cup \bigcup_{1\le x<y\le 4}(F^1G_x^1G_y^1 \cup G_x^2G_y^1 \cup G_x^1G_y^2) \cup \bigcup_{1\le x<y<z\le 4} G_x^1G_y^1G_z^1$

This subtrade contains a few of the smaller subtrades given above — this was done to simplify the trade construction in the case $m \equiv 4 \pmod 6$ — but most of the subtrade is new.

We use a subtrade of type (1) (volume 72 with 21 triples in T^c) on the triples $(F \cup G_1)^3 = F^3 \cup F^2G_1^1 \cup F^1G_1^2 \cup G_1^3$.

We use subtrades of type (2) (volume 30 with 0 triples in T^c) on the triples $F^2G_x^1$, for $x = 2, 3, 4$.

We use subtrades of type (4) (volume 10 with 0 triples in T^c) on the triples G_x^3, for $x = 2, 3, 4$.

We use subtrades of type (5) (volume 30 with 15 triples in T^c) on the triples $F^1G_x^2$, for $x = 2, 3, 4$.

The remainder of the subtrade has volume 1476, with 72 triples placed in T^c, using the triples

$$\left(\bigcup_{1\le x<y\le 4} F^1G_x^1G_y^1 \cup G_x^2G_y^1 \cup G_x^1G_y^2\right) \cup \left(\bigcup_{1\le x<y<z\le 4} G_x^1G_y^1G_z^1\right).$$

We require the following subset of $\bigcup_{1\le x<y<z\le 4} G_x^1G_y^1G_z^1$, which forms a decomposition of the pair set $\bigcup_{1\le x<y\le 4} G_x^1G_y^1$:

$$\begin{aligned} D &= \{1_x2_y3_z, 1_x2_w4_y, 1_w3_x4_y, 2_x3_w4_y \mid x, y, z \in [1,6], \\ &\quad x \equiv y \equiv z \equiv w+1 \pmod 2,\ x+y+z \equiv x+y+w \equiv 0 \pmod 3\}. \end{aligned}$$

The triple set D is an example of a 3-GDD of type 6^4, which are known to exist (see Theorem 1.22 of [7]). However the particular structure of D is important; of the 72 triples in D, the sum of the subscripts is even for 36 triples and odd for 36 triples. Although all 72 triples are discarded (placed in T^c), the absence of the odd-sum triples from T_1 and the even-sum triples from T_2 causes an imbalance which must be corrected through the disposition of the other triples.

We also need to define some pair sets before giving the subtrade. The 15 pairs on a 6-set can be partitioned into five 1-factors. For $x = 1, 2, 3, 4$, let p_x be the union of three of the 1-factors on G_x, and let q_x be the union of the remaining two 1-factors. Then p_x and q_x partition G_x^2, with the property that every point of G_x occurs in exactly three pairs from p_x and two pairs from q_x.

Finally we give a partition of the pair set $\bigcup_{1 \le x < y \le 4} G_x^1 G_y^1$ into two subsets of size 108:

$$\begin{aligned} A &= \{x_i y_j \mid x, y \in [1,3],\ y \equiv x+1 \pmod 3,\ i,j \in [1,6],\ j \text{ odd}\} \\ &\quad \cup \{4_i x_j \mid x \in [1,3],\ i,j \in [1,6],\ j \text{ even}\}, \end{aligned}$$

$$\begin{aligned} B &= \{x_i y_j \mid x, y \in [1,3],\ y \equiv x+1 \pmod 3,\ i,j \in [1,6],\ j \text{ even}\} \\ &\quad \cup \{4_i x_j \mid x \in [1,3],\ i,j \in [1,6],\ j \text{ odd}\}. \end{aligned}$$

Note that every point in $G_1 \cup G_2 \cup G_3 \cup G_4$ occurs in equal numbers of pairs from A and B (nine of each).

The subtrade is then:

