Stratification and Domination in Prisms

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Abstract

A graph G is 2-stratified if its vertex set is partitioned into two classes (each of which is a stratum or a color class). We color the vertices in one color class red and the other color class blue. Let X be a 2-stratified graph with one fixed blue vertex v specified. We say that X is rooted at v. The X-domination number of a graph G is the minimum number of red vertices of G in a red-blue coloring of the vertices of G such that every blue vertex v of G belongs to a copy of X rooted at v. In this paper we investigate the X-domination number of prisms when X is a 2-stratified 4-cycle rooted at a blue vertex.

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1 Introduction

In this paper we continue the study of stratification and domination in graphs started by Chartrand et al. [4] and studied further in [3] and elsewhere. A graph G whose vertex set has been partitioned into two sets V_1 and V_2 is called a *2-stratified graph*. The sets V_1 and V_2 are called the *strata* or sometimes the *color classes* of G. We ordinarily color the vertices of V_1 red and the vertices of V_2 blue.

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In [12], Rashidi studied a number of problems involving stratified graphs; while distance in stratified graphs was investigated in [1, 2, 5].

A set $S \subseteq V(G)$ of a graph G is a *dominating set* if every vertex not in S is adjacent to a vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of G of cardinality $\gamma(G)$ is called a $\gamma(G)$-*set*. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The book by Chartrand and Lesniak [6] includes a chapter on domination. For a more thorough study of domination in graphs, see Haynes, Hedetniemi, and Slater [10, 11].

In [4] a new mathematical framework for studying domination is presented. It is shown that the domination number and many domination related parameters can be interpreted as restricted 2-stratifications or 2-colorings, with the red vertices forming the dominating set. This framework places the domination number in a new perspective and suggests many other parameters of a graph which are related in some way to the domination number.

More precisely, let X be a 2-stratified graph with one fixed blue vertex v specified. We say that X is *rooted* at the blue vertex v. An X-*coloring* of a graph G is defined in [4] to be a red-blue coloring of the vertices of G such that every blue vertex v of G belongs to a copy of X (not necessarily induced in G) rooted at v. The X-*domination number* $\gamma_X(G)$ of G is the minimum number of red vertices of G in an X-coloring of G. In [4], an X-coloring of G that colors $\gamma_X(G)$ vertices red is called a γ_X-*coloring* of G. The set of red vertices in a γ_X-coloring is called a γ_X-set. If G has order n and G has no copy of X, then certainly $\gamma_X(G) = n$.

A *prism* is the cartesian product $G = C_n \times K_2$, $n \geq 3$, of a cycle C_n and a K_2. Throughout this paper, our prism G consists of two n-cycles $v_1, v_2, \ldots, v_n, v_1$ and $u_1, u_2, \ldots, u_n, u_1$ with u_iv_i an edge for all $i = 1, 2, \ldots, n$. Our aim is to determine the X-domination number of a prism when X is a 2-stratified cycle C_4.

For notation and graph theory terminology we in general follow [10]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. The order of G is $n = |V|$ and its size is $m = |E|$. Let v be a vertex in V. The *open neighborhood* of v is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. For a set S of vertices, the *open neighborhood* of S is defined by $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S by $N[S] = N(S) \cup S$. A vertex $w \in V$ is a *private neighbor of* v (with respect to S) if $N[w] \cap S = \{v\}$; and the *private neighbor set of* *v with respect to S*, denoted pn(v, S), is the set of all private neighbors of v.

We denote the subgraph of G induced by S by $G[S]$. The minimum degree (resp., maximum degree) among the vertices of G is denoted by $\delta(G)$ (resp., $\Delta(G)$). A *cycle* on n vertices is denoted by C_n and a *path* on n vertices by P_n.

2 Known Results

2.1 A 2-stratified P_2

If X is a K_2 rooted at a blue vertex v that is adjacent to a red vertex, then it is shown in [4] that $\gamma_X(G) = \gamma(G)$. Thus domination can be interpreted as restricted 2-stratifications or 2-colorings, with the red vertices forming the dominating set. Clearly, this X-coloring is the only well-defined one for connected graphs X with order 2.

2.2 A 2-stratified P_3

Let F be a 2-stratified P_3 rooted at a blue vertex v. The five possible choices for the graph F are shown in Figure 1. (The red vertices in Figure 1 are darkened.)

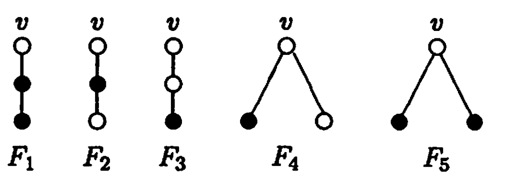

Figure 1:

The following result is established in [4].

