The Metric Dimension of Cayley Digraphs of Abelian Groups

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Abstract

A vertex w in a (di)graph G is said to resolve a pair u, v of vertices of G if the distance from u to w does not equal the distance from v to w. A set S of vertices of G is a resolving set for G if every pair of vertices of G is resolved by some vertex of S. The smallest cardinality of a resolving set for G, denoted by $dim(G)$, is called the metric dimension for G. We show that if G is the Cayley digraph $Cay(\Delta : \Gamma)$ where $\Gamma = \mathbb{Z}_m \bigoplus \mathbb{Z}_n \bigoplus \mathbb{Z}_k$ with $m \leq n \leq k$ and $\Delta = \{(1,0,0),(0,1,0),(0,0,1)\}$, then $dim(G) = n$ if $m < n$ and improve known upper bounds if $m = n$. We use these results to establish improved upper bounds for the metric dimension of Cayley digraphs of abelian groups that are expressed as a direct product of four or more cyclic groups. Lower bounds for Cayley digraphs of groups that are multiple copies of $\mathbb{Z}_n$ are established.

Key words: metric dimension, Cayley digraphs.
AMS Subject Classification Codes: 05C12, 05C20, 05C90

1 Introduction

Let G be a connected (di)graph. The *distance* from a vertex u to a vertex v in G, denoted $d(u,v)$, is the length of a shortest (directed) $u-v$ path in G. A vertex w of G resolves two vertices u and v of G if $d(u,w) \neq d(v,w)$. A set $W = \{w_1, w_2, \ldots, w_n\}$ of G resolves G if every pair of vertices of G is resolved with respect to some vertex in W. Alternatively if we fix the order of the vertices in W as listed above, then W resolves G if and only if the n-vectors $r(v|W) =$

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$(d(v,w_1), d(v,w_2), \ldots, d(v,w_n))$ and $r(u|W) = (d(u,w_1), d(u,w_2), \ldots, d(u,w_n))$, called the *representations* of v and u with respect to W, are distinct. Note that the only vertex with 0 in the i^{th} component of its representation with respect to W is w_i. Thus, in determining whether a set W is a resolving set, one need only check that all pairs of vertices in $V(G) - W$ have distinct representations with respect to W. The minimum cardinality of a resolving set of G is called the *metric dimension* or simply the *dimension* of G, and is denoted $dim(G)$. A minimum resolving set for G is also called a *basis* for G. Graph theory terminology not given here can be found in [2].

Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of the metric dimension of a graph was introduced by Slater in [13] and [14], and studied independently by Harary and Melter in [9]. It has since been studied further in [1], [3], [4], [5], [6] and [11]. Applications of this invariant to the navigation of robots in networks are discussed in [11] and applications to chemistry are given in [1]. In [11], Khuller, Raghavachari and Rosenfeld gave a construction that shows that finding the metric dimension of a graph is NP-hard (see also [8]). Thus one is motivated to find the metric dimension of classes of graphs. The metric dimension for trees was established independently in [1], [9], [11], and [13]. Some bounds for this invariant, in terms of the diameter of the graph, are given in [1]. However, very little is known about this invariant for general graphs. One may expect a correlation between the automorphism group of a graph and its metric dimension. However, the results on trees show that there are trees with arbitrarily large metric dimension having the trivial automorphism group. It was also shown in [1] that in general there is no correlation between the metric dimension of a graph and that of its subgraphs. Nevertheless there appears to be a correlation between the metric dimension and both local and global symmetries in (di)graphs that already show higher degrees of symmetry, as for example in vertex transitive (di)graphs.

