Relaxed game chromatic number of outer planar graphs

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Abstract

In a (k, d)-relaxed coloring game, two players, Alice and Bob, take turns coloring the vertices of a graph G with colors from a set C of k colors. A color i is legal for an uncolored vertex x (at a certain step) means that after coloring x with color i, the subgraph induced by vertices of color i has maximum degree at most d. Each player can only color a vertex with a legal color. Alice's goal is to have all the vertices colored, and Bob's goal is the opposite: to have an uncolored vertex without legal color. The d-relaxed game chromatic number of a graph G, denoted by $\chi_g^{(d)}(G)$ is the least number k so that when playing the (k, d)-relaxed coloring game on G, Alice has a winning strategy. This paper proves that if G is an outer planar graph, then $\chi_g^{(d)}(G) \leq 7 - d$ for $d = 0, 1, 2, 3, 4$.

1 Introduction

The game chromatic number of a graph G is defined through a two person game. Suppose $G = (V, E)$ be a graph, and C is a set of k colors. Two

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persons, say Alice and Bob, alternately color the vertices of G with colors from the color set C, with Alice having the first move. A color i is legal for an uncolored vertex x if x has no neighbour colored with color i. In each move, Alice and Bob must color an uncolored vertex with a legal color. The game is over if either all the vertices are colored or no legal color is available for the uncolored vertices. Alice's goal is to have all the vertices colored, and Bob's goal is to prevent this from happening. The game chromatic number $\chi_g(G)$ of G is the least number of colors for which Alice has a winning strategy in coloring G. The concept of the game chromatic number of a graph was introduced in [1], and has attracted some recent attention [1, 2, 5, 6, 7, 9, 10, 11, 12, 13].

The relaxed game chromatic number is a variation of the game chromatic number introduced in [3]. It is also defined through a two person game. The rules are almost the same as above. The only difference is in the definition of a legal color for an uncolored vertex.

Suppose $d \geq 0$ is an integer. In a d-relaxed coloring game (played on a graph G with color set C), a color $i \in C$ is legal for an uncolored vertex $x \in V(G)$ if by coloring x with color i, each vertex of color i is adjacent to at most d vertices of color i. In other words, color i is legal for vertex x if the following hold:

1. x has at most d neighbours that are colored by color i.

2. If y is a neighbour of x colored by color i, then y has at most $d-1$ neighbours that are colored by color i.

The *d-relaxed game chromatic number* $\chi_g^{(d)}(G)$ of G is the least cardinality of a color set C for which Alice has a winning strategy for the d-relaxed coloring game played on G with color set C. A 0-relaxed coloring game is the same as the coloring game, and the 0-relaxed game chromatic number of a graph G is the same as its game chromatic number. For convenience, we call a d-relaxed coloring game with k colors *a* (k,d)*-relaxed coloring game*.

For a class $\mathcal{C}$ of graphs, let

$$\chi_g^{(d)}(\mathcal{C}) = \max\{\chi_g^{(d)}(G) : G \in \mathcal{C}\}.$$

Let $\mathcal{F}$ be the set of all forests, and let $\mathcal{Q}$ be the class of outer planar graphs. The following results are proved in a series of papers.

Theorem 1 *[7, 3, 8]* $\chi_g(\mathcal{F}) = 4$, $\chi_g^{(1)}(\mathcal{F}) = 3$ *and* $\chi_g^{(2)}(\mathcal{F}) = 2$.

Theorem 2 *[4, 9, 3, 8]* $\chi_g(\mathcal{Q}) \leq 7$, $\chi_g^{(1)}(\mathcal{Q}) \leq 6$, $\chi_g^{(2)}(\mathcal{Q}) \leq 5$ *and* $\chi_g^{(8)}(\mathcal{Q}) \leq 2$. *Moreover,* $\chi_g^{(4)}(\mathcal{Q}) \geq 3$.

This paper generalizes the results above concerning relaxed game chromatic number of outer planar graphs and proves the following:

Theorem 3 *For* $0 \leq d \leq 4$, $\chi_g^{(d)}(\mathcal{Q}) \leq 7 - d$.

