Constrained switchings in cubic graphs

Avapa Chantasartrassmee*
Department of Mathematics, School of Science,
The University of the Thai Chamber of Commerce,
Bangkok 10400, Thailand

Narong Punnim†
Department of Mathematics, Srinakharinwirot University,
Sukhumvit 23, Bangkok 10110, Thailand
e-mail: narongp@swu.ac.th

Abstract

The graph $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$ is a graph whose vertices are the graphs with degree sequence $\mathbf{d}$, two vertices are adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. It has been shown that the graph $\mathcal{R}(\mathbf{d})$ is connected. Let $\mathcal{CR}(d)$ be the set of connected graphs with degree sequence $\mathbf{d}$. Taylor [13] proved that the subgraph of $\mathcal{R}(\mathbf{d})$ induced by $\mathcal{CR}(d)$ is connected. Several connected subgraphs of $\mathcal{CR}(3^n)$ are obtained in this paper. As an application, we are able to obtain the interpolation and extremal results for the number of maximum induced forests in the classes of connected subgraphs of $\mathcal{CR}(3^n)$.

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1 Introduction

Only finite simple graphs are considered in this paper. For the most part, our notation and terminology follows that of Bondy and Murty [2]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We will use the following notation and terminology for a typical graph G. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. As usual $|S|$ denote the cardinality of a set S and therefore we define $n = |V|$ to be the *order* of G and $m = |E|$ the *size* of G. To simplify writing, we

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write $e = uv$ for the edge e that joins the vertex u to the vertex v. A *path* of length k in a graph G, denoted by P_k, is a sequence of distinct vertices $u_1, u_2, \ldots, u_k$ of G such that for all $i = 1, 2, \ldots, k-1$, $u_i u_{i+1}$ are edges of G. A u, v-path is a path which has u as its first vertex and v as its last vertex in the path. The *degree* of a vertex v of a graph G is defined as $d_G(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|$. The maximum degree of a graph G is usually denoted by $\Delta(G)$. If $S \subseteq V(G)$, the graph $G[S]$ is the subgraph induced by S in G. For a graph G and $X \subseteq E(G)$, we denote $G - X$ the graph obtained from G by removing all edges in X. If $X = \{e\}$, we write $G - e$ for $G - \{e\}$. For a graph G and $X \subseteq V(G)$, the graph $G - X$ is the graph obtained from G by removing all vertices in X and all edges incident with vertices in X. For a graph G and $X \subseteq E(\overline{G})$, we denote $G + X$ the graph obtained from G by adding all edges in X. If $X = \{e\}$, we simply write $G + e$ for $G + \{e\}$. For two disjoint graphs G and H (i.e. $V(G) \cap V(H) = \emptyset$), we denote $G \cup H$ their union, and define pG the union of p copies of G. For a graph G and $s \in V(G)$, the *neighborhood* of s in G is defined by

$$\mathcal{N}(s) = \{v \in V(G) : sv \in E(G)\}.$$

If $S \subseteq V(G)$, then we define

$$\mathcal{N}(S) = \bigcup_{s \in S} \mathcal{N}(s).$$

If $F \subseteq V(G)$, we write $\mathcal{N}_F(S)$ for $\mathcal{N}(S) \cap F$. A graph G is said to be *regular* if all of its vertices have the same degree. A 3-regular graph is called *cubic graph.*

Let G be a graph of order n and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ is called a *degree sequence* of G, and we simply write $(d(v_1), d(v_2), \ldots, d(v_n))$ if the underlying graph G is clear from the context. A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph G. In this case, G is called a *realization* of $\mathbf{d}$.

An algorithm for determining whether or not a given sequence of non-negative integers is graphic was independently obtained by Havel [6] and Hakimi [5]. We state their results in the following theorem.

Theorem 1.1 *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a non-increasing sequence of non-negative integers and denote the sequence*

$$(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n) = \mathbf{d}'.$$

Then $\mathbf{d}$ *is graphic if and only if* $\mathbf{d}'$ *is graphic.*

■

Let G be a graph and $ab, cd \in E(G)$ be independent, where $ac, bd \notin E(G)$. Put

$$G^{\sigma(a,b;c,d)} = (G - \{ab, cd\}) + \{ac, bd\}.$$

The operation $\sigma(a, b; c, d)$ is called a *switching operation.* It is easy to see that the graph obtained from G by a switching has the same degree sequence as G. The following theorem has been shown by Havel [6] and Hakimi [5].

Theorem 1.2 *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a graphic degree sequence. If* G_1 *and* G_2 *are any two realizations of* $\mathbf{d}$*, then* G_2 *can be obtained from* G_1 *by a finite sequence of switchings.*

■

As a consequence of Theorem 1.2, Eggleton and Holton [3] defined in 1978 *the graph* $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$ whose vertices are the graphs with degree sequence $\mathbf{d}$; two vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. They obtained the following theorem.

Theorem 1.3 *The graph* $\mathcal{R}(\mathbf{d})$ *is connected.*

■

The following theorem was shown by Taylor [13] in 1980.

Theorem 1.4 *For a graphic degree sequence* $\mathbf{d}$*, let* $\mathcal{CR}(d)$ *be the set of all connected realizations of* $\mathbf{d}$*. Then the induced subgraph* $\mathcal{CR}(\mathbf{d})$ *of* $\mathcal{R}(\mathbf{d})$ *is connected.*

■

2 Connected subgraphs of $\mathcal{CR}(3^n)$

Let $\mathcal{CR}(3^n)$ be the class of connected cubic graphs of order n. Put $\mathbb{J}_1 = \mathcal{CR}(3^n)$. Let $\mathbb{J}_2$ be the class of all connected cubic K_4'-free graphs of order n, where K_4' is a graph obtained from K_4 and a subdivision to an edge. Finally, let $\mathbb{J}_3$ be the class of all connected cubic triangle-free graphs of order n. It is clear that $\mathbb{J}_3 \subseteq \mathbb{J}_2 \subseteq \mathbb{J}_1$. Let $X_n = \{\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3, \mathbb{J}_1 - \mathbb{J}_2, \mathbb{J}_1 - \mathbb{J}_3, \mathbb{J}_2 - \mathbb{J}_3\}$.