Theorem 1 ([4]) *If G is a connected graph of order at least 3, then for $i \in \{1, 2, 4, 5\}$, the parameter $\gamma_{F_i}(G)$ is given by the following table:*

i	1	2	4	5
$\gamma_{F_i}(G) =$	$\gamma_t(G)$	$\gamma(G)$	$\gamma_r(G)$	$\gamma_2(G)$

where $\gamma_t(G)$ *denotes the total domination number* (see [10]), $\gamma_r(G)$ *denotes the restrained domination number* (see [7, 10]), *and* $\gamma_2(G)$ *denotes the 2-domination number* (see [8, 10]).

2.3 A 2-stratified K_3

The two 2-stratified graphs K_3 rooted at a blue vertex v are shown in Figure 2, where the red vertices are indicated by darkened vertices.

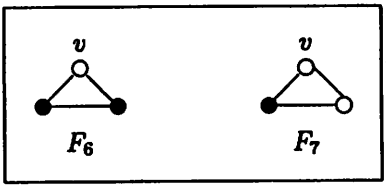

Figure 2:

Obviously, in any F_6-coloring and F_7-coloring of G, every vertex not on a triangle of G must be colored red. F_6-coloring and F_7-coloring of graphs are studied in [4] and [?].

3 A 2-stratified C_4

Let X be a 2-stratified C_4 rooted at a blue vertex v. The five possible choices for the graph X are shown in Figure 3. (The red vertices in Figure 3 are darkened.)

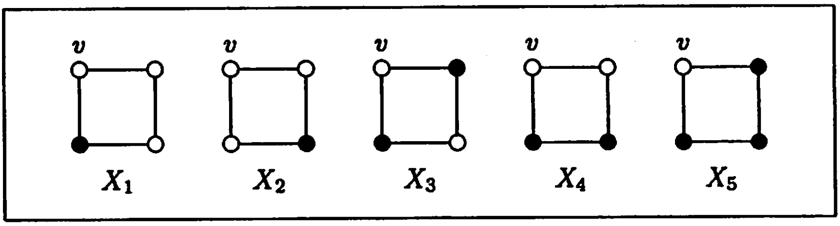

Figure 3:

4 Stratification in Prisms

In this section, we determine the X-domination number of a prism when X is a 2-stratified cycle C_4. We shall prove:

Theorem 2 *For $n \geq 3$, let $G = C_n \times K_2$. Then for $i \in \{1,2,3,4,5\}$, the parameter $\gamma_{X_i}(G)$ is given by the following table:*

i	$\gamma_{X_i}(G)$
1	$\lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$
2	$\begin{cases} 2 & \text{if } n = 4 \\ 2n & \text{otherwise} \end{cases}$
3	n
4	$2\left\lceil \frac{n}{3} \right\rceil$
5	$\left\lceil \frac{4n}{3} \right\rceil$

Throughout Section 4, we let $G = C_n \times K_2$. The proof of Theorem 2 follows from Propositions 3, 4, 5, 6 and 7. We have selected two of the more interesting and informative proofs in this section and simply stated the remaining results without proof.

Proposition 3 *For $n \geq 3$, $\gamma_{X_1}(G) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.*

Proof. The desired result follows from Claims 1 and 2.

Claim 1 $\gamma_{X_1}(G) \geq \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Proof. In any X_1-coloring of a graph, every vertex colored blue is rooted at a copy of X_1. Hence as an immediate consequence of the definition of an X_1-coloring, any X_1-coloring of G colors at least one vertex from every 4-cycle red.

Suppose n is odd. Consider any given X_1-coloring of G. Renaming vertices if necessary, we may assume v_1 is colored red. Since $G - \{u_1, v_1\}$ contains $(n-1)/2$ disjoint 4-cycles, each of which contains at least one red vertex, our given X_1-coloring contains at least $(n+1)/2$ red vertices. Thus, $\gamma_{X_1}(G) \geq (n+1)/2$.

Suppose n is even. Then, G has $n/2$ disjoint 4-cycles, and therefore has at least $n/2$ red vertices. Thus, $\gamma_{X_1}(G) \geq n/2$. Further, suppose $n \equiv 2 \pmod 4$ and that exactly $n/2$ vertices are colored red. Then, every 4-cycle in G contains exactly one red vertex. In particular, v_1 is the only red vertex in the 4-cycle v_1, u_1, u_2, v_2, v_1. Since u_2 is rooted in a copy of X_1, the vertex u_3 is colored red, and so u_3 is the only red vertex in the 4-cycle u_3, v_3, v_4, u_4, u_3. Since v_4 is rooted in a copy of X_1, the vertex v_5 is colored red, and so v_5 is the only red vertex in the 4-cycle v_5, u_5, u_6, v_6, v_5. Proceeding in this manner, v_{n-1} is the only red vertex in the 4-cycle $v_{n-1}, u_{n-1}, u_n, v_n, v_{n-1}$. But then u_n is not rooted at a copy of X_1, a contradiction. Hence, if $n \equiv 2 \pmod 4$, then at least $n/2+1$ vertices are colored red. □