2 Alice's strategy

Let G be a 2-connected triangulated outer planar graph (i.e., each inner face of G is a triangle). We produce an ordering of the vertices of G as follows: choose an edge incident to the infinite face and label its two end vertices v_1, v_2. Suppose we have labeled vertices $v_1, v_2, \cdots, v_i$, and there are unlabeled vertices. Then choose a triangle which contains only one unlabeled vertex and label it v_{i+1}. This method produces a labeling $v_1, v_2, \cdots, v_n$ of $V(G)$ such that for each j ($3 \leq j \leq n$), v_j is adjacent to two labeled vertices v_{j_1}, v_{j_2} with $j_1 < j_2 < j$. We call v_j a *parent* of v_i if $v_i \sim v_j$ and $j < i$. The ordering constructed above has the following properties:

1. For $i \geq 3$, the vertex v_i has exactly two parents and these two parents are adjacent.

2. If $i \neq j$, then v_i and v_j have at most one parent in common.

For each $i \geq 3$, suppose v_{i_1}, v_{i_2} are the two parents of v_i. If $i_1 < i_2$, we call v_{i_1} the *major parent* of v_i, and call v_{i_2} the *minor parent* of v_i. The vertex v_i is called a *major child* of v_{i_1} and a *minor child* of v_{i_2}. For each vertex $x \neq v_1, v_2$, we shall denote by $f(x)$ and $m(x)$ the major parent and minor parent of x, respectively.

Note that if two vertices of G are joined by an edge, then one is a parent of the other. If w is a minor child of x, then $f(w)$ is a parent of x. By Property (2) of outerplanar graphs, two minor children of x have different major parents. Therefore x can have at most two minor children, one with major parent $f(x)$ and the other with major parent $m(x)$.

Any outer planar graph G is a subgraph of a triangulated outer planar graph. First we shall only consider 2-connected triangulated outer planar graph, and describe a winning strategy for Alice for such graphs. Later we shall explain that the same strategy works for all outer planar graphs.

For $d = 0, 1, \cdots, 4$, Alice's strategies for the (s, d)-relaxed coloring game are the same, except that the color sets are of different sizes. In the following, we assume that $s = 3, 4, 5, 6, 7$ and $d \geq 7 - s$. Alice and Bob are playing the (s, d)-relaxed coloring game on a 2-connected triangulated outer planar graph G. The vertices of G are $\{v_1, v_2, \cdots, v_n\}$, where the ordering is constructed as in the previous section.

We shall first describe the strategy for Alice to pick the vertex to be colored next. Let U denote the set of uncolored vertices. Alice maintains a subset $A \subseteq V$ of *active* vertices. Initially $A = \emptyset$. When a new vertex is put into A, we say x is activated. Once a vertex is activated, it remains active forever. Initially, Alice colors v_1 and activates v_1. Now suppose that Bob has colored the vertex b. Then b is activated if it not active yet. Alice updates A and chooses the next vertex x to be colored by using the following strategy:

Alice will jump from vertex to vertex until she finds the vertex she wants to color. The so called "jumps" are done by applying the following rules successively:

First Alice jumps to b. Assume Alice has jumped to a vertex x.

Rule 1. If $x = v_1$ or v_2, then Alice chooses the least uncolored vertex, activates it (if it is not active yet) and colors it;

Rule 2. If x is active and uncolored, then she colors x;

Rule 3. If x is inactive, uncolored, and both $f(x)$ and $m(x)$ are colored, then she activates x and colors x;

Rule 4. If none of the above is true, then Alice activates x (if x is inactive), and jumps to $f(x)$ or $m(x)$ (by following the Jumping Rule below) and returns to Rule 1.

Jumping Rule: *If $f(x)$ is uncolored, or $f(x)$ and x are colored the same color, then jump to $f(x)$. Otherwise, jump to $m(x)$.*

After choosing the vertex x to be colored, Alice finds a legal color for x as follows: if the colored neighbours of x use at most $s-1$ colors, then she colors x with a color not used by its colored neighbours; if the colored neighbours of x use s colors, then she colors x with a legal color which is not used by its parents and is used the least number of times among its children. Since $s \geq 3$ and x has at most two parents, so there is always a color not used by its parents. If $s = 2$, then this strategy does not apply: some vertices must be colored the same color as one of their parents. In [4], a slightly different strategy was used to prove that $\chi_g^{(8)}(Q) \leq 2$.

In the remainder of this section, we assume that $s = 3, 4, 5, 6, 7$ and $d \geq 7 - s$, Alice and Bob play an (s, d)-relaxed coloring game on a triangulated outer planar graph G and Alice uses the strategy described as above. Observe that Alice only colors active vertices, and a vertex colored by Bob is activated immediately.

