Asymptotic Normality of the (r,β)-Stirling Numbers

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Abstract

In this paper , the unimodality of (r,β)-Stirling numbers and certain asymptotic approximation of (r,β)-Bell numbers are established. Together with these results and the most general form of Central Limit Theorem, viz. Bounded Variance Normal Convergence Criterion, the (r,β)-Stirling numbers are shown to be asymptotically normal.

1 Introduction

The normal distribution plays a very important role in a wide variety of applications of probability theory to physical problems and to problems of statistics. One of the reasons for this is found in a class of limit theorems known generically as the Central Limit Theorem. A class of independent variable may individually have distributions which are quite different from the normal distribution. But when these are summed and standardized in an approximate manner, the resulting random variable has a distribution which is approximately normal.

Harper [13] introduced a method for proving that a sequence of numbers satisfy the Local Limit Theorem (LLT) and, together with the asymptotic approximation of Moser and Wyman [16] for the Bell numbers, used the method in proving that the Stirling numbers of the second kind satisfy the LLT. This method was later generalized by Bender [2] and the generalization was improved by Canfield [3]. Rucinski and Voigt [18] used the improved generalized method in proving that LLT is satisfied by the numbers $S_k^n(\mathbf{a})$ defined by the relation

$$x^n = \sum_{k=0}^{n} S_k^n(\mathbf{a}) p_k^{\mathbf{a}}(x)$$

where $\mathbf{a}$ is the sequence $(a, a+r, a+2r, ...)$, for $a \geq 0$, $r \geq 0$ and

$$p_k^{\mathbf{a}}(x) = \prod_{i=0}^{k}[x - (a+ir)].$$

The numbers $S_k^n(\mathbf{a})$ when $\mathbf{a}$ is an arithmetic progression coincide with the (r, β)-Stirling numbers.

In this paper, we used the method of Moser and Wyman [16] to establish an asymptotic approximation of the (r, β)-Bell numbers. With this asymptotic approximation and following the method of Harper we proved that the (r, β)-Stirling numbers satisfy the LLT which is also known as the Bounded Variance Normal Convergence Criterion.

2 The (r, β)-Stirling Numbers

The (r, β)-**Stirling numbers**, denoted by $\left\langle{n \atop k}\right\rangle_{\beta,r}$ were defined in [8] by means of the following 'linear transformation':

$$t^n = \sum_{k=0}^{n} \left\langle{n \atop k}\right\rangle_{\beta,r} (t-r)_{\beta,k},$$

where $(t-r)_{\beta,k} = \prod_{i=0}^{k-1}(t - r - i\beta)$. The parameters r and β may be real or complex numbers. However, in this paper, we restrict r and β to be nonnegative real numbers.

Several properties of the (r, β)-Stirling numbers were given in [8]. To mention a few, we have the triangular recurrence relation

$$\left\langle{n+1 \atop k}\right\rangle_{\beta,r} = \left\langle{n \atop k-1}\right\rangle_{\beta,r} + (k\beta + r)\left\langle{n \atop k}\right\rangle_{\beta,r}, \tag{1}$$

the exponential generating function

$$\sum_{k=0}^{n} \left\langle{n \atop k}\right\rangle_{\beta,r} \frac{x^n}{n!} = \frac{1}{\beta^k k!} e^{rt} \left(e^{\beta x} - 1\right)^k, \tag{2}$$

and the explicit formulas

$$\left\langle {n \atop k} \right\rangle_{\beta,r} = \sum_{c_o+c_1+\ldots+c_k=n-k} \prod_{j=0}^{k} (\beta j + r)^{c_j}$$

$$\beta^k k! \left\langle {n \atop k} \right\rangle_{\beta,r} = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (\beta j + r)^n.$$

Moreover, the (r, β)-Stirling numbers were interpreted combinatorially based on the above explicit formulas. For the second formula, the (r, β)-Stirling numbers with multiplier $\beta^k k!$ were interpreted as follows:

Consider $k+1$ distinct cells the first k of which each has β compartments and the last cell with r distinct compartments. Suppose we distribute n distinct balls into $k+1$ cells one ball a time such that

(A1) the capacity of each compartment is unlimited;

(B1) the first k cells are nonempty.