Claim 2 $\gamma_{X_1}(G) \leq \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Proof. If $n = 3$, then $\{v_1, u_3\}$ is an X_1-coloring of G, and the desired upper bound follows. Hence we may assume $n \geq 4$. Suppose first that $n \not\equiv 2 \pmod 4$. Let

$$S = \bigcup_{i=0}^{\lfloor n/4 \rfloor - 1} \{v_{4i+1}, u_{4i+3}\}.$$

If $n \equiv 0 \pmod 4$, let $D = S$. If $n \equiv 1 \pmod 4$, let $D = S \cup \{v_n\}$. If $n \equiv 3 \pmod 4$, let $D = S \cup \{u_n, v_{n-2}\}$. In all cases, coloring the vertices in D red and coloring all remaining vertices blue, produces an X_1-coloring of G, and so $\gamma_{X_1}(G) \leq |D| = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Suppose, secondly, that $n \equiv 2 \pmod 4$. If $n = 6$, let $S = \emptyset$, while if $n \geq 10$, let

$$S = \bigcup_{i=0}^{\lfloor n/4 \rfloor - 2} \{v_{4i+1}, u_{4i+3}\}.$$

Let $R = \{v_{n-5}, v_{n-4}, u_{n-2}, u_{n-1}\}$. Coloring the vertices in $R \cup S$ red and coloring all remaining vertices blue, produces an X_1-coloring of G, and so $\gamma_{X_1}(G) \leq |R| + |S| = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$. □

We omit the proof of the next three propositions.

Proposition 4 *For $n \geq 3$, $\gamma_{X_2}(G) = 2n$, unless $n = 4$ in which case $\gamma_{X_2}(G) = 2$.*

Proposition 5 *For $n \geq 3$, $\gamma_{X_3}(G) = n$.*

Proposition 6 *For $n \geq 3$, $\gamma_{X_4}(G) = 2\left\lceil \frac{n}{3} \right\rceil$.*

Proposition 7 *For $n \geq 3$, $\gamma_{X_5}(C_n \times K_2) = \left\lceil \frac{4n}{3} \right\rceil$.*

Proof. In any X_5-coloring of a graph, every vertex colored blue is rooted at a copy of X_5. Hence as an immediate consequence of the definition of an X_5-coloring, any X_5-coloring of G colors at least four vertices from every subgraph $H = P_3 \times K_2$ of G red. Furthermore, if it colors a vertex v blue, then v lies on a 4-cycle with three red vertices.

Consider any given X_5-coloring of G. If every vertex of G is colored red, then the required lower bound follows. Hence, renaming vertices if necessary, we may assume that our given X_5-coloring of G colors v_1 blue. Thus, v_1 lies on a 4-cycle in which the other three vertices are colored red. Renaming vertices if necessary, we may therefore assume that the vertices u_1, u_2 and v_2 are all colored red.

If $n \equiv 0 \pmod 3$, then G contains $n/3$ disjoint copies of H, each of which contains at least four red vertices, and so our given X_5-coloring contains at least $4n/3 = \lceil 4n/3 \rceil$ red vertices. If $n \equiv 1 \pmod 3$, then $G - \{u_2, v_2\}$ can be partitioned into $(n-1)/3$ disjoint copies of H, each of which contains at least four red vertices, and so our given X_5-coloring of G colors at least $2+4(n-1)/3 = (4n+2)/3 = \lceil 4n/3 \rceil$ vertices red. Finally, if $n \equiv 2 \pmod 3$, then $G - \{u_1, u_2, v_1, v_2\}$ can be partitioned into $(n-2)/3$ disjoint copies of H, each of which contains at least four red vertices, and so our given X_5-coloring of G colors at least $3 + 4(n-2)/3 = (4n+1)/3 = \lceil 4n/3 \rceil$ vertices red.

In all three cases, our given X_5-coloring of G colors at least $\lceil 4n/3 \rceil$ vertices red. Thus, $\gamma_{X_5}(G) \geq \lceil 4n/3 \rceil$. We show next that $\gamma_{X_5}(G) \leq \lceil 4n/3 \rceil$. Let

$$S = \bigcup_{i=0}^{\lfloor n/3 \rfloor - 1} \{v_{3i+2}, v_{3i+3}\}.$$

If $n \not\equiv 2 \pmod 3$, let $D = V(G) - S$. If $n \equiv 2 \pmod 3$, let $D = V(G) - (S \cup \{v_n\})$. Then coloring the vertices of D red and coloring all remaining vertices of G blue produces an X_5-coloring of G. Thus, $\gamma_{X_5}(G) \leq |D| = \lceil 4n/3 \rceil$. □

5 Domination Parameters in Prisms

In this section, we determine the relationship between the X-domination numbers of a prism and domination type parameters. In all but one of the five possible choices for a 2-stratified C_4 (see Figure 3), the red vertices form a dominating set in the graph. Hence we have the following observation.