$\{f_i x_j y_k \mid x_j y_k \in A,\ i \in \{1,2,3\}\} \subseteq T_1,$	(324 triples)
$\{f_i x_j y_k \mid x_j y_k \in B,\ i \in \{4,5\}\} \subseteq T_1,$	(216 triples)
$\{x_i y_j z_k \mid 1 \le x < y < z \le 4,\ i,j,k \in [1,6],$ $i+j+k \text{ odd}\} \setminus D \subseteq T_1,$	(396 triples)
$G_1^2 G_2^1, G_2^2 G_3^1, G_3^2 G_1^1 \subseteq T_1,$	(270 triples)
$\{x_i x_j 4_k \mid x \in [1,3],\ x_i x_j \in p_x,\ k \in \{2,4,6\}\} \subseteq T_1,$	(81 triples)
$\{x_i x_j 4_k \mid x \in [1,3],\ x_i x_j \in q_x,\ k \in \{1,3,5\}\} \subseteq T_1,$	(54 triples)
$\{4_i 4_j x_k \mid x \in [1,3],\ 4_i 4_j \in p_4,\ k \in \{1,3,5\}\} \subseteq T_1,$	(81 triples)
$\{4_i 4_j x_k \mid x \in [1,3],\ 4_i 4_j \in q_4,\ k \in \{2,4,6\}\} \subseteq T_1,$	(54 triples)
$\{f_i x_j y_k \mid x_j y_k \in A,\ i \in \{4,5\}\} \subseteq T_2,$	(216 triples)
$\{f_i x_j y_k \mid x_j y_k \in B,\ i \in \{1,2,3\}\} \subseteq T_2,$	(324 triples)
$\{x_i y_j z_k \mid 1 \le x < y < z \le 4,\ i,j,k \in [1,6],$ $i+j+k \text{ even}\} \setminus D \subseteq T_2,$	(396 triples)
$G_1^1 G_2^2, G_2^1 G_3^2, G_3^1 G_1^2 \subseteq T_2,$	(270 triples)
$\{x_i x_j 4_k \mid x \in [1,3],\ x_i x_j \in p_x,\ k \in \{1,3,5\}\} \subseteq T_2,$	(81 triples)
$\{x_i x_j 4_k \mid x \in [1,3],\ x_i x_j \in q_x,\ k \in \{2,4,6\}\} \subseteq T_2,$	(54 triples)
$\{4_i 4_j x_k \mid x \in [1,3],\ 4_i 4_j \in p_4,\ k \in \{2,4,6\}\} \subseteq T_2,$	(81 triples)
$\{4_i 4_j x_k \mid x \in [1,3],\ 4_i 4_j \in q_4,\ k \in \{1,3,5\}\} \subseteq T_2,$	(54 triples)
$D \subseteq T^c.$	(72 triples)

4 Results

The combined results of [4] and this paper are summarised in the following theorem:

Theorem 4.1 *The maximal possible volume of a simple, non-Steiner* (3, 2) *trade of foundation size v is given below for all* $v \in \mathbb{N}$*:*

Foundation size (v)	*Maximal trade volume*
$v \leq 5$	$vol(T_M(v)) = 0$
$v \equiv 0 \pmod 4$	$vol(T_M(v)) = v(v+1)(v-4)/12$
$v \equiv 2 \pmod 4$	$vol(T_M(v)) = v(v-1)(v-2)/12$
$v = 7$	$vol(T_M(v)) = 12$
$v \equiv 1, 3 \pmod 6$, $v > 7$	$vol(T_M(v)) = v(v-1)(v-3)/12$
$v \equiv 5 \pmod 6$, $v > 5$	$vol(T_M(v)) = (v(v-1)(v-3) - 16)/12$

Proof. Follows from Theorem 1.1 (from [4]), and from Theorems 2.1, 2.2, 2.3, and 2.4. □

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References

[1] E.J. Billington, *Combinatorial trades: a survey of recent results*, Chapter 3 in: Designs 2002: Further combinatorial and constructive design theory (ed. W.D. Wallis), Kluwer Academic Press, Norwell, Mass., USA. (to appear).

[2] A.S. Hedayat, *The theory of trade-off for t-designs*, Coding Theory and Design Theory, Part II, IMA Vol. Math. Appl., **21**, 101–126, Springer, New York, 1990.

[3] G.B. Khosrovshahi, H.R. Maimani and R. Torabi, *On trades: an update*, Discrete Applied Mathematics **95** (1999), 361–376.

[4] G.B. Khosrovshahi and R. Torabi, *Maximal Trades*, Ars Combinatoria **51** (1999), 211–223.

[5] T.P. Kirkman, *On a problem in combinations*, Cambridge and Dublin Math. Journal **2** (1847), 191–204.

[6] C.C. Lindner and C.A. Rodger, *Design Theory*, CRC Press, Florida, Boca Raton, 1997.

[7] R.C. Mullin and H.-D.O.F. Gronau, *PBDs and GDDs: The Basics*, Section III.1 in: CRC Handbook of Combinatorial Designs (eds. C. J. Colbourn and J. H. Dinitz), CRC Press, Boca Raton, 1996, pp. 185–193.

[8] R.M. Wilson, *Some partitions of all triples into Steiner triple systems*, in Hypergraph seminar (Ohio State University 1972), Lecture Notes in Mathematics **411**, 267–277 (Springer, Berlin, Heidelberg, New York 1974).