The Restricted Edge-Connectivity of Kautz Undirected Graphs*

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Abstract

A connected graph is said to be super edge-connected if every minimum edge-cut isolates a vertex. The restricted edge-connectivity λ' of a connected graph is the minimum number of edges whose deletion results in a disconnected graph such that each connected component has at least two vertices. A graph G is called λ'-optimal if $\lambda'(G) = \min\{d_G(u) + d_G(v) - 2 : uv \text{ is an edge in } G\}$. This paper proves that for any d and n with $d \geq 2$ and $n \geq 1$ the Kautz undirected graph $UK(d, n)$ is λ'-optimal except $UK(2, 1)$ and $UK(2, 2)$ and, hence, is super edge-connected except $UK(2, 2)$.

Keywords: Edge-connectivity, Restricted edge-connectivity, Super edge-connected, Kautz graphs

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1 Introduction

Throughout this paper, a graph $G = (V, E)$ always means a simple connected graph with a vertex-set V and an edge-set E. We follow [5, 18] for

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graph-theoretical terminology and notation not defined here. A set of edges S of G is called an edge-cut if $G - S$ disconnected. The edge-connectivity $\lambda(G)$ of G is defined as the minimum cardinality of an edge-cut S.

It is well known that when the underlying topology of an interconnection network is modelled by a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network, the edge-connectivity $\lambda(G)$ is an important parameter to measure the fault-tolerance of the network [17]. This parameter, however, has an obvious deficiency, that it is tacitly assumed that all edges incident with a vertex of G can potentially fail at the same time. In other words, in the definition of $\lambda(G)$, absolutely no conditions or restrictions are imposed either on the minimum edge-cut S or on the components of $G - S$.

To compensate for this shortcoming, in 1981, Bauer *et al* [1] proposed the concept of super edge-connectedness. A connected graph is said to be super edge-connected if every minimum edge-cut isolates a vertex. The study of super edge-connected graphs has a particular significance in the design of reliable networks, mainly due to the fact that attaining super edge-connectedness implies minimizing the numbers of minimum edge-cuts (see [4]). A quite natural problem is that if G is super edge-connected then how many edges must be removed to disconnect G such that every component of the resulting graph contains no isolated vertices.

In 1988, Esfahanian and Hakimi [8] proposed the concept of the restricted edge-connectivity. The restricted edge-connectivity of G, denoted by $\lambda'(G)$, is defined as the minimum number of edges whose deletion results in a disconnected graph and contains no isolated vertices. In general, $\lambda'(G)$ does not always exist for a connected graph G. For example, $\lambda'(G)$ does not exist if G is a star $K_{1,n}$ or a complete graph K_3. We write $\lambda'(G) = \infty$ if $\lambda'(G)$ does not exist. In [8], Esfahanian and Hakimi showed that if G has at least four vertices then $\lambda'(G)$ does not exist if and only if G is a star and that if $\lambda'(G)$ exists then

$$\lambda'(G) \leq \xi(G), \tag{1}$$

where the symbol $d_G(x)$ denotes the degree of the vertex x in G and $\xi(G) = \min\{d_G(u) + d_G(v) - 2 : uv \text{ is an edge in } G\}$.

A graph G is called λ'-optimal if $\lambda'(G) = \xi(G)$. Several sufficient conditions for graphs to be λ'-optimal were given for example by Wang and Li [14], Hellwig and Volkmann [9] for graphs with diameter 2, Ueffing and Volkmann [12] for the cartesian product of graphs, Xu and Xu [19] for transitive graphs. It is clear that G is super edge-connected if $\lambda'(G) > \lambda(G)$. Recently, Hellwig and Volkmann [10] have showed that a λ'-optimal graph G is super edge-connected if its minimum degree $\delta(G) \geq 3$.

This new parameter λ' in conjunction with λ can provide more accurate measures for fault tolerance of a large-scale parallel processing system and, thus, has received much attention of many researchers (see, for example, [6] ~ [17], [19]).

In this paper, we consider λ' for the Kautz undirected graph $UK(d,n)$. The following theorem completely determines $\lambda'(UK(d,n))$ for any d and n with $d \geq 2$ and $n \geq 1$.