Observation 8 *For $i \in \{1, 3, 4, 5\}$ and for any graph G, $\gamma(G) \leq \gamma_{X_i}(G)$.*

Given a graph $G = (V, E)$ and a subset $S \subseteq V$, we call the coloring of G that colors the vertices of S red and the vertices of $V - S$ blue the *red-blue coloring associated with S*. We shall prove:

Theorem 9 *For $n \geq 3$, let $G = C_n \times K_2$. Then for $i \in \{1, 3, 4, 5\}$, the parameter $\gamma_{X_i}(G)$ is given by the following table:*

i	$\gamma_{X_i}(G) =$
1	$\gamma(G)$
3	$\gamma_2(G)$
4	$\begin{cases} \gamma_t(G) + 1 & \text{if } n \equiv 1 \pmod 6 \\ \gamma_t(G) & \text{otherwise.} \end{cases}$
5	$\begin{cases} \gamma^t_{\times 2}(G) - 1 & \text{if } n \equiv 2 \pmod 6 \\ \gamma^t_{\times 2}(G) & \text{otherwise.} \end{cases}$

where $\gamma_2(G)$ denotes the 2-domination number, $\gamma_t(G)$ denotes the total domination number, and $\gamma^t_{\times 2}(G)$ denotes the double total domination number (which we define in Subsection 5.4).

Throughout Section 5, we let $G = C_n \times K_2$. The proof of Theorem 9 follows from Propositions 10, 12, 14 and 16.

5.1 The domination number

A dominating set S in a graph is a minimal dominating set if and only if for each $v \in S$, we have $\text{pn}(v, S) \neq \emptyset$.

Proposition 10 *For $n \geq 3$, $\gamma(G) = \gamma_{X_1}(G)$.*

Proof. By Observation 8, $\gamma(G) \leq \gamma_{X_1}(G)$. Hence it suffices for us to show that $\gamma(G) \geq \gamma_{X_1}(G)$. Among all $\gamma(G)$-sets, let S be chosen so that

(1) $G[S]$ has minimum size.
(2) Subject to (1), the red-blue coloring associated with S contains the maximum
number of blue vertices that are rooted at a copy of X_1.

We proceed further by proving three claims.

Claim 3 *$|N(v) \cap S| \leq 1$ for all $v \in S$.*

Proof. Suppose there exists a vertex $v_i \in S$ such that $|N(v_i) \cap S| \geq 2$. If $u_i \in S$, then by symmetry we may assume that $v_{i+1} \in S$. But then $(S - \{u_i, v_i\}) \cup \{u_{i-1}\}$ is a dominating set of G of cardinality less than $\gamma(G)$, which is impossible. Hence, $u_i \notin S$; that is, $\{v_{i-1}, v_{i+1}\} \subset S$. Then, $u_i \in \text{pn}(v_i, S)$, and so $u_{i-1} \notin S$ and $u_{i+1} \notin S$. Hence, $(S - \{v_i\}) \cup \{u_i\}$ is a $\gamma(G)$-set that induces a subgraph of G with fewer edges than $G[S]$, contradicting our choice of S. □

Claim 4 *$|\{u_i, v_i\} \cap S| \leq 1$ for $i = 1, 2, \ldots, n$.*

Proof. Suppose that $\{u_i, v_i\} \subseteq S$ for some i, $1 \leq i \leq n$. By Claim 3, $S \cap \{u_{i-1}, v_{i-1}, u_{i+1}, v_{i+1}\} = \emptyset$. By the minimality of S, $\text{pn}(v_i, S) \subseteq \{v_{i-1}, v_{i+1}\}$ and $\text{pn}(u_i, S) \subseteq \{u_{i-1}, u_{i+1}\}$. Suppose that $v_{i-1} \in \text{pn}(v_i, S)$ and $u_{i+1} \in \text{pn}(u_i, S)$. Then, $S \cap \{u_{i+2}, v_{i-2}\} = \emptyset$. Hence, $(S - \{u_i, v_i\}) \cup \{u_{i+1}, v_{i-1}\}$ is a $\gamma(G)$-set that induces a subgraph of G with fewer edges than $G[S]$, contradicting our choice of S. Similarly we have a contradiction if $v_{i+1} \in \text{pn}(v_i, S)$ and $u_{i-1} \in \text{pn}(u_i, S)$. Hence, by symmetry, we may assume $\text{pn}(v_i, S) = \{v_{i+1}\}$ and $\text{pn}(u_i, S) = \{u_{i+1}\}$. Hence, $\{u_{i-2}, v_{i-2}\} \subset S$ while $S \cap \{u_{i+2}, v_{i+2}\} = \emptyset$. But then $(S - \{v_i\}) \cup \{v_{i+1}\}$ is a $\gamma(G)$-set that induces a subgraph of G with fewer edges than $G[S]$, contradicting our choice of S. □