Let Ω be the set of all possible ways of distributing n balls under restriction (A1). Then the number of outcomes in Ω satisfying (B1) is $\beta^k k! \left\langle {n \atop k} \right\rangle_{\beta,r}$ with $\beta, r \geq 0$.

3 Unimodality

One important concept in probability theory that may be helpful in analyzing the behavior of a certain distribution is unimodality. This concept will give us an idea whether a distribution will approach to normality. Here, we will show that the behavior of (r, β)-Stirling numbers is unimodal.

The following theorem in [6] is very useful in determining the unimodality of a real sequence.

Theorem 3.2 *If the generating polynomial*

$$P(x) = \sum_{k=0}^{n} v_k x^k, \; v_n \neq 0$$

of real nonnegative sequence v_k, $0 \leq k \leq n$ has only real roots ≤ 0, then v_k is unimodal, either with a peak or a plateau.

The following Lemma is also essential to the proof of the unimodality as well as the asymptotic normality of the (r, β)-Stirling numbers.

Lemma 3.3 *Let $Q_n(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{\beta,r} x^k$. Then the roots of Q_n are distinct real and nonpositive for all $n \geq 1$.*

Proof. Using the explicit formula of the (r,β)-Stirling numbers with $n=1$ we have $Q_1(x)=r+x$. Assume that $Q_{n-1}(x)$ has $n-1$ distinct nonpositive real roots, $n>2$. Now using the triangular recurrence relation (1) of the (r,β)-Stirling numbers, we have

$$Q_n(x)=x\beta\frac{d}{dx}\,Q_{n-1}(x)+(x+r)Q_{n-1}(x). \tag{3}$$

Let $\widehat{Q}_n(x)=Q_n(x)e^{\frac{x}{\beta}+\frac{r\ln x}{\beta}}$. Then differentiating $\widehat{Q}_{n-1}$ with respect to x and simplifying, we have

$$\frac{d}{dx}\,\widehat{Q}_{n-1}(x)=\frac{1}{\beta x}\left[\beta x\frac{d}{dx}\,Q_{n-1}(x)e^{\frac{x}{\beta}+\frac{r\ln x}{\beta}}+(x+r)Q_{n-1}(x)e^{\frac{x}{\beta}+\frac{r\ln x}{\beta}}\right].$$

Again, using (1), we get

$$\begin{aligned}\frac{d}{dx}\,\widehat{Q}_{n-1}(x)&=\frac{1}{\beta x}Q_n(x)e^{\frac{x}{\beta}+\frac{r\ln x}{\beta}}\\&=\beta x\frac{d}{dx}\,\widehat{Q}_{n-1}(x)=Q_n(x)e^{\frac{x}{\beta}+\frac{r\ln x}{\beta}}.\end{aligned}$$

Hence

$$\widehat{Q}_n(x)=\beta x\frac{d}{dx}\,\widehat{Q}_{n-1}(x).$$

Since $\widehat{Q}_{n-1}(x)$ has n distinct nonpositive real roots, by Rolle's Theorem, $\frac{d}{dx}\,\widehat{Q}_{n-1}(x)$ has $n-1$ distinct negative real roots. This implies that $\widehat{Q}_n(x)$ has n distinct nonpositive real roots. Therefore, the roots of Q_n must also be distinct real and nonpositive for all integer $n\geq 1$. □

Finally, the following result gives the unimodality of the (r,β)-Stirling numbers. This follows from Theorem 3.2 and Lemma 3.3 by taking $v_k=\left\langle {n \atop k} \right\rangle_{\beta,r}$.

Theorem 3.4 *The sequence* $\left\langle {n \atop k} \right\rangle_{\beta,r}$ *of the* (r,β)*-Stirling numbers, n fixed* (≥ 3)*, k variable* $(k\leq n)$*, is unimodal.*

4 An Asymptotic Formula for the (r,β)-Bell numbers

An Asymptotic Expansion

In this section, we will establish an asymptotic formula for the (r,β)-Bell numbers $G_{n,r,\beta}$ defined by

$$G_{n,r,\beta} = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{\beta,r}.$$

We will be using the method of Moser and Wymann [16] in obtaining the asymptotic formula for the Bell numbers. Here we assume that the parameter β is a positive real number.