Theorem For the Kautz undirected graph $UK(d,n)$ with $d \geq 2$ and $n \geq 1$,

$$\lambda'(UK(d,n)) = \begin{cases} \infty, & \text{for } n=1,\ d=2; \\ 3, & \text{for } n=d=2; \\ 2d-2, & \text{for } n=1,\ d\geq 3; \\ 4d-4, & \text{for } n\geq 2,\ d\geq 3 \\ & \quad \text{or } n\geq 3,\ d\geq 2. \end{cases}$$

Corollary The Kautz undirected graph $UK(d,n)$ is λ'-optimal except $UK(2,1)$ and $UK(2,2)$ and, hence, is super edge-connected except $UK(2,2)$.

The proofs of the theorem and the corollary are in Section 3. In Section 2, the definition and some properties of the Kautz undirected graph $UK(d,n)$ are given.

2 Properties of Kautz Graphs

The well-known Kautz digraph is an important class of graphs and widely used in the design and analysis of interconnection networks [3]. Let d and n be two given integers with $n \geq 1$ and $d \geq 2$.

The Kautz digraph, denoted by $K(d,n)$, is a digraph with the vertex-set $V = \{x_1x_2\cdots x_n : x_i \in \{0,1,\ldots,d\},\ x_{i+1} \neq x_i,\ i=1,2,\ldots,n-1\}$ and the directed edge-set E, where for $x,y \in V$, if $x = x_1x_2\cdots x_n$, then $(x,y) \in E$ if and only if $y = x_2x_3\cdots x_n\alpha$, where $\alpha \in \{0,1,\ldots,d\} \setminus \{x_n\}$.

The Kautz undirected graph, denoted by $UK(d,n)$, is a simple undirected graph obtained from $K(d,n)$ by deleting the orientation of all edges and omitting multiple edges.

¿From definitions, $K(d,1)$ is a complete digraph of order $d+1$ and $UK(d,1)$ is a complete undirected graph of order $d+1$. Thus, $\lambda(UK(d,1)) = d$. It has been shown that $K(d,n)$ is d-regular and has connectivity d. It is clear that $UK(d,2)$ is $(2d-1)$-regular, and $UK(d,n)$ has the minimum degree $\delta = 2d-1$ and the maximum degree $\Delta = 2d$ for $n \geq 3$. Furthermore, Bermond *et al* [2] proved that the connectivity of $UK(d,n)$ is $2d-1$ for $n \geq 2$, which implies that $\lambda(UK(d,n)) = 2d-1$ for $n \geq 2$. For more properties of $K(d,n)$ and $UK(d,n)$, the reader is referred to the new book by Xu [17].

A pair of directed edges is said to be *symmetric* if they have the same end-vertices but different orientations. The Kautz digraph contains symmetric edges. If there is a pair of symmetric edges between two vertices x and y in $K(d,n)$, then it is not difficult to see that the coordinates of x are alternately in two different components a and b. It follows that the Kautz digraph $K(d,n)$ contains exactly $\binom{d+1}{2}$ pairs of symmetric edges. Clearly, directed distance between two end-vertices in different pairs of symmetric edges in $K(d,n)$ is equal to either $n-1$ or n. Moreover, two end-vertices in different pairs of symmetric edges have no vertices in common if and only if $n \geq 2$. An edge in $UK(d,n)$ is said to be *singular* if it corresponds a pair of symmetric edges in $K(d,n)$.

Let X and Y be two disjoint nonempty subsets of vertices in a digraph G. Use the symbol $E(X,Y)$ to denote the set of directed edges from X to Y in G. The following property on a regular digraph is useful, and the detail proof can be found in Example 1.4.1 in [18].

Lemma 2.1 Let X and Y be two disjoint nonempty subsets of vertices in a connected regular digraph G. Then $|E(X,Y)| = |E(Y,X)|$. ∎

For two end-vertices x and y of a pair of symmetric edges in $K(d,n)$, let

$$A_x^- = N^-(x)\setminus\{y\}, \qquad A_x^+ = N^+(x)\setminus\{y\},$$
$$A_y^+ = N^+(y)\setminus\{x\}, \qquad A_y^- = N^-(y)\setminus\{x\}.$$

Lemma 2.2 Let xy be a singular edge in $UK(d,n)$, where $n \geq 2$ and $d \geq 2$.