Claim 5 *The red-blue coloring associated with S is an X_1-coloring of G.*

Proof. Suppose not. Then, renaming vertices if necessary, we may assume that v_1 is a blue vertex that is not rooted at a copy of X_1 in the red-blue coloring associated with S. Since S is a dominating set, at least one neighbor of v_1 is in S. If $v_2 \in S$, then by Claim 4, $u_2 \notin S$. Since v_1 is not rooted at a copy of X_1 in the red-blue coloring associated with S, we must have $u_1 \in S$. Similarly, if $v_n \in S$, then $u_1 \in S$. Hence, $u_1 \in S$.

If $S \cap \{v_2, v_n\} = \emptyset$, then $\{u_2, u_n\} \subset S$, and so $|N(u_1) \cap S| = 2$, contradicting Claim 3. Hence at least one of v_2 and v_n is in S. By symmetry, we may assume $v_2 \in S$.

By Claim 4, $u_2 \notin S$. If $u_n \in S$, then $S - \{u_1\}$ is a dominating set of cardinality less than $\gamma(G)$, which is impossible. Hence, $u_n \notin S$, and so $v_n \in S$ (since v_1 is not rooted at a copy of X_1). If $v_3 \in S$, then $S - \{v_2\}$ is a dominating set, which is impossible. If $u_3 \in S$, then $(S - \{u_1, v_2\}) \cup \{u_2\}$ is a dominating set of cardinality less than $\gamma(G)$, which is impossible. Hence, $S \cap \{u_3, v_3\} = \emptyset$. In order to dominate u_3, we have $u_4 \in S$. Thus by Claim 4, $v_4 \notin S$.

By Claim 4, $|S \cap \{u_5, v_5\}| \leq 1$. If $u_5 \notin S$ and $v_5 \in S$, then $(S - \{u_1, u_4, v_2\}) \cup \{u_2, v_4\}$ is a dominating set of cardinality less than $\gamma(G)$, which is impossible. If $u_5 \in S$ and $v_5 \notin S$, then $(S - \{u_1, u_4, v_2\}) \cup \{u_2, v_3\}$ is a dominating set of cardinality less than $\gamma(G)$, which is impossible. Hence, $S \cap \{u_5, v_5\} = \emptyset$. Let $S' = (S - \{v_2\}) \cup \{v_3\}$. Then, S' is a $\gamma(G)$-set such that $G[S']$ has the same size as $G[S]$ and the red-blue coloring associated with S' contains one more blue vertex that is rooted at a copy of X_1 than does the red-blue coloring associated with S. This contradicts our choice of the set S. □

By Claim 5, the red-blue coloring associated with S is an X_1-coloring of G. Hence, $\gamma_{X_1}(G) \leq \gamma(G)$, thus completing the proof of Proposition 10. □

As a consequence of the proof of Proposition 10, we have the following result.

Corollary 11 *For $n \geq 3$, there exists a $\gamma(G)$-set whose associated red-blue coloring is a minimum X_1-coloring in G.*

5.2 The 2-domination number

Let S be a dominating set in a graph $G = (V, E)$. We say that a vertex $v \in V$ is *double dominated* by S if $|N[v] \cap S| \geq 2$. The set S is a 2-*dominating set* of G if every vertex in $V - S$ is double dominated by S. The 2-*domination number*, denoted by $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set in G (see [8, 10]). A 2-dominating set of G of cardinality $\gamma_2(G)$ is called a $\gamma_2(G)$-*set*.

Proposition 12 *For* $n \geq 3$, $\gamma_2(G) = \gamma_{X_3}(G)$.

Proof. The red vertices in any X_3-coloring of G form a 2-dominating set of G, and so $\gamma_2(G) \leq \gamma_{X_3}(G)$. Hence it suffices for us to show that $\gamma_2(G) \geq \gamma_{X_3}(G)$. Among all $\gamma_2(G)$-sets, let S be chosen so that the red-blue coloring associated with S contains the maximum number of blue vertices that are rooted at a copy of X_3.

Claim 6 *The red-blue coloring associated with* S *is an* X_3-*coloring of* G.

Proof. Suppose not. Then, renaming vertices if necessary, we may assume that v_1 is a blue vertex that is not rooted at a copy of X_3 in the red-blue coloring associated with S. Since S is a 2-dominating set, at least two neighbors of v_1 are in S.