Lemma 1 *If Alice has just finished her move and v is an uncolored vertex, then v has at most 3 active children. Moreover, v has at most 1 active major child.*

Proof. By our strategy, each time a major child w of v is activated, Alice will jump from w to v. The first time Alice jumps to a vertex, she activates it; the second time she jumps to it, she colors that vertex. So if v is uncolored, then v has at most 1 active major child. As v has at most two minor children, so v has at most 3 active children. ∎

For any vertex v, we denote by $Ma(v)$ the set of major children of v.

Lemma 2 *Suppose at a certain moment, Alice has just finished her move. Assume that the following hold*

- *i is a color not used by any parent of x.*
- *u is a child of x colored by color i.*
- *u has 3 neighbours, say w_1, w_2, w_3, colored by color i.*

Then x is a colored vertex.

Proof. Assume to the contrary that all the above hold, but x is uncolored. We consider two cases.

Case 1 u is a major child of x.

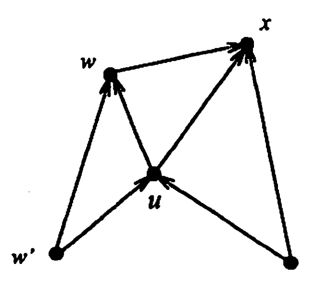

Figure 1: u is a major child of x

Refer to Figure 1. The vertex u has at most three neighbours, namely x, w and w', which are not an element of $Ma(x) \cup Ma(u)$. As x is uncolored, at least one of w_1, w_2, w_3 belong to $Ma(x) \cup Ma(u)$. Without loss of generality, assume that $w_1 \in Ma(x) \cup Ma(u)$.

If u is colored by Bob, then after Bob colors u, Alice jumps once to x. When w_1 is activated, Alice jumps once to x (either directly or through u). So when u and w_1 are both colored, Alice has jumped to x twice, and hence x is colored.

Assume u is colored by Alice. If u is colored before w_1, then w_1 is colored by Bob, because Alice never colors a vertex the same color as a parent of that vertex. When u is activated Alice jumps to x once. When w_1 is colored, Alice jumps to x again. So if u, w_1 are all colored, Alice has jumped to x twice, and hence x is colored. Assume u is colored after w_1. When Alice colors u, all the s colors are used by the neighbours of u, for otherwise Alice will not color u the same color as w_1. Moreover, the color i is used the least number of times among the children of u.

Let B be the set of colored neighbours of u at the moment u is colored. Since B contains vertices of s colors and w_1, w_2, w_3 are of the same color, we have $|B \cup \{w_1, w_2, w_3\}| \geq s + 2 \geq 5$. Therefore

$$|(B \cup \{w_1, w_2, w_3\}) \cap (Ma(u) \cup Ma(x))| \geq 3.$$

If $v \in B \cap (Ma(u) \cup Ma(x))$, then at the time v is activated, Alice jumps once to $\{x, u\}$. If $v \in (\{w_1, w_2, w_3\} - B) \cap (Ma(u) \cup Ma(x))$, then v must be colored by Bob, as v is colored after u and has the same color as u. When Bob colors v, Alice jumps once to $\{x, u\}$. Note that at the time u is activated, Alice jumps once from u to x. After u is colored, then whenever Alice jumps to u, Alice jumps again from u to x. Therefore if all of $B \cup \{w_1, w_2, w_3, u\}$ are colored, then Alice will have jumped to x at least twice, and hence x is colored.

Case 2 u is a minor child of x.

Then $f(u)$ is a parent of x. So $f(u)$ is either uncolored or colored by a color different from i.

If u is colored by Bob, then each time w_1, w_2, w_3 is activated, Alice jumps once to the set $\{f(u), x\}$ (either directly or through u). When u is colored, Alice jumps once to the set $\{f(u), x\}$. Note that at any time, $f(u)$ is either uncolored or colored with a color different from i. By the jumping rule, Alice will not jump from $W = \{u, w_1, w_2, w_3\}$ to $f(u)$ after $f(u)$ is colored (as x is uncolored). Therefore Alice jumps from W to $f(u)$ at most twice, and hence jumps to x at least twice. So x will be colored.

If u is colored by Alice, but colored before w_1, w_2, w_3, then each of w_1, w_2, w_3 is colored by Bob. Each time Bob colors a w_i, Alice jumps once from W to $\{f(u), x\}$. Moreover, at the time u is activated, Alice jumps once to the set $\{f(u), x\}$. When W is colored, Alice jumps from W to $\{f(u), x\}$ at least 4 times, and hence jumps to x at least twice. So x will be colored.