The notation $\mathcal{CR}(3^n), \mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3$, and X_n as described above will be used throughout this paper.

Let $\mathbb{J} \in X_n$ and $G \in \mathbb{J}$. A switching σ is called a $\mathbb{J}$*-switching with respect to* G if $G^\sigma \in \mathbb{J}$. A sequence $\sigma_1, \sigma_2, \ldots, \sigma_t$ of switchings is called a *sequence of* $\mathbb{J}$*-switchings with respect to* G if for all $i = 1, 2, \ldots, t, G^{\sigma_1\sigma_2\cdots\sigma_i} \in \mathbb{J}$.

As a consequence of Taylor [13], one can see that if $G_1, G_2 \in \mathbb{J}_1$, then there exists a sequence of $\mathbb{J}_1$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to G_1 such that $G_1^{\sigma_1\sigma_2\cdots\sigma_t} = G_2$. Thus the subgraph of $\mathcal{R}(3^n)$ induced by $\mathbb{J}_1$ is connected.

Let G be a connected graph of order at least three. A vertex $v \in V(G)$ is called a *cut vertex* of G if $G - v$ contains at least 2 components. An edge $e \in E(G)$ is called a *cut edge* of G if $G - e$ contains at least 2 components. A connected graph G is 2-*connected* if G contains no cut vertices and it is called 2-*edge connected* if it contains no cut edges. A maximal 2-connected subgraph of a connected graph G is called a *block* of G. Thus a 2-connected graph is a block. It is well known that if G is a connected graph containing two distinct blocks B_1 and B_2, then $E(B_1) \cap E(B_2) = \emptyset$ and $|V(B_1) \cap V(B_2)| \leq 1$. If B_1 and B_2 have a vertex v in common, then v is a cut vertex of G. Furthermore if B is a block of order at least three, for any two distinct vertices of B, there is a cycle in B containing the two vertices. Let G be a graph and r be a positive integer. An r-regular spanning subgraph of G is called an *r-factor* of G. Petersen [8] proved the following theorem.

Theorem 2.1 *If G be a 2-connected cubic graph, then G contains a 2-factor.*

■

For a 2-connected cubic graph G, G may contain several 2-factors. If we choose one such 2-factor F of G, then $F' = (V(G), E(G) - E(F))$ is a 1-factor of G.

A connected cubic graph G which contains K_4' as its subgraph is not 2-connected. We first consider the class $\mathbb{J}_1 - \mathbb{J}_2$ separately. Observe further that $\mathbb{J}_1 - \mathbb{J}_2 \neq \emptyset$ if and only if n is even and $n \geq 10$.

Theorem 2.2 *Let $G_1, G_2 \in \mathbb{J} = \mathbb{J}_1 - \mathbb{J}_2$. Then $G_1 \cong G_2$ or there exists a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to G_1 such that $G_1^{\sigma_1\sigma_2\cdots\sigma_t} \cong G_2$.*

Proof. Let $\mathbb{J} = \mathbb{J}_1 - \mathbb{J}_2$ and $G_1, G_2 \in \mathbb{J}$. Since G_1 and G_2 contain K_4' as induced subgraph, there exist $e \in G_1$ and $f \in G_2$ such that $G_1 - e = G_1' \cup K_4'$ and $G_2 - f = G_2' \cup K_4'$. It turns out that the graphs G_1' and G_2' are connected and have the same degree sequence. Thus $G_1' \cong G_2'$ or by Taylor [13], there exists a sequence of switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that $G_1'^{\sigma_1\sigma_2\cdots\sigma_t} \cong G_2'$. The sequence of switchings can be considered as a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to G_1 and $G_1^{\sigma_1\sigma_2\cdots\sigma_t} \cong G_2$. Thus the proof is complete. ■

Since the class of cubic graphs of order $n \leq 10$ are known in [12] and they are easy to verify as our purpose, we will consider from now on only the connected cubic graphs of order n where $n \geq 12$. Let $\mathbb{J} \in X_n$ and $\mathbb{J} \neq \mathbb{J}_1 - \mathbb{J}_2$. We have the following theorems.

Theorem 2.3 *Let $\mathbb{J} \in X_n$, $\mathbb{J} \neq \mathbb{J}_1 - \mathbb{J}_2$, and $G \in \mathbb{J}$. If G contains a cut vertex, then there exists a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to G such that $G^{\sigma_1\sigma_2\cdots\sigma_t}$ is 2-connected.*

Proof. Let $G \in \mathbb{J}$ and G is not 2-connected. Since G is cubic, G contains at least three blocks. Let S be the set of all cut vertices of G and $v \in S$. Since $d_G(v) = 3$, there exists a component G_i of $G - v$ such that v is adjacent to exactly one vertex u of G_i and this vertex is also a cut vertex of G. Thus $|S| \geq 2$ and $|S| = 2$ if and only if G contains exactly three blocks. We will proceed by induction on the number of blocks, $b(G)$, of a graph G. If $b(G) = 3$, there exists a cut set of vertices $S = \{u, v\}$ of G. Thus $uv \in E(G)$ and the blocks of G are B_1, B_2, and $B_3 = G[\{u, v\}]$ such that B_1 and B_2 contain u and v respectively. It is clear that B_1 and B_2 have order at least 5.