We show that $S \cap \{u_1, v_2, v_n\} = \{v_2, v_n\}$. Suppose $\{u_1, v_2\} \subset S$. Since v_1 is not rooted at a copy of X_3 in the red-blue coloring associated with S, we must have $u_2 \in S$. If $v_n \in S$, then $u_n \in S$. But then $S - \{u_1\}$ is a 2-dominating set of G, contradicting the minimality of S. Hence, $v_n \notin S$. In order to double dominate v_n, we must have $\{u_n, v_{n-1}\} \subset S$. But then $(S - \{u_1\}) \cup \{v_1\}$ is a $\gamma_2(G)$-set such that the red-blue coloring associated with this set contains at least one more blue vertex, namely u_1, that is rooted at a copy of X_3 than does the red-blue coloring associated with S. This contradicts our choice of the set S. Hence, $\{u_1, v_2\} \not\subset S$. Similarly, $\{u_1, v_n\} \not\subset S$. Hence, $S \cap \{u_1, v_2, v_n\} = \{v_2, v_n\}$. In order to double dominate u_1, we must have $\{u_2, u_n\} \subset S$.

We show next that $S \cap \{u_3, v_3\} = \emptyset$ while $\{u_4, v_4\} \subset S$. If $v_3 \in S$, then $(S - \{v_2\}) \cup \{v_1\}$ is a $\gamma_2(G)$-set such that the red-blue coloring associated with this set contains at least two more blue vertices that are rooted at a copy of X_3 than does the red-blue coloring associated with S, a contradiction. If $u_3 \in S$, then by considering the set $(S - \{u_2\}) \cup \{u_1\}$ we produce a similar contradiction. Hence, $S \cap \{u_3, v_3\} = \emptyset$. In order to double dominate u_3 and v_3, we must have $\{u_4, v_4\} \subset S$, as claimed.

Continuing in this way, we have that $S \cap \{u_i, v_i\} = \emptyset$ for all i odd while $\{u_i, v_i\} \subset S$ for all i even. As observed earlier, in order to double dominate u_1 and v_1 we have $\{u_n, v_n\} \subset S$. Hence, n is even. But then $(S - \{v_2, v_4, \ldots, v_n\}) \cup \{v_1, v_3, \ldots, v_{n-1}\}$ is a $\gamma_2(G)$-set such that every blue vertex in the red-blue coloring associated with this set is rooted at a copy of X_3, contrary to our choice of S. □

By Claim 6, the red-blue coloring associated with S is an X_3-coloring of G. Hence, $\gamma_{X_3}(G) \le \gamma_2(G)$, thus completing the proof of Proposition 12. □

As a consequence of the proof of Proposition 12, we have the following result.

Corollary 13 *For $n \ge 3$, there exists a $\gamma_2(G)$-set whose associated red-blue coloring is a minimum X_3-coloring in G.*

5.3 The total domination number

A set $S \subseteq V$ in a graph $G = (V, E)$ is a *total dominating set* (TDS) if every vertex is adjacent to at least one vertex of S. Equivalently, S is a TDS of G if for every vertex $v \in V$, $|N(v) \cap S| \ge 1$. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a TDS of G. A TDS of cardinality $\gamma_t(G)$ we call a $\gamma_t(G)$-set.

Proposition 14 *For $n \ge 3$,*

$$\gamma_{X_4}(G) = \begin{cases} \gamma_t(G) + 1 & \text{if } n \equiv 1 \pmod{6} \\ \gamma_t(G) & \text{otherwise.} \end{cases}$$

Proof. Any TDS of G contains at least two vertices from every subgraph $H = P_3 \times K_2$ of G (since the two vertices of degree 3 in H have disjoint open neighborhoods, each of which contains at least one vertex from any TDS). Let S be a $\gamma_t(G)$-set.

Suppose, first, that $n \equiv 1 \pmod{6}$. Renaming vertices if necessary, we may assume $v_1 \notin S$. To dominate v_1, the set S contains at least one neighbor of v_1. If $u_1 \in S$, then $G - \{u_1, v_1\}$ can be partitioned into $(n-1)/3$

disjoint copies of H, each of which contains at least two vertices of S, and so $|S| \geq 1 + 2(n-1)/3 = (2n+1)/3$. If $v_2 \in S$, then $G - \{u_2, v_2\}$ can be partitioned into $(n-1)/3$ disjoint copies of H, and so once again $|S| \geq (2n+1)/3$. Similarly, if $v_n \in S$, then $|S| \geq (2n+1)/3$. Hence, $\gamma_t(G) \geq (2n+1)/3 = 2\lceil n/3 \rceil - 1$. On the other hand, the set

$$\left(\bigcup_{i=0}^{(n-7)/6} \{u_{6i+2}, u_{6i+3}, v_{6i+5}, v_{6i+6}\} \right) \cup \{u_1\}$$

is a TDS of G of cardinality $(2n+1)/3$, and so $\gamma_t(G) \leq (2n+1)/3 = 2\lceil n/3 \rceil - 1$. Consequently, $\gamma_t(G) = 2\lceil n/3 \rceil - 1$, and so, by Theorem 2, $\gamma_t(G) = \gamma_{X_4}(G) - 1$.