We begin by obtaining a generating function for the (r,β)-Bell numbers.

Theorem 4.1 *For $\beta \neq 0$, (r,β)-Bell numbers $G_{n,r,\beta}$ have the following exponential generating function:*

$$\sum_{n=0}^{\infty} G_{n,r,\beta}\frac{x^n}{n!} = \exp\left[rx + \frac{e^{\beta x}-1}{\beta}\right]$$

where both e^x and $\exp x$ *denote the exponential function.*

Proof. Using the definition of $G_{n,r,\beta}$ and the exponential generating function for (r,β)-Stirling numbers, we can easily obtain the above generating function. □

By making use of the exponential generating function for $G_{n,r,\beta}$ and Cauchy's Theorem for integrals we obtain the integral representation

$$G_{n,r,\beta} = \frac{n!}{2\pi i}\int_{\gamma} \frac{\exp\left[rz + \frac{e^{\beta z}-1}{\beta}\right]}{z^{n+1}}dz,$$

where γ is the circle $z = Re^{i\theta}$, $-\pi \leq \theta \leq \pi$. Contour integration yields

$$G_{n,r,\beta} = \frac{n!}{2\pi i R^n}\int_{-\pi}^{\pi} \exp\left(\beta^{-1}e^{\beta Re^{i\theta}} + rRe^{i\theta} - in\theta - \beta^{-1}\right)d\theta,$$

which can be written into the compact form

$$G_{n,r,\beta} = A\int_{-\pi}^{\pi} \exp(F(\theta))d\theta, \tag{4}$$

where

$$A = \frac{n!\exp\left(rR + \beta^{-1}e^{\beta R} - \beta^{-1}\right)}{2\pi R^n},$$

and

$$F(\theta) = \beta^{-1}e^{\beta Re^{i\theta}} + rRe^{i\theta} - in\theta - (rR + \beta^{-1}e^{\beta R}). \tag{5}$$

Define $\epsilon = e^{\frac{-3R}{8}}$ and let

$$J_1 = \int_{-\pi}^{\epsilon} \exp(F(\theta))d\theta \quad \text{and} \quad J_2 = \int_{\epsilon}^{\pi} \exp(F(\theta))d\theta.$$

Thus (3) can be written as

$$G_{n,r,\beta} = AJ_1 + A\int_{\epsilon}^{\epsilon} \exp(F(\theta))d\theta + AJ_2.$$

Lemma 4.2 *There exists a constant $k > 0$ such that*

$$|J_2| < e^{-k\beta^{-1}e^{\beta R}}(\pi - \epsilon).$$

Proof. It can be shown that

$$|\exp(F(\theta))| = e^{-\left[(rR+\beta^{-1}e^{\beta R})+\beta^{-1}\cos(\beta R\sin\theta)e^{\beta R\cos\theta}\right]}.$$

Since $\cos\theta < 1$ for $0 < \epsilon < \theta \leq \pi$, we have

$$|\exp(F(\theta))| = e^{-\beta^{-1}e^{\beta R}}[1 - \cos(\beta R \sin\theta)].$$

Since $[1 - \cos(\beta R\sin\theta)] > 0$ for $\cos\theta < 1$ for $0 < \epsilon < \theta \leq \pi$, there exists a constant $k > 0$ such that $[1 - \cos(\beta R \sin\theta)] < k$. Hence

$$|J_2| < e^{-k\beta^{-1}e^{\beta R}}(\pi - \epsilon). \ \Box$$

It will be seen later that $R \to \infty$ as $n \to \infty$. With the result in Lemma 4.2 we see that J_1 and J_2 will tend to zero as $n \to \infty$. Hence

$$G_{n,r,\beta} \sim A\int_{-\epsilon}^{\epsilon} \exp(F(\theta))d\theta. \tag{6}$$

Observe that $F(\theta)$ is analytic at $\theta = 0$.Thus $F(\theta)$ has a Maclaurin series expansion about $\theta = 0$. This Maclaurin expansion can be written in the form

$$\begin{aligned} F(\theta) &= (Re^{\beta R} + rR - n)i\theta + \frac{1}{2}(\beta R^2 + Re^{\beta R} + rR)i^2\theta \qquad (7) \\ &\quad + \sum_{k=3}^{\infty} \left[\beta^{-1}\rho^k(e^{\beta R}) + rR\right](i\theta)^k. \end{aligned}$$

where we define ρ to be the operator $\rho = R\frac{d\theta}{dR}$. Choose R such that $Re^{\beta R} + rR - n = 0$, that is, R satisfies the equation $xe^{\beta R} + rx - n = 0$. This R is shown to exist in the following lemma.