If $\mathbb{J} \in \{\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3\}$, then choose $e = ab \in E(B_1)$ and $f = cd \in E(B_2)$ such that e is not adjacent to u and f is not adjacent to v. Thus $\sigma(a, b; c, d)$ is a $\mathbb{J}$-switching with respect to G and $G^{\sigma(a,b;c,d)}$ is 2-connected.

If $\mathbb{J} \in \{\mathbb{J}_1 - \mathbb{J}_3, \mathbb{J}_2 - \mathbb{J}_3\}$ and $G \in \mathbb{J}$, then at least one of B_i must contain a triangle as its subgraph. Suppose B_1 contains a triangle T. Choose $e = ab \in E(B_1)$ and $f = cd \in E(B_2)$ such that $e \notin E(T)$. Thus $\sigma(a, b; c, d)$ is a $\mathbb{J}$-switching with respect to G and $G^{\sigma(a,b;c,d)}$ is a 2-connected graph containing a triangle.

Let k be an integer greater than 2 and suppose that the theorem is true for all connected cubic graphs H with $b(H) < k$. Let G be a connected cubic graph with $b(G) = k$. Thus there exist two vertex disjoint blocks B_1 and B_2 of G, each of which has order at least 5. We can analogously define the switching $\sigma_1 = \sigma(a, b; c, d)$ as described above and the resulting graph $G_1 = G^{\sigma_1}$ reduces the number of blocks of G by one. By induction, there exists a sequence of $\mathbb{J}$-switchings $\sigma_2, \sigma_3, \ldots, \sigma_t$ with respect to G_1 such that $G^{\sigma_1\sigma_2\cdots\sigma_t} = G_1^{\sigma_2\sigma_3\cdots\sigma_t}$ is 2-connected. This completes the proof. ■

Theorem 2.4 *Let $\mathbb{J} \in X_n$, $\mathbb{J} \neq \mathbb{J}_1 - \mathbb{J}_2$, and $G \in \mathbb{J}$. If G is 2-connected, then G is hamiltonian or there exists a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to G such that $G^{\sigma_1\sigma_2\cdots\sigma_t}$ is hamiltonian.*

Proof. Let G be a 2-connected cubic graph. By Theorem 2.1, there is a 2-factor F of G. Let $c(F)$ be the number of cycle in F. Thus we may choose F of G with minimum $c(F)$. If $c(F) = 1$, then G is hamiltonian. Suppose $c(F) = 2$ and $F = C_1 \cup C_2$.

If $\mathbb{J} \in \{\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3\}$, then choose $e = ab \in E(C_1)$ and $f = cd \in E(C_2)$ such that a is not adjacent to c and b is not adjacent to d. Thus $\sigma(a, b; c, d)$ is a $\mathbb{J}$-switching with respect to G and $G^{\sigma(a,b;c,d)}$ is a hamiltonian graph.

If $\mathbb{J} \in \{\mathbb{J}_1 - \mathbb{J}_3, \mathbb{J}_2 - \mathbb{J}_3\}$ and $G \in \mathbb{J}$, then at least one of C_i must contain three vertices which induces a triangle in G. Suppose C_1 contains three vertices which induces a triangle T and C_1 has order at least 4. Choose $e = ab \in E(C_1)$ and $f = cd \in C_2$ such that $e \notin E(T)$. Thus $G^{\sigma(a,b;c,d)}$ is hamiltonian containing a triangle. Suppose C_1 has order 3 and $V(C_1) = \{x, y, z\}$. Then C_2 has order at least 9. Let $\{x', y', z'\} \in V(C_2)$ such that $xx', yy', zz' \in E(G)$. Since G is not hamiltonian, $x'y', y'z', x'z' \notin E(C_2)$. Let $u \in V(C_2)$ and $ux' \in E(C_2)$. Thus there is a vertex $v \in V(C_2)$ such that $uv \in E(G)$. Since $x'y', x'z' \notin E(C_2)$, $u \notin \{y', z'\}$.

Case 1. Suppose that $y'v \notin E(G)$ or $z'v \notin E(G)$. Without loss of generality, we may assume that $y'v \notin E(G)$. Then $G^{\sigma(y,y';u,v)}$ is a hamiltonian graph containing a triangle.

Case 2. Suppose that $y'v \in E(G)$ and $z'v \in E(G)$. Since $\mathcal{N}(v) = \{u, y', z'\}$, there is at least one edge $e \in \{y'v, z'v\}$ satisfying $e \in E(C_2)$. Without loss of generality, we may assume that $z'v \in E(C_2)$. Therefore if $y'u \in E(G)$, then $G^{\sigma(x,x';u,v)}$ is a hamiltonian graph containing a triangle and if $y'u \notin E(G)$, then $G^{\sigma(y,y';v,u)}$ is a hamiltonian graph containing a triangle.

Suppose that $c(F) = k \geq 3$, we can analogously apply an appropriate $\mathbb{J}$-switching or a sequence of $\mathbb{J}$-switchings with respect to G to obtain a 2-connected cubic graph G' containing a 2-factor F' with $c(F') < c(F)$. This completes the proof. ■

Let n be an integer and $n \geq 6$. Let $G(2n)$ be a graph of order $2n$ with $V(G(2n)) = \{v_0, v_1, \ldots, v_{2n-1}\}$ and $E(G(2n)) = \{v_i v_{i+1} : i = 0, 1, \ldots, 2n-1 \pmod{2n}\} \cup \{v_0 v_n, v_1 v_{n+1}\} \cup \{v_j v_{2n-j+1} : j = 2, 3, \ldots, n-1\}$.