(i) $E(A_x^-, A_y^+) \cap E(A_y^-, A_x^+) = \emptyset$, and

$$|E(A_x^-, A_y^+)| = |E(A_y^-, A_x^+)| = (d-1)^2.$$

(ii) If $n = 2$, then for any $u \in (A_x^- \cup A_x^+)$ there is some $v \in (A_y^- \cup A_y^+)$ such that uv is a singular edge in $UK(d,2)$.

(iii) There exist $2d-1$ internally disjoint xy-paths in $UK(d,n)$ such that one of which is of length one, otherwise of length three.

Proof We may suppose that $x = abab\cdots ab$ if n is even and $x = abab\cdots aba$ if n is odd. Without loss of generality, we suppose that n is even. Then $y = bab\cdots ba$, where $a,b \in \{0,1,\ldots,d\}$ and $a \neq b$.

(i) For any $u \in A_x^-$ and $v \in A_y^+$, they can be expressed as $u = cabab\cdots aba$ and $v = abab\cdots bae$, where $c,e \in \{0,1,\ldots,d\}$ and $c,e \notin \{a,b\}$. Then $u \neq v$ and (u,v) is a directed edge in $K(d,n)$ for $d \geq 2$. Clearly, $A_x^- \cap A_y^+ = \emptyset$ and

$$E(A_x^-, A_y^+) = \{(u,v):\ c,e \in \{0,1,\ldots,d\} \setminus \{a,b\}\}.$$

Thus, $|E(A_x^-, A_y^+)| = (d-1)^2$.

Similarly, For any $z \in A_x^+$ and $w \in A_y^-$, they can be expressed as $z = bab\cdots abg$ and $w = hbab\cdots bab$, where $g, h \in \{0, 1, \ldots, d\}$ and $g, h \notin \{a, b\}$. Then $z \neq w$ and (w, z) is a directed edge in $K(d, n)$ for $d \geq 2$. Clearly, $A_x^+ \cap A_y^- = \emptyset$ and

$$E(A_y^-, A_x^+) = \{(z, w) : \ h, g \in \{0, 1, \ldots, d\} \setminus \{a, b\}\}.$$

Thus, $|E(A_y^-, A_x^+)| = (d-1)^2$.

Since $u \neq w$ and $v \neq z$, $A_x^- \cap A_y^- = \emptyset$ and $A_x^+ \cap A_y^+ = \emptyset$. Also since two end-vertices in different pairs of symmetric edges have no vertex in common if $n \geq 2$, $A_x^- \cap A_x^+ = \emptyset$ and $A_y^+ \cap A_y^- = \emptyset$, and so $A_x^-, A_x^+, A_y^-, A_y^+$ are pairwise disjoint. Thus, $E(A_x^-, A_y^+) \cap E(A_y^-, A_x^+) = \emptyset$.

(ii) Since $n = 2$, we may assume $x = ab$, $y = ba$, where $a, b \in \{0, 1, \ldots, d\}$ and $a \neq b$. If $u \in A_x^-$, then for $d \geq 2$ we may assume $u = ca$ $(c \neq a, b)$, and so $v = ac \in A_y^+$. If $u \in A_x^+$, we may assume $u = bc$ $(c \neq a, b)$, and so $v = cb \in A_y^-$, where $c \in \{0, 1, \ldots, d\}$. Thus, (u, v) and (v, u) are a pair of symmetric edges in $K(d, 2)$, and so uv is a singular edge in $UK(d, 2)$.

The assertion (iii) holds clearly from (i). ∎

Two directed walks from $\{x, y\}$ to $\{u, v\}$ in $K(d, n)$ is said internally disjoint, if they have common vertices only in $\{x, y\}$ or $\{u, v\}$.

Lemma 2.3 Let xy and uv be nonadjacent edges in $UK(d, n)$ where $d \geq 2$ and $n \geq 2$. If xy is singular, then there are $(2d-2)$ internally-disjoint directed paths from $\{x, y\}$ to $\{u, v\}$ in $K(d, n)$.