Lemma 4.3 *There exists a unique positive real solution to the equation* $xe^{\beta R} + rx - n = 0$.

Proof. We can rewrite the given equation into the form

$$\frac{x}{n - rx} = e^{-\beta x}$$

The desired solution is the x-coordinate of the intersection of the functions $h(x) = \frac{x}{n-rx}$ and $g(x) = e^{-\beta x}$. □

It can be seen from the preceding Lemma that $R \to \infty$ as $n \to \infty$. With this choice of R, we have

$$F(\theta) = -\frac{1}{2}(\beta R^2 + Re^{\beta R} + rR)\theta + \sum_{k=3}^{\infty} \left[\beta^{-1}\rho^k(e^{\beta R}) + rR\right] (i\theta)^k.$$

We now introduce the following notations:

$$\begin{aligned} \phi &= \left[\frac{1}{2}(\beta R^2 e^{\beta R} + Re^{\beta R} + rR)^{1/2}\right] \theta \\ a_k &= \frac{\left[\beta^{-1}e^{-\beta R}\rho^{k+2}(e^{\beta R}) + rRe^{-\beta R}\right] (i\phi)^{k+2}}{(k+1)!\left[\frac{1}{2}(\beta R^2 + R + rRe^{-\beta R})\right]^{\frac{k+2}{2}}} \\ z &= e^{-\frac{\beta R}{2}} \\ f(z) &= \sum_{k=1}^{\infty} a_k z^k. \end{aligned} \tag{8}$$

Then $F(\theta) = -\phi^2 + f(z)$ and

$$G_{n,r,\beta} \sim C \int_{-h}^{h} \exp[-\phi^2 + f(z)]dz \tag{9}$$

where $h = \frac{1}{2}(\beta R^2 e^{\beta R} + Re^{\beta R} + rR)^{1/2} e^{\frac{-3R}{8}}$ and $C = \frac{A}{\left[\frac{1}{2}(\beta R^2 e^{\beta R} + rR)\right]^{1/2}}$.

We have defined z as a function of R. However, for the moment we consider z to be an independent variable and expand $e^{f(z)}$ into a convergent Maclaurin series expansion of the form

$$e^{f(z)} = \sum_{k=0}^{\infty} b_k z^k \tag{10}$$

where $b_0 = e^{f(0)} = 1, \;\; b_1 = e^{f(0)} f'(0) = a_1, \;\; b_2 = a_2 + \frac{a_1^2}{2}$.

Lemma 4.4 *There is a constant R_o such that for all $R > R_o$,*

$$|a_k| < |2\phi|^{k+2}. \tag{11}$$

Proof. We see that

$$|a_k| = \frac{R^{k+2}\left[1+o(R^{k+2})\right](2)^{\frac{k+2}{2}}}{(k+2)!(\beta R^2)^{\frac{k+2}{2}}\left[1+o(R^2)\right]}|\phi|^{k+2}$$

which tends to

$$\frac{2^{\frac{k+2}{2}}}{(k+2)!} < 2^{k+2}|\phi|^{k+2}$$

as $R \to \infty$. From this, it follows that there is a constant R_o satisfying (11). □

Now, it will follow from Lemma 4.4 that the radius of convergence of (10) becomes large when θ is near zero. Thus, $z = e^{-\frac{\beta R}{2}}$ is within the domain of convergence.