Assume that u is colored by Alice, and at the time Alice colors u, some w_i is colored. By assumption, this colored w_i is not a parent of u, hence is a child of u. So when Alice colors u, all the s colors are used among the neighbours of u, for otherwise Alice will not color u the same color as a child. Let B be the set of colored neighbours of u at the time Alice colors

u. Similarly as in Case 1, we have

$$|B \cup \{w_1, w_2, w_3\}| \geq 5.$$

Each time a vertex of $B \cup \{w_1, w_2, w_3\}$ is activated, Alice jumps once to the set $\{f(u), x\}$, except at the time Alice colors u or Bob colors $f(u) \in B$. Therefore either Alice has jumped to the set $\{f(u), x\}$ at least 4 times after u, w_1, w_2, w_3 are colored or $f(u)$ is colored by Bob and Alice has jumped to the set $\{f(u), x\}$ at least 3 times. Similarly as above, this implies that Alice has jumped to x at least twice, so x is colored. ∎

3 The strategy is a winning strategy for Alice

Theorem 4 *Suppose $G = (V, E)$ is an outer planar graph, $s = 3, 4, 5, 6, 7$, and $d \geq 7 - s$. For the (s, d)-relaxed coloring game on G, the strategy described in the previous section is a winning strategy.*

Proof. First we consider the case that G is triangulated. It suffices to prove that any uncolored vertex has a legal color not used by its parents. Assume to the contrary that there is an uncolored vertex x such that every color is either used by a parent of x, or is not legal for x.

Assume $s = 3$ and $d \geq 4$. By Lemma 1, x has at most 4 colored children, and hence at most 6 colored neighbours (if x has 4 colored children, then one of them is colored by Bob in his last move). Since any color not used by the parents of x is not a legal color for x, each color is used by a neighbour of x. This implies that each color is used at most 4 times, and hence assigning any color to x would not violate Condition (1).

Let i be a color not used by any parent of x. By assumption, i is not a legal color for x. Hence Condition (2) is violated, i.e., x has a neighbour, say u, colored with color i, and u has $d \geq 4$ neighbours, say $w_1, w_2, \cdots, w_d$, of color i. The vertex u cannot be colored by Bob in his last move. Otherwise, before u is coloured, u has four neighbours w_1, w_2, w_3, w_4 of colour i. By Lemma 1, one of these four neighbours, say w_1, is a parent of u. As i is not used by any parent of x, w_1 is a child of x. This implies that u is a major child of x. Now w_2, w_3, w_4 are coloured children of u. By Lemma 1, two of w_2, w_3, w_4 are minor children of u and one is a major child of u. Assume w_2, w_3 are minor children of u and w_4 is a major child of u. Then one of w_2, w_3, say w_2, is a major child of x. As u has a major active child w_4, u is active. So x has two active major children: u and w_2. This is in contrary to Lemma 1. Thus before Bob's last move, u is colored with color i and has at least 3 neighbours of colour i. But this is in contrary to Lemma 2.

Next assume that $s = 4$ and $d \geq 3$. Similarly as above, x has at most 4 colored children and hence 6 colored neighbours. By our assumption, each color is used by a neighbour of x. Hence each color is used at most 3 times, and hence assigning any color to x would not violate Condition (1).

Let i, j be two colors not used by any parent of x. Similarly as above, x has a neighbour u colored with color i, and u has $d \geq 3$ neighbours $w_1, w_2, \cdots, w_d$ of color i; and x also has a neighbour u' of color j, and u' has $d \geq 3$ neighbours $w'_1, w'_2, \cdots, w'_d$, colored with color j. Without loss of generality, we assume that the last colored vertex of $\{u', w'_1, w'_2, w'_3\}$ is colored after the last colored vertex of $\{u, w_1, w_2, w_3\}$. By Lemma 2, at the time u and w_1, w_2, w_3 are all colored, Alice will color x in the next step. So it is impossible that $\{u', w'_1, w'_2, w'_3\}$ are all colored and x is still uncolored.

The proofs for $s = 5, 6, 7$ are similar. The cases $d = 6, 7$ is quite straightforward and the case $s = 5$ is a little bit involved. Since the conclusion of Theorem 3 for $s = 5, 6, 7$ was proved in [9, 3, 8], we omit the details for these cases.

If G is not triangulated, then let G' be a triangulation of G. When playing the coloring game on G, Alice simply pretend she is playing the game on G'. By examining the proof of Lemma 2, it is easy to see that the same argument shows that the strategy is still a winning strategy for Alice. ∎

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