Proof Let $x = x_1 x_2 \cdots x_n$, where $x_i \in \{a, b\} \subseteq \{0, 1, \ldots, d\}$ and $a \neq b$. Then $y = x_2 x_3 \cdots x_n \alpha$, where $\alpha = x_1$ if n is even and $\alpha = x_2$ if n is odd. Let $u = u_1 u_2 \cdots u_n$. Then $v = u_2 u_3 \cdots u_n u_{n+1}$, where $u_1, \ldots, u_{n+1} \in \{0, 1, \ldots, d\}$ and $u_i \neq u_{i+1}, i = 1, 2, \ldots, n$. Choose $(2d-2)$ directed walks $W_1, W_2, \ldots, W_{d-1}, T_1, T_2, \ldots, T_{d-1}$ from $\{x, y\}$ to $\{u, v\}$ as follows: For $1 \leq i \leq d-1$,

$$\begin{aligned} W_i &= x_1 x_2 x_3 \cdots x_n, x_2 x_3 \cdots x_n w_i, x_3 \cdots x_n w_i u_1, \ldots, \\ &\quad w_i u_1 u_2 \cdots u_{n-1}, u_1 u_2 \cdots u_{n-1} u_n; \quad \text{if } w_i \neq u_1; \\ W_i &= x_1 x_2 x_3 \cdots x_n, x_2 x_3 \cdots x_n w_i, x_3 \cdots x_n w_i u_2, \ldots, \\ &\quad w_i u_2 u_3 \cdots u_{n-1}, u_1 u_2 \cdots u_{n-1} u_n, \quad \text{if } w_i = u_1, \end{aligned}$$

and for $1 \leq j \leq d-1$,

$$\begin{aligned} T_j &= x_2 x_3 x_4 \cdots x_n \alpha, x_3 x_4 \cdots x_n \alpha t_j, x_4 \cdots x_n \alpha t_j u_2, \ldots, \\ &\quad t_j u_2 u_3 \cdots u_n, u_2 u_3 \cdots u_n u_{n+1} \quad \text{if } t_j \neq u_2; \end{aligned}$$

$$T_j = x_2x_3x_4\cdots x_n\alpha, x_3x_4\cdots x_n\alpha t_j, x_4\cdots x_n\alpha t_j u_3, \ldots,$$
$$\alpha t_j u_3 u_4 \cdots u_n, t_j u_3 u_4 \cdots u_n u_{n+1} \quad \text{if } t_j = u_2,$$

where $w_i, t_j \in \{0,1,\ldots,d\}\backslash\{a,b\}$ and $w_1, w_2, \ldots, w_{d-1}$ are pairwise distinct and $t_1, t_2, \ldots, t_{d-1}$ are pairwise distinct. We now show that these directed walks are internally disjoint.

Assume that there are i and j $(1 \le i \ne j \le d-1)$ such that W_i and W_j are internally joint. Without loss of generality, we may suppose that $w_i \ne u_1$ and let z be the first internal vertex of W_i and W_j in common. Let the length of the section $W_i(x,z)$ be k and the length of the section $W_j(x,z)$ be t. Then $2 \le k,\ t \le n-1$. Since z can reach x along W_i by k steps and along W_j by t steps, we can express z as

$$\begin{aligned} z &= x_{k+1}\cdots x_n w_i u_1 \cdots u_{k-1} \\ &= \begin{cases} x_{t+1}\cdots x_n w_j u_1 \cdots u_{t-1}, & w_j \ne u_1, \\ x_{t+1}\cdots x_n w_j u_2 \cdots u_t, & w_j = u_1. \end{cases} \end{aligned}$$

Since $w_i \ne w_j$, we have $k \ne t$. If $k < t$, there is some h with $k+1 \le h \le n$ such that $w_j = x_h \in \{a,b\}$, a contradiction. If $k > t$, there is some l with $t+1 \le l \le n$ such that $w_i = x_l \in \{a,b\}$, a contradiction. Therefore, $W_1, W_2, \ldots, W_{d-1}$ are internally disjoint.

Similarly, we can show that $T_1, T_2, \ldots, T_{d-1}$ are internally disjoint.