With $z = e^{-\frac{\beta R}{2}}$,

$$G_{n,r,\beta} \sim C\sum_{k=0}^{s-1}\left(\int_{-h}^{h} e^{-\phi^2} b_k d\phi\right) z^k + Q_s \tag{12}$$

where

$$Q_s = \int_{-h}^{h}\left(\sum_{k=s}^{\infty} e^{-\phi^2} b_k z^k\right) d\phi.$$

Note that $R \to \infty$ as $n \to \infty$. Furthermore with

$$\begin{aligned} h &= \frac{1}{2}(\beta R^2 e^{\beta R} + R e^{\beta R} + rR)^{1/2} e^{\frac{-3R}{8}} \\ &= \frac{1}{2}(\beta R^2 + R + rRe^{-\beta R})^{1/2} e^{\frac{R(4\beta-3)}{8}} \end{aligned}$$

$h \to \infty$ as $R \to \infty$. From these facts and the known asymptotic expansion of the function of the form

$$\int_{-h}^{h} e^{-\phi^2}(\text{polynomial in } |\phi|)d\phi,$$

the replacement of h by ∞ in (12) is easily justified (see [8]). Hence

$$G_{n,r,\beta} \sim C\sum_{k=0}^{s-1}\left(\int_{-\infty}^{\infty} e^{-\phi^2} b_k d\phi\right) z^k + Q_s. \tag{13}$$

It remains to show that $Q_s = o(|z|^s)$ as $R \to \infty$ that is, as $z \to 0$. From a Lemma in [4], $|b_k| \leq |2\phi|^{k+2}(1+|2\phi|^2)^{k-1}$. Thus,

$$\left|\sum_{k=s}^{\infty} b_k z^k\right| \leq \left[|2\phi|^{s+2}(1+|2\phi|^2)^{s-1}|z|^s\right]\left[1+\mu+\mu^2+...\right],$$

where $\mu = |2\phi|(1+|2\phi|^2)|z|$.

Now, for $\mu < 1$, we have

$$\left|\sum_{k=s}^{\infty} b_k z^k\right| \leq \frac{|2\phi|^{s+2}(1+|2\phi|^2)^{s-1}|z|^s}{1-|z||2\phi|(1+|2\phi|^2)}. \tag{14}$$

Let M and $P_s(|\phi|)|z|^s$ denote the denominator and the numerator, respectively in (14). Since $|\phi| \leq h$ and $z = e^{-\frac{\beta R}{2}}$, we have

$$|\phi^3||z| \leq \frac{1}{8}\left(\beta R^2 + R + rRe^{-\beta R}\right)^{3/2} e^{\frac{-3R}{8}} \to 0 \; as \; R \to \infty.$$

Hence for sufficiently large R, $M \geq 1/2$. Moreover,

$$\int_{-\infty}^{\infty} e^{-\phi^2} P_s(|\phi|) d\phi$$

exists and tends to zero as $R \to \infty$. Therefore,

$$\frac{|Q_s|}{|z|^s} \leq \int_{-\infty}^{\infty} \frac{e^{-\phi^2} P_s(|\phi|)}{M} d\phi.$$

Thus, $|Q_s| = o(|z|)^s$. Consequently,

$$G_{n,r,\beta} \sim C \sum_{k=0}^{\infty}\left(\int_{-\infty}^{\infty} e^{-\phi^2} b_k d\phi\right) e^{\frac{-k\beta R}{2}}. \tag{15}$$

Since $\int_{-\infty}^{\infty} e^{-x^2} x^n = 0$ for odd n, and b_{2k+1}, as a polynomial in ϕ, contain only odd powers of ϕ, it follows that

$$G_{n,r,\beta} \sim C \sum_{k=0}^{\infty}\left(\int_{-\infty}^{\infty} e^{-\phi^2} b_{2k} d\phi\right) e^{-k\beta R}. \tag{16}$$

An Asymptotic Approximation

Calculation yields

$$a_1 = \frac{\beta R^3 + 3R^2 + \beta^{-1}R + rRe^{-\beta R}}{3!\left[\frac{1}{2}(\beta R^2 + R + rRe^{-\beta R})\right]^{3/2}}(i\phi)^3$$

$$a_2 = \frac{\beta R^4 + 6\beta R^3 + 7R^2 + \beta^{-1}R + rRe^{-\beta R}}{4!\left[\frac{1}{2}(\beta R^2 + R + rRe^{-\beta R})\right]^2}(i\phi)^4.$$

Taking the first two terms of the asymptotic expansion of (15), we have

$$G_{n,r,\beta} \sim C\int_{-\infty}^{\infty} e^{-\phi^2} b_o d\phi + Cz^2 \int_{-\infty}^{\infty} e^{-\phi^2} b_2 d\phi.$$