It is clear that $G(2n)$ is a cubic triangle-free graph containing a hamiltonian cycle C with $E(C) = \{v_i v_{i+1} : i = 0, 1, \ldots, 2n-1 \pmod{2n}\}$.

Let $\mathbb{J} \in \{\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3\}$ and let $G \in \mathbb{J}$. Suppose further that G is hamiltonian. Up to isomorphism, we may assume that G contains C as its hamiltonian cycle. Thus the following sets are uniquely determined.

$RV(G) = \{v_i \in V(G) : i = 1, 2, 3, \ldots, n\}$,
$LV(G) = V(G) - RV(G)$,
$[L, R](G) = \{uv \in E(G) : u \in LV(G) \text{ and } v \in RV(G)\} - E(C)$,
$[R, L](G) = \{uv \in E(G) : u \in RV(G) \text{ and } v \in LV(G)\} - E(C)$,
$[L, L](G) = \{uv \in E(G) : u, v \in L(G)\} - E(C)$, and
$[R, R](G) = \{uv \in E(G) : u, v \in RV(G)\} - E(C)$.

Since $e \in [L,R](G)$ if and only if $e \in [R,L](G)$, we have $[L,R](G) = [R,L](G)$ and $|[L,L](G)| = |[R,R](G)|$. Let $e = v_iv_j \in [L,L](G)$. Then there exists $f = v_sv_t \in [R,R](G)$. If $i \in \{0, 2n-1\}$ and $s \in \{1,2\}$, then $\sigma_1 = \sigma_1(v_i,v_j;v_t,v_s)$ is a $\mathbb{J}$-switching with respect to G. Thus $|[L,L](G^{\sigma_1})| = |[R,R](G^{\sigma_1})| = |[L,L](G)| - 1$. If $i \in \{n+1, n+2\}$ and $s \in \{n-1, n\}$, then $\sigma_1 = \sigma_1(v_i,v_j;v_t,v_s)$ is a $\mathbb{J}$-switching with respect to G. Thus $|[L,L](G^{\sigma_1})| = |[R,R](G^{\sigma_1})| = |[L,L](G)| - 1$. In other case, a switching $\sigma_1 = \sigma_1(v_i,v_j;v_s,v_t)$ is a $\mathbb{J}$-switching with respect to G. Thus the number of $[L,L](G^{\sigma_1})$ can be decreased by 2 from the number of $[L,L](G)$. With this observation, there exists a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that $[R,R](G^{\sigma_1\sigma_2\cdots\sigma_t}) = [L,L](G^{\sigma_1\sigma_2\cdots\sigma_t}) = \emptyset$.

Theorem 2.5 *Let $\mathbb{J} \in \{\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3\}$. If $G \in \mathbb{J}$ and $[R,R](G) = \emptyset$, then $G \cong G(2n)$ or there exists a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to G such that $G^{\sigma_1\sigma_2\cdots\sigma_t} \cong G(2n)$.*

Proof. If $v_0v_n \notin E(G)$, then there exist $i \in \{3, 4, \ldots, n-1\}$ and $j \in \{n+3, n+4, \ldots, 2n-1\}$ such that $v_0v_i, v_nv_j \in E(G)$. Then $G^{\sigma_1} = G^{\sigma(v_0,v_i;v_n,v_j)}$ is a $\mathbb{J}$-switching with respect to G and $v_0v_n \in [L,R](G^{\sigma(v_0,v_i;v_n,v_j)})$. Similarly if $v_1v_{n+1} \notin E(G^{\sigma(v_0,v_i;v_n,v_j)})$, then there exists a $\mathbb{J}$-switching σ_2 with respect to $G_1 = G^{\sigma_1}$ such that $G_2 = G^{\sigma_1\sigma_2}$ and $v_0v_n, v_1v_{n+1} \in G_2$. If $v_2v_i \in E(G_2)$ and $i \neq 2n-1$, then there exists j, $3 \leq j \leq n-2$, such that $v_{2n-1}v_j \in E(G_2)$ and $v_2v_{2n-1} \in E(G_3) = E(G_2^{\sigma_3})$, where $\sigma_3 = \sigma(v_2, v_i; v_{2n-1}, v_j)$. Let j be smallest integer in $\{3, 4, \ldots, n-1\}$ such that $v_jv_{2n-j+1} \notin E(G_3)$. Then there exist i and k such that $j < k < n$, $n+1 < i < 2n-j+1$ and $v_jv_i, v_kv_{2n-j+1} \in E(G_3)$. Thus $v_jv_{2n-j+1} \in E(G_4) = E(G_3^{\sigma_4})$, where $\sigma_4 = \sigma(v_j, v_i; v_{2n-j+1}, v_k)$. Similarly there exists a sequence of $\mathbb{J}$-switchings $\sigma_5, \sigma_6, \ldots, \sigma_t$ with respect to G_4 such that $G(2n) \cong G_4^{\sigma_5\sigma_6\cdots\sigma_t} = G^{\sigma_1\sigma_2\cdots\sigma_t}$. This completes the proof. ■

By Theorem 2.5, we have the following theorem.

Theorem 2.6 *Let $\mathbb{J} \in \{\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3\}$. Then the subgraph of $\mathcal{CR}(3^n)$ induced by $\mathbb{J}$ is connected.*

■

Let n be an integer and $n \geq 6$. Let $H(2n)$ be a graph of order $2n$ with $V(H(2n)) = \{v_0, v_1, \ldots, v_{2n-1}\}$ and $E(H(2n)) = \{v_iv_{i+1} : i = 0, 1, \ldots, 2n-1 (\text{mod } 2n)\} \cup \{v_0v_2, v_1v_{2n-1}\} \cup \{v_{n-1}v_{n+1}, v_nv_{n+2}\} \cup \{v_jv_{2n-j+1} : j = 3, 4, \ldots, n-2\}$.