Assume that there are i and j $(1 \le i,j \le d-1)$ such that W_i and T_j are internally joint. Let z be the first internal vertex of W_i and T_j in common. Let the length of the section $W_i(x,z)$ be k and the length of the section $T_j(y,z)$ be t. Then $2 \le k,\ t \le n-1$. Thus, we can express z as

$$z = \begin{cases} x_{k+1}\cdots x_n w_i u_1 \cdots u_{k-1} = x_{t+2}\cdots x_n \alpha t_j u_2 \cdots u_t, \\ \qquad \text{if } w_i \ne u_1, t_j \ne u_2; \\ x_{k+1}\cdots x_n w_i u_2 \cdots u_k = x_{t+2}\cdots x_n \alpha t_j u_2 \cdots u_t, \\ \qquad \text{if } w_i = u_1, t_j \ne u_2; \\ x_{k+1}\cdots x_n w_i u_1 \cdots u_{k-1} = x_{t+2}\cdots x_n \alpha u_2 \cdots u_{t+1}, \\ \qquad \text{if } w_i \ne u_1, t_j = u_2; \\ x_{k+1}\cdots x_n w_i u_2 \cdots u_k = x_{t+2}\cdots x_n \alpha u_2 \cdots u_{t+1}, \\ \qquad \text{if } w_i = u_1, t_j = u_2. \end{cases}$$

We can obtain w_i or $t_j \in \{a,b\}$, a contradiction. Therefore, W_i and T_j are internally disjoint for any i and j with $1 \le i,j \le d-1$. Since any directed walk from $\{x,y\}$ to $\{u,v\}$ contains a directed path from $\{x,y\}$ to $\{u,v\}$, the lemma follows immediately. ∎

Lemma 2.4 Let xy and uv be two distinct singular edges in $UK(d,2)$ that have no end-vertices in common. Then there are $(4d-4)$ internally disjoint paths between $\{x,y\}$ and $\{u,v\}$ in $UK(d,2)$.

Proof Let $x = x_1x_2$ and $u = u_1u_2$, where $x_1, x_2, u_1, u_2 \in \{0, 1, \ldots, d\}$, $x_1 \neq x_2, u_1 \neq u_2$. Then $y = x_2x_1$ and $v = u_2u_1$. Choose $4d-4$ directed walks $W_1, W_2, \ldots, W_{d-1}, T_1, T_2, \ldots, T_{d-1}, P_1, \ldots, P_{d-1}, Q_1, Q_2, \ldots, Q_{d-1}$ in $K(d,2)$ from $\{x,y\}$ to $\{u,v\}$ or from $\{u,v\}$ to $\{x,y\}$ as follows.

$$W_i = \begin{cases} x_1x_2, x_2w_i, w_iu_1, u_1u_2, & w_i \neq u_1, \\ x_1x_2, x_2w_i, w_iu_2, & w_i = u_1; \end{cases}$$

$$T_j = \begin{cases} x_2x_1, x_1t_j, t_ju_2, u_2u_1, & t_j \neq u_2, \\ x_2x_1, x_1t_j, t_ju_1, & t_j = u_2; \end{cases}$$

$$P_i = \begin{cases} u_1u_2, u_2p_i, p_ix_1, x_1x_2, & p_i \neq x_1, \\ u_1u_2, u_2p_i, p_ix_2, & p_i = x_1; \end{cases}$$

$$Q_j = \begin{cases} u_2u_1, u_1q_j, q_jx_2, x_2x_1, & q_j \neq x_2, \\ u_2u_1, u_1q_j, g_jx_1, & q_j = x_2, \end{cases}$$

where $w_i, t_j \in \{0, 1, \ldots, d\} \backslash \{x_1, x_2\}$, $w_1, w_2, \ldots, w_{d-1}$ are pairwise different, $t_1, t_2, \ldots, t_{d-1}$ are pairwise different, $p_i, q_j \in \{0, 1, \ldots, d\} \backslash \{u_1, u_2\}$, $p_1, p_2, \ldots, p_{d-1}$ are pairwise different, $q_1, q_2, \ldots, q_{d-1}$ are pairwise different. It is easy to check that $W_1, W_2, \ldots, W_{d-1}, T_1, T_2, \ldots, T_{d-1}, P_1, P_2, \ldots, P_{d-1}, Q_1, Q_2, \ldots, Q_{d-1}$ are internally disjoint, and each of them must contain a directed path from $\{x,y\}$ to $\{u,v\}$ or from $\{u,v\}$ to $\{x,y\}$ as its subgraph. ∎

3 Proof of Theorem

In this section, we give the proofs of the theorem and the corollary stated in Introduction.