Since $b_2 = a_2 + \frac{a_1^2}{2}$ and $b_o = 1$,

$$G_{n,r,\beta} \sim C\int_{-\infty}^{\infty} e^{-\phi^2} d\phi + Cz^2 \int_{-\infty}^{\infty} a_2 e^{-\phi^2} d\phi + C\frac{z^2}{2}\int_{-\infty}^{\infty} e^{-\phi^2} a_1^2 d\phi. \quad (17)$$

Let I_1, I_2, I_3, denote, respectively the integrals in (17). Then evaluating the last two integrals by parts and since $\int_{-\infty}^{\infty} e^{-\phi^2} d\phi = \sqrt{\pi}$, we obtain

$$I_1 = C\sqrt{\pi}$$

$$I_2 = \frac{Ce^{-R}\sqrt{\pi}(\beta R^3 + 6\beta R^2 + \beta^{-1} + re^{-\beta R})}{8R(\beta R + 1 + re^{-\beta R})^2}$$

$$I_3 = \frac{-5Ce^{-R}\sqrt{\pi}(\beta R^2 + 3\beta^{-1}R^2 + +re^{-\beta R})^2}{24R(\beta R + 1 + re^{-\beta R})^3}.$$

Substituting the results in (17) and simplifying, we obtain

$$G_{n,r,\beta} \sim C\sqrt{\pi}(1 + \frac{D+E}{F}), \quad (18)$$

where

$$D = (3\beta^2 R^3 + 8\beta R^3 + 3\beta R + 3 - 10\beta^{-1} - 2re^{-\beta R})re^{-\beta R} \quad (19)$$

$$E = (3\beta^3 - 5\beta^2)R^4 + (21\beta^2 - 30\beta)R^3 + (39\beta - 55)R^2 + (24 - 30\beta^{-1})R + (3\beta^{-1} - 5\beta^{-2}) \quad (20)$$

$$F = 24Re^{\beta R}(\beta R + 1 + re^{-\beta R})^3. \quad (21)$$

Since $Re^{\beta R} = (n - rR)\beta^{-1}$ and $R^n = n^n(\beta e^{\beta R} + r)^{-n}$,

$$C = \frac{n!\exp(rR + \beta^{-1}e^{\beta R} - \beta)}{\pi\left[n^n(\beta e^{\beta R} + r)^{-n}\right]\left[2(n - rR)\beta^{-1}\right]^{1/2}(\beta R + 1 + re^{-\beta R})^{1/2}}.$$

Using Stirling's approximation for $n!$, namely,

$$n! \sim (2\pi)e^{-n}n^{n+\frac{1}{2}}(1+\frac{1}{12n}),$$

we obtain

$$C \sim \frac{n^{1/2}(1+\frac{1}{12n})\exp(rR+\beta^{-1}e^{\beta R}-\beta)(\beta^{\beta R}+r)^n}{\pi^{1/2}[(n-rR)\beta^{-1}]^{\frac{1}{2}}(\beta R+1+re^{-\beta R})^{\frac{1}{2}}e^n}. \tag{22}$$

Finally, we have

$$G_{n,r,\beta} \sim \frac{n^{1/2}(1+\frac{1}{12n})\exp(rR+\beta^{-1}e^{\beta R}-\beta-n)(\beta^{\beta R}+r)^n}{[(n-rR)\beta^{-1}]^{\frac{1}{2}}(\beta R+1+re^{-\beta R})^{\frac{1}{2}}}\left(1+\frac{D+E}{F}\right). \tag{23}$$

A consequence of the asymptotic formula is the following lemma.

Lemma 4.5 $\frac{G_{n+2,r,\beta}}{G_{n,r,\beta}} - \left(\frac{G_{n+1,r,\beta}}{G_{n,r,\beta}}\right)^2 \to \infty$ *as* $n \to \infty$.