It is clear that $H(2n)$ is a cubic hamiltonian graph of order $2n$ containing a triangle. Again put C as its hamiltonian cycle with $E(C) = \{v_iv_{i+1} : i = 0, 1, \ldots, 2n-1 (\text{mod } 2n)\}$.

Let $\mathbb{J} \in \{\mathbb{J}_1 - \mathbb{J}_3, \mathbb{J}_2 - \mathbb{J}_3\}$ and let $G \in \mathbb{J}$. Suppose further that G is hamiltonian. Up to isomorphism, we may suppose that G contains C as its hamiltonian cycle. Thus the following sets are uniquely determined.

$RV(G) = \{v_i \in V(G) : i = 1, 2, 3, \ldots, n\}$,
$LV(G) = V(G) - RV(G)$,
$[L, R](G) = \{uv \in E(G) : u \in LV(G) \text{ and } v \in RV(G)\} - E(C)$,
$[R, L](G) = \{uv \in E(G) : u \in RV(G) \text{ and } v \in LV(G)\} - E(C)$,
$[L, L](G) = \{uv \in E(G) : u, v \in L(G)\} - E(C)$, and
$[R, R](G) = \{uv \in E(G) : u, v \in RV(G)\} - E(C)$.

Since $e \in [L, R](G)$ if and only if $e \in [R, L](G)$, we have $[L, R](G) = [R, L](G)$ and $|[L, L](G)| = |[R, R](G)|$. Suppose that $v_0v_2 \notin E(G)$. Thus there exist i, j such that $v_0v_i, v_2v_j \in E(G)$ and $G_1 = G^{\sigma(v_0,v_i;v_2,v_j)}$ contains v_0v_2 as its edge. By using the same argument as described previously, there exists a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to G such that $G_t = G^{\sigma_1\sigma_2\cdots\sigma_t}$ and $[L, L](G_t) = \emptyset$. Again by using the same argument as described in the proof of Theorem 2.5 we have the following theorem.

Theorem 2.7 *Let $\mathbb{J} \in \{\mathbb{J}_1 - \mathbb{J}_3, \mathbb{J}_2 - \mathbb{J}_3\}$. If $G \in \mathbb{J}$ and $[R, R](G) = \emptyset$, then $G = H(2n)$ or there exists a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to G such that $G^{\sigma_1\sigma_2\cdots\sigma_t} \cong H(2n)$.*

■

By Theorem 2.7, we have the following theorem.

Theorem 2.8 *Let $\mathbb{J} \in \{\mathbb{J}_1 - \mathbb{J}_3, \mathbb{J}_2 - \mathbb{J}_3\}$. Then the subgraph of $\mathcal{CR}(3^n)$ induced by $\mathbb{J}$ is connected.*

■

Combining results in this section we can conclude the following theorem.

Theorem 2.9 *Let $\mathbb{J} \in X_n$. Then the subgraph of $\mathcal{CR}(3^n)$ induced by $\mathbb{J}$ is connected.*

■

3 Interpolation and Extremal Results

Let $\mathbb{G}$ be the class of all simple graphs, a function $f : \mathbb{G} \to \mathbb{Z}$ is called a *graph parameter* if $G \cong H$, then $f(G) = f(H)$. If f is a graph parameter and $\mathbb{J} \subseteq \mathbb{G}$, f is called an *interpolation graph parameter with respect to* $\mathbb{J}$ if there exist integers x and y such that

$$\{f(G) : G \in \mathbb{J}\} = [x, y] = \{k \in \mathbb{Z} : x \leq k \leq y\}.$$

Studying interpolation theorems for graph parameters may be divided into two parts, the first part deals with the question that given a graph parameter f and a subset $\mathbb{J}$ of $\mathbb{G}$, does f interpolate over $\mathbb{J}$? If f interpolates over $\mathbb{J}$, then $\{f(G) : G \in \mathbb{J}\}$ is uniquely determined by

$$\min(f, \mathbb{J}) = \min\{f(G) : G \in \mathbb{J}\} \text{ and } \max(f, \mathbb{J}) = \max\{f(G) : G \in \mathbb{J}\}.$$

Thus the second part of the interpolation theorems for graph parameters is to find the values of $\min(f, \mathbb{J})$ and $\max(f, \mathbb{J})$ for the corresponding interpolation graph parameter f and the set $\mathbb{J}$ and this part falls into the category of the extremal problems in graph theory.

Acyclic graph is a graph containing no cycle as its subgraph. An acyclic graph is called a forest. Therefore, each component of an acyclic graph is a tree.

Erdős et al. [4] first defined a graph parameter t as follows. Let G be a graph and $F \subseteq V(G)$. F is called an *induced forest* of G if $G[F]$ is acyclic. An induced forest F of G is *maximal* if for every $v \in V(G) - F$, $F \cup \{v\}$ is not an induced forest of G. Let $t(G)$ be defined as

$$t(G) := \max\{|F| : F \text{ is an induced forest of } G\}.$$

Thus $t(G)$ is the maximum cardinality of induced forests of G. An induced forest F of G with $|F| = t(G)$ is call a *maximum induced forest* of G.

The second author proved in [9] that if G is a graph and σ is a switching on G, then $|t(G) - t(G^\sigma)| \leq 1$. With this result and the results in Section 2 we have the following theorem.