Suppose, then, that $n \not\equiv 1 \pmod 6$. The red vertices in any X_4-coloring of G form a TDS of G, and so $\gamma_t(G) \leq \gamma_{X_4}(G)$. Hence it suffices for us to show that $|S| = \gamma_t(G) \geq \gamma_{X_4}(G)$.

Suppose $n \equiv 0 \pmod 3$. Then, G contains $n/3$ disjoint copies of H, each of which contains at least two vertices of S, and so $|S| \geq 2n/3 = 2\lceil n/3 \rceil$. Hence by Theorem 2, $\gamma_t(G) \geq \gamma_{X_4}(G)$.

Suppose $n \equiv 2 \pmod 3$. Renaming vertices if necessary, we may assume $v_1 \notin S$. If $u_1 \in S$, then to totally dominate u_1 we may assume by symmetry that $u_2 \in S$, and so the 4-cycle $C'\colon v_1, v_2, u_2, u_1, v_1$ contains at least two vertices of S. On the other hand, if $u_1 \notin S$, then we may assume by symmetry that $v_2 \in S$ (to dominate v_1). To totally dominate v_2, at least one of u_2 or v_3 is in S, and so the 4-cycle $C'\colon v_2, v_3, u_3, u_2, v_2$ contains at least two vertices of S. In both cases the cycle C' contains at least two vertices of S and $G - V(C')$ can be partitioned into $(n-2)/3$ disjoint copies of H, each of which contains at least two vertices of S, and so $|S| \geq 2 + 2(n-2)/3 = 2(n+1)/3 = 2\lceil n/3 \rceil$. Hence by Theorem 2, $\gamma_t(G) \geq \gamma_{X_4}(G)$.

We show next that if $n \equiv 4 \pmod 6$, then $\gamma_t(G) \geq 2\lceil n/3 \rceil$ (and so, by Theorem 2, $\gamma_t(G) \geq \gamma_{X_4}(G)$). We proceed by induction on $n \geq 4$. If $n = 4$, then $\gamma_t(G) = 4 = 2\lceil n/3 \rceil$. This establishes the base case. Assume, then, that $n \geq 10$ and that for all integers $n' \equiv 4 \pmod 6$ with $4 \leq n' < n$ that $\gamma_t(C_{n'} \times K_2) \geq 2\lceil n'/3 \rceil$. Among all $\gamma_t(G)$-sets, let S be chosen to contain as many pairs $\{u_i, v_i\}$ as possible. We show that S contains at least one such pair. Assume, to the contrary, that $|S \cap \{u_i, v_i\}| \leq 1$ for all $i = 1, 2, \ldots, n$. Let C be the red-blue coloring associated with S. If every blue vertex in C is rooted at a copy of X_4, then $\gamma_t(G) \geq \gamma_{X_4}(G)$, as desired. Hence we may assume, renaming vertices if necessary, that v_1 is a blue vertex that is not rooted at a copy of X_4 in C. If $u_1 \in S$, then to totally dominate u_1, we may assume $u_2 \in S$. By assumption, $|S \cap \{u_2, v_2\}| \leq 1$, and so $v_2 \notin S$. But then v_1 is rooted at a copy of X_4, a contradiction. Hence, $u_1 \notin S$.

By symmetry, we may assume $v_2 \in S$ (to dominate v_1), implying that $v_3 \in S$ and $S \cap \{u_2, u_3\} = \emptyset$. To dominate u_1, it follows from our choice of the set S that $S \cap \{u_{n-1}, u_n, v_{n-1}, v_n\} = \{u_{n-1}, u_n\}$. If $u_4 \in S$ or if $v_5 \in S$, then $(S - \{v_3\}) \cup \{u_2\}$ is a $\gamma_t(G)$-set that contains the pair $\{u_2, v_2\}$, contrary to our choice of S. Hence, $S \cap \{u_4, v_5\} = \emptyset$.

Claim 7 $v_4 \notin S$.