A set of edges F in G is called a nontrivial edge-cut if $G - F$ is disconnected and contains no isolated vertices. A nontrivial edge-cut F is called a λ'-cut if $|F| = \lambda'(G)$.

Proof of Theorem It is clear that $\lambda'(UK(d,1))$ does not exist for $d = 2$ and $\lambda'(UK(d,1)) = 2d-2$ for $d \geq 3$ since $UK(d,1) = K_{d+1}$. Clearly $\lambda'(UK(2,2)) = 3$, we only consider $n = 2$, $d \geq 3$ or $n \geq 3$, $d \geq 2$. Under this hypothesis, $UK(d,n)$ has vertices more than four and, hence, by (1) we have

$$\lambda'(UK(d,n)) \leq \xi(UKd,n)) = 2\delta(UK(d,n)) - 2 = 4d - 4.$$

In order to complete the proof of the theorem, we only need to prove $\lambda'(UK(d,n)) \geq 4d-4$.

Let F be a λ'-cut of $UK(d,n)$. Then $UK(d,n) - F$ has exactly two connected components, say G_1 and G_2. Let $X = V(G_1)$ and $Y = V(G_2)$. Then

$$|F| = |E(X,Y) \cup E(Y,X)| = \lambda'(UK(d,n)).$$

We now show that $|F| \geq 4d-4$ by considering two case according to the values of n and d.

Case 1 $n=2$ and $d \geq 3$.

If G_1 and G_2 both contain singular edges then $|F| \geq 4d-4$ by Lemma 2.4. Without loss of generality, assume that G_1 contains no singular edges. Since every vertex in $UK(d,2)$ is incident with a singular edge, every vertex in G_1 is incident with a singular edge in F.

If there is some vertex $x \in X$ such that $(A_x^- \cup A_x^+) \subset X$, where xy is a singular edge in F and $y \in Y$, then $(A_y^+ \cup A_y^-) \subset Y$, for otherwise, there is a singular edge in G_1 by Lemma 2.2 (ii), which contradicts the hypothesis that G_1 contains no singular edges. It follows from Lemma 2.2 (i) that, for $d \geq 3$,

$$\begin{aligned} |F| &\geq |E(A_x^-, A_y^+)| + |E(A_y^-, A_x^+)| + 1 \\ &\geq 2(d-1)^2 + 1 > 4d-4. \end{aligned}$$

If $(A_x^- \cup A_x^+) \not\subseteq X$ for any $x \in X$, then $(A_x^- \cup A_x^+) \cap Y \neq \emptyset$, which implies that every vertex in X is incident with at least two edges in F. Thus, if $|X| \geq 2d-1$ then

$$|F| \geq 2|X| \geq 2(2d-1) = 4d-2 > 4d-4.$$

Assume $t = |X| \leq 2d-2$ below. Noting that $UK(d,2)$ is $(2d-1)$-regular and $|E(G_1)| \leq \frac{1}{2}t(t-1)$, we have

$$|F| \geq (2d-1)t - t(t-1) = 2dt - t^2 \geq 4d-4,$$

since the function $f(t) = 2dt - t^2$ is convex on the interval $[2, 2d-2]$ and $f(t) \geq f(2) = f(2d-2) = 4d-4$.

Case 2 $n \geq 3$ and $d \geq 2$.

If F contains no singular edges, then either G_1 or G_2 must contain a singular edge. By Lemma 2.1 and Lemma 2.3, we have that

$$\begin{aligned} |F| &= |E(X,Y)| + |E(Y,X)| = 2|E(X,Y)| \\ &\geq 2(2d-2) = 4d-4. \end{aligned}$$

If F contains at least two singular edges, then it is easy to see that the end-vertices of any two singular edges have no common neighbors if $n \geq 4$ and have at most two common neighbors if $n=3$. It follows from Lemma 2.2 (iii) that $|F| \geq 2(2d-1) - 2 = 4d-4$.