Theorem 3.1 *Let $\mathbb{J} \in X_n$. Then t is an interpolation graph parameter with respect to $\mathbb{J}$.*

■

We now answer the second part of interpolation theorem. In other words, we will find $\min(t, \mathbb{J})$ and $\max(t, \mathbb{J})$ for all $\mathbb{J} \in X_n$.

Let G be a graph and X, Y be disjoint nonempty subsets of $V(G)$. Denote by $e(X)$ the number of edges in $G[X]$ and $e(X, Y)$ the number of edges in G connecting vertices in X to vertices in Y.

Let G be a cubic graph of order n and F be a maximum induced forest of G. Let $|F| = f$ and therefore $G - F$ has order $n - f$. An upper bound of $\max(t, \mathbb{J}_1)$ can be obtained by the following obvious identities.

1. $\frac{3n}{2} = |E(G)| = e(F) + e(F, V(G - F)) + e(V(G - F))$
2. $3(n - f) = e(F, V(G - F)) + 2e(V(G - F))$.

It follows that
$\frac{3n}{2} = e(F) + 3(n-f) - e(V(G-F)) \le f - 1 + 3(n-f)$.
Therefore $f \le \frac{3n-2}{4}$.
Since $\mathbb{J}_3 \subseteq \mathbb{J}_2 \subseteq \mathbb{J}_1$, $\max(t, \mathbb{J}_3) \le \max(t, \mathbb{J}_2) \le \max(t, \mathbb{J}_1) \le \frac{3n-2}{4}$.

We will show in the next theorem that $\max(t, \mathbb{J}_3) = \lfloor \frac{3n-2}{4} \rfloor$ by constructing a connected cubic triangle-free graph G of order n and $t(G) = \lfloor \frac{3n-2}{4} \rfloor$. Note that several classes of graphs can be constructed to reach the bound even if we restrict that the graph is planar and $n \ge 8$. We now turn to consider the graph $G(2n)$ as we have constructed in Section 2. $G(2n)$ is a connected cubic triangle-free graph of order $2n$ if $n \ge 3$. We have the following theorem.

Theorem 3.2 *If $n = 2m \ge 6$, then $t(G(2m)) = \lfloor \frac{3n-2}{4} \rfloor$.*

Proof. Let $G = G(2m) - \{v_0v_m, v_1v_{m+1}\}$ and $S = \{v_1, v_3, \ldots, v_m\}$ if m is odd and $S = \{v_1, v_3, \ldots, v_{m-1}, v_m\}$ if m is even. Thus $F = G - S$ is a tree of order $\lfloor \frac{3m-1}{2} \rfloor = \lfloor \frac{3n-2}{4} \rfloor$. Since $v_1, v_m \in S$, adding edges v_0v_m, v_1v_{m+1} to G will not create any cycle in F. Therefore $t(G(2m)) = \lfloor \frac{3m-1}{2} \rfloor = \lfloor \frac{3n-2}{4} \rfloor$. ■

Corollary 3.3 $\max(t, \mathbb{J}_3) = \max(t, \mathbb{J}_2) = \max(t, \mathbb{J}_1) = \lfloor \frac{3n-2}{4} \rfloor$. ■

Let G be a connected cubic graph of order n containing K_4' as its subgraph. Put $G' = G - K_4'$. Thus G' has order $n-5$ and size $\frac{3n}{2} - 8$. Moreover $t(G) = t(G') + t(K_4') = t(G') + 3$. Let F' be a maximum induced forest of G' and $|F'| = f'$. We have
$\frac{3n}{2} - 8 = e(V(G')) \le (f'-1) + 3(n-5-f') = 3n - 16 - 2f'$.
It follows that $f' \le \frac{3n-16}{4}$ and $t(G) \le \frac{3n-16}{4} + 3 = \frac{3n-4}{4}$.

Theorem 3.4 $\max(t, \mathbb{J}_1 - \mathbb{J}_2) = \lfloor \frac{3n-4}{4} \rfloor$.

Proof. It is clear that $\mathbb{J}_1 - \mathbb{J}_2 \ne \emptyset$ if and only if n is even and $n \ge 10$. If $n = 10$, we can take G as a graph obtained from two copies of K_4' and joining the two vertices of degree 2. Let $n - 6 = 2m \ge 6$ and $H = G(2m)$. Thus by Theorem 3.2, we have $t(H) = \lfloor \frac{3(n-6)-2}{4} \rfloor$. In the proof of Theorem 3.2, let $e = v_0v_1 \in E(H)$ and define a graph K with $V(K) = V(H) \cup \{x\}$ and $E(K) = (E(H) - e) \cup \{v_0x, xv_1\}$ and finally let G be a graph obtained from K and a copy of K_4' by joining the two vertices of degree 2. Thus $t(G) = \lfloor \frac{3(n-6)-2}{4} \rfloor + 1 + 3 = \lfloor \frac{3(n-6)-2}{4} \rfloor + 4 = \lfloor \frac{3n-4}{4} \rfloor$. ■

Since $\mathbb{J}_2 - \mathbb{J}_3 \subseteq \mathbb{J}_1 - \mathbb{J}_3$, $\max(t, \mathbb{J}_2 - \mathbb{J}_3) \le \max(t, \mathbb{J}_1 - \mathbb{J}_3) \le \lfloor \frac{3n-2}{4} \rfloor$.

Theorem 3.5 $\max(t, \mathbb{J}_2 - \mathbb{J}_3) = \max(t, \mathbb{J}_1 - \mathbb{J}_3) = \lfloor \frac{3n-2}{4} \rfloor$.