Proof. Suppose $v_4 \in S$. If $u_5 \in S$, then $(S - \{v_4\}) \cup \{v_5\}$ is a $\gamma_t(G)$-set that contains the pair $\{u_5, v_5\}$, contrary to our choice of S. Hence, $u_5 \notin S$, and so $u_6 \in S$ (to dominate u_5). Further, $u_7 \in S$ to totally dominate u_6. By our choice of S, $S \cap \{v_6, v_7\} = \emptyset$. If $u_8 \in S$, then $(S - \{v_4, u_6\}) \cup \{u_5, v_5\}$ is a $\gamma_t(G)$-set that contains the pair $\{u_5, v_5\}$, contrary to our choice of S. Hence, $u_8 \notin S$. If $v_8 \in S$, then $(S - \{u_7\}) \cup \{v_6\}$ is a $\gamma_t(G)$-set that contains the pair $\{u_6, v_6\}$, contrary to our choice of S. Hence, $v_8 \notin S$, implying that $S \cap \{u_9, u_{10}, v_9, v_{10}\} = \{v_9, v_{10}\}$. If $u_{11} \in S$, then $(S - \{v_{10}\}) \cup \{u_9\}$ is a $\gamma_t(G)$-set that contains the pair $\{u_9, v_9\}$, a contradiction. Hence, $u_{11} \notin S$. If $v_{11} \in S$, then $(S - \{v_4, u_7, v_9\}) \cup \{u_5, u_8, v_8\}$ is a $\gamma_t(G)$-set that contains the pair $\{u_8, v_8\}$, a contradiction. Hence, $v_{11} \notin S$, implying that $S \cap \{u_{12}, v_{12}, u_{13}, v_{13}\} = \{u_{12}, u_{13}\}$. Continuing in this way, we have that for each i where $1 \leq i \leq (n-4)/6$,

$$S \cap \left(\bigcup_{j=-1}^{4} \{u_{6i+j}, v_{6i+j}\} \right) = \{u_{6i}, u_{6i+1}, v_{6i+3}, v_{6i+4}\}.$$

This implies that $S \cap \{u_{n-1}, v_{n-1}, u_n, v_n\} = \{v_{n-1}, v_n\}$. But then the vertex u_1 is not dominated by S, a contradiction. □

By Claim 7, $v_4 \notin S$, implying that $S \cap \{u_4, v_4, u_5, v_5, u_6, v_6\} = \{u_5, u_6\}$. If $v_7 \in S$ or if $u_8 \in S$, then $(S - \{u_6\}) \cup \{v_5\}$ is a $\gamma_t(G)$-set that contains the pair $\{u_5, v_5\}$, contrary to our choice of S. Hence, $S \cap \{v_7, u_8\} = \emptyset$. Thus if $u_7 \notin S$, then $u_8 \in S$ to dominate v_7. Let $V' = \{u_1, v_1, u_2, v_2, \ldots, u_6, v_6\}$. Then, $S' = S \cap V' = \{v_2, v_3, u_5, u_6\}$, and $\{u_{n-1}, u_n\} \subset S$. Let G' be the prism $C_{n-6} \times K_2$ obtained from $G - V'$ by adding the edges v_7v_n and u_7u_n. Since S is a TDS of G, the set $S - S'$ is a TDS of G'. Thus, by the induction hypothesis, $|S| - 4 = |S - S'| \geq \gamma_t(G') \geq 2\lceil (n-6)/3 \rceil$, and so $|S| \geq 2\lceil n/3 \rceil$, as desired. Hence by Theorem 2, if $n \equiv 4 \pmod 6$, then $\gamma_t(G) \geq \gamma_{X_4}(G)$. □

Since the red vertices in any X_4-coloring of G form a TDS of G, as an immediate consequence of Proposition 14 we have the following result.

Corollary 15 *For $n \geq 3$ with $n \not\equiv 1 \pmod 6$, there exists a $\gamma_t(G)$-set whose associated red-blue coloring is a minimum X_4-coloring in G.*

5.4 The double total domination number

In this subsection, we consider a generalization of total domination in graphs which we call double total domination (defined in a similar way as that of double domination introduced by Harary and Haynes [9]). Let $G = (V, E)$ be a graph and let $S \subseteq V$. We say that a vertex $v \in V$ is *double totally dominated* by S if $|N(v) \cap S| \geq 2$. If every vertex of V is double totally dominated by S, then we call S a *double total dominating set* (DTDS) of G. The *double total domination number* $\gamma^t_{\times 2}(G)$ is the minimum cardinality of a DTDS of G. A DTDS of cardinality $\gamma^t_{\times 2}(G)$ we call a $\gamma^t_{\times 2}(G)$-set. We omit a proof of the next two results (the interested reader can obtain a proof directly from the authors).

Proposition 16 *For $n \geq 3$,*

$$\gamma_{X_5}(G) = \begin{cases} \gamma^t_{\times 2}(G) - 1 & \text{if } n \equiv 2 \pmod 6 \\ \gamma^t_{\times 2}(G) & \text{otherwise.} \end{cases}$$

Proposition 17 *For $n \geq 3$ with $n \not\equiv 2$ or $3 \pmod 6$, there exists a $\gamma^t_{\times 2}(G)$-set whose associated red-blue coloring is a minimum X_5-coloring in G.*

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