We now assume that xy is the only singular edge in F, where $x \in X$ and $y \in Y$. If we can show that

$$|E(Y,X)| \geq 2d-1, \tag{2}$$

then, by Lemma 2.1 and (2), we have

$$\begin{aligned} |F| &= |E(X,Y) \cup E(Y,X)| = 2|E(Y,X)| - 1 \\ &\geq 2(2d-1) - 1 = 4d - 3 > 4d - 4, \end{aligned}$$

as required. We now show the inequality (2).

Since $d \geq 2$, $K(d,n)$ contains at least three symmetric edges, and so G_1 or G_2 contains a singular edge. Without loss of generality, assume that G_2 contains a singular edge uv.

If $|X| = 2$, then the only edge in G_1 is not singular, and so $|E(Y,X)| = 2d-1$ clearly. Assume now that $|X| \geq 3$. If any two distinct vertices $w, t \in X \backslash \{x\}$ are not adjacent in G_1, then

$$\begin{aligned} |E(Y,X)| &\geq |E(Y, X\backslash\{x\})| + 1 \\ &\geq (d-1)|X\backslash\{x\}| + 1 \\ &\geq 2(d-1) + 1 = 2d - 1. \end{aligned}$$

Now, let us suppose that there exist $w, t \in X\backslash\{x\}$ such that they are adjacent in G_1. By Lemma 2.3, there are $2d-2$ internally disjoint directed paths from $\{u,v\}$ to $\{w,t\}$ in $K(d,n)$. Let B be the set of edges of these paths that are in (Y,X). Then $|B| = 2d-2$.

Clearly, $|E(Y,X)| \geq |B| + |(y,x)| = 2d-1$ if $(y,x) \notin B$. Assume $(y,x) \in B$ below.

If $A_y^+ \cap X \neq \emptyset$ then, since $\{(y,z) : z \in A_y^+ \cap X\}$ is not in B, we have

$$|E(Y,X)| \geq |B| + |A_y^+ \cap X| \geq 2d - 2 + 1 = 2d - 1.$$

Assume $A_y^+ \cap X = \emptyset$. If $A_x^- \cap Y = \emptyset$, then $E(A_x^-, A_y^+) \subseteq E(X,Y)$. If $d \geq 3$, by Lemma 2.2 (i), we have

$$\begin{aligned} |E(Y,X)| &= |E(X,Y)| \geq |E(A_x^-, A_y^+)| + 1 \\ &= (d-1)^2 + 1 \geq 2d - 1. \end{aligned}$$

If $d = 2$, noting that $E(A_y^-, A_x^+)$ has only one edge e and $d^-(y) = 2$. If $e \in E(Y,X)$ then $e \notin B$, we have

$$|E(Y,X)| \geq |B| + 1 = (2d-2) + 1 = 2d - 1.$$

If $e \notin E(Y,X)$ then $(A_x^+ \cup A_y^-) \subset X$ or $(A_x^+ \cup A_y^-) \subset Y$. By Lemma 2.2 (i), we have that

$$\begin{aligned} |E(Y,X)| &= |E(X,Y)| \geq |E(A_x^-, A_y^+)| + 2 \\ &\geq (d-1)^2 + 2 \geq 2d - 1. \end{aligned}$$

If $A_x^- \cap Y \neq \emptyset$ then, since $\{(w,x): w \in A_x^- \cap Y\}$ is not in B, we have

$$|E(Y,X)| \geq |B| + |A_x^- \cap Y| \geq 2d - 2 + 1 = 2d - 1.$$

Thus, all cases imply that $|E(Y,X)| \geq 2d-1$ and so the proof of the theorem is complete. ∎

Proof of Corollary It is a simple observation from the theorem and the definition of $UK(d,n)$ that $\lambda'(UK(d,n)) = \xi(UK(d,n))$ except $UK(2,1)$ and $UK(2,2)$ and, hence, $UK(d,n)$ is λ'-optimal.

Note $\lambda(UK(d,n)) = \delta(UK(d,n))$, $\delta(UK(d,1)) = d$, $\delta(UK(d,n)) = 2d-1$ for $n \geq 2$. By the theorem, $\lambda'(UK(d,n)) > \lambda(UK(d,n))$ except $UK(2,2)$ and, hence, $UK(d,n)$ is super edge-connected. ∎

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