Proof. Let m be an integer such that $n = 2m$. The graph $H(2m)$ contains a triangle. Let $S = \{v_1, v_3, \ldots, v_m\}$ if m is odd and $S = \{v_1, v_3, \ldots, v_{m-1}, v_m\}$ if m is even. Thus $F = G - S$ is a tree of order $\lfloor \frac{3n-2}{4} \rfloor$. Thus $\max(t, \mathbb{J}_2 - \mathbb{J}_3) = \max(t, \mathbb{J}_1 - \mathbb{J}_3) = \lfloor \frac{3n-2}{4} \rfloor$.

■

Thus the values of $\max(t, \mathbb{J})$ are obtained for all $\mathbb{J} \in X_n$.

The problem of finding lower bounds of $t(G)$, where G runs over a class of cubic graphs, have been investigated in the literature. First observe that if G is a cubic graph of order n and F is a maximum induced forest of G, then $G - F$ is a forest. Thus $|F| \geq \frac{n}{2}$. The bound is sharp if and only if n is a multiple of 4. The second author proved in [10] that if $n = 4q+t,\ t = 0, 2$, then $\min(t, \mathcal{R}(3^n)) = 2q+t$. Liu and Zhao [7] proved that $t(G) \geq \frac{5}{8}n - \frac{1}{4}$ for any connected cubic graph G of order n. This means that for any connected cubic graph of order n, the bound of the order of maximum induced forest can be improved from 50 percent of the number of vertices to 60 percent.

The second author proved in [11] the following result.

Theorem 3.6 *Let n be an even integer with $n \geq 12$. Then*

$$\min(t, \mathbb{J}_1) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & \text{if } n \equiv 2 \pmod 8, \\ \lceil \frac{5}{8}n \rceil & \text{otherwise.} \end{cases}$$

■

Note that a graph G that has been constructed in such a way that $t(G) = \min(t, \mathbb{J}_1)$ and G contains K_4' as its subgraph. Thus we have the following corollary.

Corollary 3.7 *Let n be an even integer with $n \geq 12$. If $\mathbb{J} \in \{\mathbb{J}_1, \mathbb{J}_1 - \mathbb{J}_2, \mathbb{J}_1 - \mathbb{J}_3\}$, then*

$$\min(t, \mathbb{J}) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & \text{if } n \equiv 2 \pmod 8, \\ \lceil \frac{5}{8}n \rceil & \text{otherwise.} \end{cases}$$

■

Zheng and Lu [14] proved that $t(G) \geq \frac{2n}{3}$ for any connected cubic graph G of order n without triangle, except for two cubic graphs with $n = 8$ and $t(G) = 5$. This means that if we look at the class of connected cubic graphs of order n containing no triangle, then the bound can be improved from 60 percent of the number of vertices to 66 percent.

It is easy to see that there exists cubic graph G of order n containing triangles and $t(G) \geq \frac{2n}{3}$.

We are able to extend the result of Zheng and Lu [14] by proving that $t(G) \geq \frac{2n}{3}$ for any connected cubic K_4'-free graph G of order $n \geq 10$.

Let H be a graph. A graph G is called an *H-free graph* if G does not contain H as an induced subgraph. Let X be a set of graphs. Then a graph G is called an *X-free graph* if for every $H \in X$, G is an H-free graph. In [12], five connected cubic graphs of order 8 are given, all of which have maximum induced forests of order 5. Alon et al. [1] proved that let G be a $\{K_4, K_4'\}$-free graph with maximum degree 3. If G is of order n and of size m, then $t(G) \geq n - \frac{m}{4}$. Consequently, if G is a connected cubic K_4'-free graph of order $n \geq 10$, then $t(G) \geq \frac{5n}{8}$.

Observe that if G is a connected K_4'-free graph of order 8 and $\Delta(G) = 3$, then $t(G) \geq 5$ and $t(G) = 5$ if and only if G is a cubic graph.

Theorem 3.8 *Let $X = \mathcal{CR}(3^8) \cup \{K_4, K_4'\}$ and let G be an X-free graph of order n with $\Delta(G) = 3$. Then $t(G) \geq \frac{2n}{3}$.*

Proof. Let $X = \mathcal{CR}(3^8) \cup \{K_4, K_4'\}$ and let G be an X-free graph of order n. By calculation we found that $\lceil \frac{5n}{8} \rceil = \lceil \frac{2n}{3} \rceil$ for all n with $4 \leq n \leq 10$ and $n \neq 8$. For $n = 8$ we also have $t(G) \geq \frac{2n}{3}$. Thus the theorem holds for all $4 \leq n \leq 10$. Now suppose $n \geq 11$ and G contains a triangle T with $V(T) = \{x, y, z\}$. If there exists a vertex in $V(T)$, say x, such that $d_G(x) = 2$, then by induction on n there exists a maximum induced forest F_1 of $G - T$ with $|F_1| \geq \frac{2(n-3)}{3}$. Hence $F = F_1 \cup \{x, y\}$ is an induced forest of G and $|F| \geq \frac{2n}{3}$. Suppose for all triangles $T = \{x, y, z\}$ of G, $d_G(x) = d_G(y) = d_G(z) = 3$. Since G is a K_4-free graph, $|N_G(T)| \geq 2$.

Case 1. Suppose x and y have a common neighbor u and v is a neighbor of z. Since G is a K_4'-free graph, u and v are not adjacent in G. Thus by induction on n, $G - T$ contains an induced forest of order at least $\frac{2(n-3)}{3}$. Since $d_{G-T}(u) = 1$, any maximum induced forest of $G - T$ must contain u. If there is a maximum induced forest F_1 of $G - T$ which does not contain u, v-path, then $F_1 \cup \{y, z\}$ is an induced forest of G of order at least $\frac{2n}{3}$. Suppose for any maximum induced forest F_1 of $G - T$, F_1 contains u, v-path. Since $G' = G - T + uv$ satisfies conditions of the theorem, there is a maximum induced forest F' of G' of order at least $\frac{2(n-3)}{3}$. If $uv \notin E(F')$, then $F = F' \cup \{y, z\}$ is a maximum induced forest of G of order at least $\frac{2n}{3}$. If $uv \in E(F')$, then $F = (F' - uv) \cup \{y, z\}$ is a maximum induced forest of G of order at least $\frac{2n}{3}$.