Proof: Using the above asymptotic formula for the (r,β)-Bell numbers and the fact that

$$\frac{(n+1)(1+\frac{1}{12n})^2(n-rR)}{n(1+\frac{1}{12n})^2} \sim 1$$

as $n \to \infty$, and $\beta Re^{\beta R}+rR=n$, we have

$$\frac{G_{n+2,r,\beta}}{G_{n,r,\beta}} - \left(\frac{G_{n+1,r,\beta}}{G_{n,r,\beta}}\right)^2 \sim \frac{n}{R^2}+o(1).$$

Finally, letting $n \to \infty$, we obtain the desired result. □

5 The Asymptotic Normality

The following theorem is essential to the proof of the main theorem, that is, on the asymptotic normality of the (r,β)-Stirling numbers. This result is one of the most general forms of the central limit theorem that gives conditions under which the sum of a large number of random variables has a probability distribution that is approximately normal.

Theorem 5.1 [11] (Bounded Variance Normal Convergence Criterion) *Let the independent summands* $\{X_{n_k}\}_{k=1}^{m_n}$*, centered at expectations, be such that*

$$\sum var(X_{n_k}) = 1, \ \textit{for all } n.$$

Let F_{n_k} be the (cumulative) distribution function of X_{n_k}. Then $S_n = \sum_k X_{n_k}$ converges normally with mean =0 and variance=1 and the

$max_k var(X_{nk}) \to 0$ if and only if for all $\epsilon > 0$,

$$\lim_{n\to\infty} g_n(\epsilon) = \lim_{n\to\infty} \sum_k \int_{|x|>\epsilon} X^2 dF_{n_k} = 0.$$

The following lemma is also needed.

Lemma 5.1 *If $-x_{n_{s_i}}$, $i = 1, 2, ..., n$ are the distinct nonpositive real roots of $Q_n(x)$, then*

$$\left\langle {n \atop j} \right\rangle_{\beta,r} = \sum_{1\le s_1<s_2<...<s_{n-j}\le n} x_{n_{s_1}} x_{n_{s_2}} ... x_{n_{s_{n-j}}}$$

where $\left\langle {n \atop j} \right\rangle_{\beta,r} = 1$ if $n = j$.

Proof. This can be proved by induction on n and using Lemma 3.3.

Lemma 5.2 *The distribution whose density function is*

$$\left\{ \frac{1}{G_{n,r,\beta}} \left\langle {n \atop j} \right\rangle_{\beta,r} \right\}_{j=0}^{n}$$

is the distribution of a sum of independent random variables taking on only the values 0 and 1.

Proof Consider an experiment consisting of n independent trials. Suppose that each trial results in an outcome that may be classified as a success or a failure (we take the value 1 for success and 0 for failure). If X'_{n_k} is a random variable that assumes the values 0 and 1 in the kth trial of the experiment, then $\sum_{k=1}^{n} X'_{n_k}$ represents the number of successes that occur in the experiment. In particular, if in the kth trial the weight of failure is x_{n_k} and the weight of success is 1, then the kth trial can be generated by the linear factor $x + x_{n_k}$. Thus the entire experiment can be generated by the product

$$(x + x_{n_1})(x + x_{n_2}) \cdots (x + x_{n_n}).$$

Moreover, if $-x_{n_k}s$ are the roots of

$$Q_n(x) = \sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle_{\beta,r} x^j,$$

then

$$\sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle_{\beta,r} x^j = (x + x_{n_1})(x + x_{n_2}) \cdots (x + x_{n_n}). \tag{24}$$

By Lemma 3.3, the $-x_{n_k}s$ are distinct and nonpositive.
Now, we can define

$$Pr[X'_{n_k} = y] = \begin{cases} \frac{x_{n_k}}{1+x_{n_k}}, & \text{if } y = 0 \\ \frac{1}{1+x_{n_k}}, & \text{if } y = 1. \end{cases}$$

Then

$$\begin{aligned} Pr\left[\sum_{k=1}^{n} X'_{n_k} = j\right] &= \sum \prod_{i=1}^{n} Pr\left[X'_{n_i} = y_i\right]; \; y_i = 0, 1 \\ &= \sum \prod_{i=1}^{n} \frac{x_{n_i}^{1-y_i}}{1+x_{n_i}}, \quad (\text{define } 0^0 = 1) \end{aligned}$$

where the sum is taken over all possible binary sequence with $n-j$ zeros and j ones. Thus,

$$Pr\left[\sum_{k=1}^{n} X'_{n_k} = j\right] = \sum_{1 \le s_1 < s_2 < \ldots < s_{n-j} \le n} \frac{x_{n_{s_1}} x_{n_{s_2}} \cdots x_{n_{s_{n-j}}}}{(1+x_{n_1})(1+x_{n_2}) \cdots (1+x_{n_n})}.$$

Letting $x = 1$ in (24) and using Lemma 5.1, we prove the lemma completely. □

Let us consider now the main result.