Case 2. Suppose x, y, z have different neighbors u, v and w respectively. Since $n \geq 11$, $G[\{u, v, w\}]$ is not a triangle. Thus there exist two vertices in $\{u, v, w\}$, say u, v, such that $uv \notin E(G)$. We consider the following subcases.

Subcase 2.1 Suppose that $G-T+uv$ contains K_4. Let H be the copy of K_4 in $G-T+uv$. Thus $uv \in E(H)$ and $vw \notin E(G)$. Since $G' = G-T-V(H)$ satisfies the condition of the theorem, $t(G') \geq \frac{2(n-7)}{3}$. Let F' be a maximum induced forest of G' and F_1 be a maximum induced forest of $H-uv$. Since $|F_1| = 3$ and there exists an induced forest $F = F' \cup \{y,z\} \cup F_1$ of G, $t(G) \geq |F| = |F' \cup \{y,z\} \cup F_1| \geq \frac{2(n-7)}{3} + 2 + 3 = \frac{2n+1}{3} \geq \frac{2n}{3}$.

Subcase 2.2 Suppose that $G-T+uv$ contains K_4'. Let H be the copy of K_4' in $G-T+uv$. Thus $uv \in E(H)$. Since $G' = G-T-V(H)-\{w\}$ satisfies the condition of the theorem, $t(G') \geq \frac{2(n-9)}{3}$. Let F' be a maximum induced forest of G' and F_1 be a maximum induced forest of $H-uv$. Since $|F_1| = 4$ and there exists an induced forest $F = F' \cup \{y,z\} \cup F_1$ of G, $t(G) \geq |F| = |F' \cup \{y,z\} \cup F_1| \geq \frac{2(n-9)}{3} + 2 + 4 = \frac{2n}{3}$.

Subcase 2.3 Suppose that $G-T+uv$ contains a cubic graph of order 8. Let H be the cubic graph of order 8 in $G-T+uv$. Thus $uv \in E(H)$. Since $G' = G-T-V(H)$ satisfies the condition of the theorem, $t(G') \geq \frac{2(n-11)}{3}$. Let F' be a maximum induced forest of G' and F_1 be a maximum induced forest of $H-uv$. Since $|F_1| = 6$ and there exists an induced forest $F = F' \cup \{y,z\} \cup F_1$ of G, $t(G) \geq |F| = |F' \cup \{y,z\} \cup F_1| \geq \frac{2(n-11)}{3} + 2 + 6 = \frac{2n+2}{3} > \frac{2n}{3}$.

Subcase 2.4 Suppose that $G-T+uv$ satisfies the condition of the theorem. Thus there exists a maximum induced forest F' of $G-T$ of order at least $\frac{2(n-3)}{3}$. If $uv \notin E(F')$, then $F = F' \cup \{x,y\}$ is an induced forest of G of order at least $\frac{2n}{3}$. If $uv \in E(F)$, then $F = (F_1 - uv) \cup \{x,y\}$ is an induced forest of G of order at least $\frac{2n}{3}$.

Thus the proof is complete. ■

The following corollary is an immediate consequence of Theorem 3.8.

Corollary 3.9 *Let G be a connected cubic K_4'-free graph of order $n \neq 8$. Then $t(G) \geq \frac{2n}{3}$.*

■

According to the result of Zheng and Lu [14] that $t(G) \geq \frac{2n}{3}$ for any connected cubic graph G of order n without triangles, except for two cubic graphs with $n = 8$. They mentioned in their paper that the lower bound is best possible but no proof of this was given in the paper.

We now construct a class of graphs to show that $\min(t, \mathbb{J}_3) = \min(t, \mathbb{J}_2) = \lceil \frac{2n}{3} \rceil$.

First observe the following facts:

1. $t(K_{3,3}) = 4$.
2. $t(Q_3) = 5$, where Q_3 is the 3-cube.

3. There is a switching σ such that $(2K_4')^\sigma$ is a connected triangle-free graph and $t((2K_4')^\sigma) = 7$. Let $K = (2K_4')^\sigma$.

4. If $e \in E(K_{3,3})$ and $f \in E(Q_3)$, then $t(K_{3,3}-e) = 4$ and $t(Q_3-f) = 6$. Let $P = K_{3,3} - e$ and $Q = Q_3 - f$.

5. Let n be an even integer with $n \geq 12$. Write $n = 6q + t,\ t = 0, 2, 4$ and construct a connected cubic triangle-free graph according to the values of t:

   (a) If $t = 0$, construct graph G of order $6q$ by taking q copies of P and joining q appropriate edges between the q copies of P.

   (b) If $t = 2$, construct graph G of order $6q+2$ by taking $q-1$ copies of P and a copy of Q and then joining q appropriate edges between them.

   (c) If $t = 4$, construct a graph G of order $6q + 4$ by taking $q - 1$ copies of P and a copy of K and then joining q appropriate edges between them.

It is easy to check that the graphs G constructed above satisfying $t(G) = \lceil \frac{2n}{3} \rceil$. Thus we have the following theorem.

Theorem 3.10 $\min(t, \mathbb{J}_2) = \min(t, \mathbb{J}_3) = \min(t, \mathbb{J}_2 - \mathbb{J}_3) = \lceil \frac{2n}{3} \rceil$.

■

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