Theorem 5.3 *The (r, β)-Stirling numbers are asymptotically normal in the sense that*

$$\sum_{j=1}^{x_n} \frac{1}{G_{n,r,\beta}} \left\langle {n \atop j} \right\rangle_{\beta,r} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \text{ as } n \to \infty,$$

where

$$x_n = \sqrt{\frac{G_{n+2,r,\beta}}{G_{n,r,\beta}} - \left(\frac{G_{n+1,r,\beta}}{G_{n,r,\beta}}\right)^2} \, x + \left(\frac{G_{n+1,r,\beta}}{G_{n,r,\beta}} - 1\right).$$

Proof From Lemma 3.3 , the roots of the polynomial Q_n are distinct real and nonpositive. This is equivalent to the fact that Q_n can be factored into linear terms with real nonnegative coefficients. That is, $Q_n = (x+x_{n_1})(x+x_{n_2}) \cdots (x+x_{n_n})$. Let X'_{n_k} be a random variable that takes only the values 0 and 1. By Lemma 5.2, we have

$$Pr\left[\sum X'_{n_k} = j\right] = \frac{1}{G_{n,r,\beta}} \left\langle {n \atop j} \right\rangle_{\beta,r}.$$

Let $S'_n = \sum_k X'_{n_k}$. Using the triangular recurrence relation for the (r, β)-Stirling numbers, we calculate the expectation and variance of S'_n as follows

$$E(S'_n) = \frac{1}{\beta}\left\{\frac{G_{n,r+1,\beta}}{G_{n,r,\beta}} - (r+1)\right\}, \quad Var(S'_n) = \frac{1}{\beta^2}\left[\frac{G_{n+2,r,\beta}}{G_{n,r,\beta}} - \left(\frac{G_{n+1,r,\beta}}{G_{n,r,\beta}}\right)^2\right] - \frac{1}{\beta}.$$

We normalize and let $S_n = \sum_k X_{n_k}$, where $X_{n_k} = \frac{X'_{n_k} - E(X'_{n_k})}{\sqrt{Var(S'_n)}}$. Since $0 \leq X'_{n_k} \leq 1$ and $0 \leq E(X'_{n_k}) \leq 1$, $-1 \leq X'_{n_k} - E(X'_{n_k}) \leq 1$. Now, using Lemma 4.5, we have $\lim_{n\to\infty} Var(S'_n) = \infty$. Thus, $\lim_{n\to\infty} X_{n_k} = 0$. Hence by the definition of a limit of a sequence, given $\epsilon > 0$, there exists a natural number N such that $|X_{n_k}| < \epsilon$ for all $n \geq N$. This implies that,

$$E(X_{n_k}^2) = \int_{-\infty}^{\infty} X^2 dF_{n_k} = \int_{-\epsilon}^{\epsilon} X^2 dF_{n_k}$$

where F_{n_k} is the cumulative probability distribution of X_{n_k}. Thus,

$$\int_{|x|>\epsilon} X^2 dF_{n_k} = 0.$$

Therefore ,

$$\sum_k \int_{|x|>\epsilon} X^2 dF_{n_k} = 0, \; for \; all \; n \geq N,$$

That is, for all $\epsilon > 0$,

$$\lim_{n\to\infty} g_n(\epsilon) = \lim_{n\to\infty} \sum_k \int_{|x|>\epsilon} X^2 dF_{n_k} = 0.$$

Since X_{n_k}'s are independent random variables, $Var(S_n) = \sum_k Var(X_{n_k})$. But S_n is a normalized form of S'_n, so $Var(S_n) = 1$. This implies that $\sum_k Var(X_{n_k}) = 1$. Thus the hypotheses of the normal convergence criterion are fulfilled. Therefore, S_n converges normally with mean $= 0$, and variance $= 1$ and $(\max_k Var(X_{n_k})) \to 0$. This finally proves the theorem. □

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