The literature abounds with results on Cayley (di)graphs that depend on their highly symmetric structure. Cayley (di)graphs with minimal sets of generators are also natural models for interconnection networks in computer design as they represent sparse (di)graphs with relatively small diameter. One of the simplest Cayley (di)graphs, namely the n-cube, has led to a multitude of deep and interesting problems [10]. Bounds for the metric dimension of the n-cube are given in [1]. Only recently the exact asymptotic value for the metric dimension of the n-cube, namely $2n/log_2 n$, was found in [12] by showing that the problem for these graphs is equivalent to a combinatorial search problem for counterfeit coins. The metric dimension of Cayley digraphs was first studied in [6] to explore relationships between this invariant and higher degrees of symmetry in vertex transitive (di)graphs. Cayley digraphs have the added advantage that distances between pairs of vertices can be described algebraically, thus lending themselves more readily to the use of algebraic tools when computing 'distance related' invariants. In [6] sharp bounds on the metric dimension of certain types of Cayley digraphs are presented. In this paper, we reexamine these graphs and improve these bounds for Cayley digraphs of direct products of three cyclic groups.

First, recall the definition of a Cayley digraph (see [7]).

Let Γ be a finite group and Δ a set of generators for Γ. The *Cayley digraph of* Γ *with generating set* Δ, denoted by $Cay(\Delta : \Gamma)$, is defined as follows:
1. The vertices of $Cay(\Delta : \Gamma)$ are precisely the elements of Γ.
2. For u and v in Γ, there is an arc from u to v if and only if $ug = v$ for some generator $g \in \Delta$.

Note that for a given finite group Γ and a specified set of generators Δ of Γ, every element of the group can be expressed as a product of generators in Γ. Hence, in the digraph $G = Cay(\Delta : \Gamma)$, there exists a path from any vertex to every other vertex. Thus, any Cayley digraph is strongly connected, and the metric dimension of any Cayley digraph is thus defined. It is not difficult to see that the metric dimension of the Cayley digraph of the cyclic group with one generator is 1.

In [6] the following two results are established.

Theorem 1.1 *Let m and n be positive integers. Let H be the Cayley digraph for the group $\mathbb{Z}_m \bigoplus \mathbb{Z}_n$ with generating set $\{(1,0),(0,1)\}$. Then $dim(H) = min(m,n)$.*

Let $k \geq 2$ be an integer. Then $e_i^{(k)}$ denotes the k-vector whose i^{th} entry is 1 and all of whose other entries are 0.

Theorem 1.2 *Let $k, n_1, n_2, \ldots, n_k$ be positive integers, each of which is at least 2, such that $n_1 \leq n_2 \leq n_3 \leq \ldots \leq n_k$. Let $\Gamma = \mathbb{Z}_{n_1} \bigoplus \mathbb{Z}_{n_2} \bigoplus \cdots \bigoplus \mathbb{Z}_{n_k}$ and $\Delta = \{e_i^{(k)} | 1 \leq i \leq k\}$. If $G = Cay(\Delta : \Gamma)$, then*

$$n_{k-1} \leq dim(G) \leq n_{k-1} + \sum_{i=1}^{k-2}(n_i - 1).$$

In [6] the metric dimensions of Cayley digraphs of groups of order at most 125 that are direct products of three cyclic groups are determined. These results show that both the upper and lower bounds of Theorem 1.2 can be attained and support the belief that higher degrees of symmetry, in vertex transitive graphs, have some correlation with higher metric dimension. In this paper we show that the lower bound of Theorem 1.2 is achieved for Cayley digraphs of groups that are products of three cyclic groups $\mathbb{Z}_m, \mathbb{Z}_n, \mathbb{Z}_k$ where $m < n \leq k$. We also improve the upper bound of this theorem if $m = n$.

2 The Converse Metric Dimension of a Graph

In order to simplify our arguments, we introduce some terminology and related observations. A vertex v *conversely resolves* a pair a, b in a (strong) digraph if $d(v,a) \neq d(v,b)$. A set S of vertices is a *converse resolving set* for a digraph H if every two vertices of H are conversely resolved by some vertex of S. The smallest cardinality of a converse resolving set for H is called the *converse metric dimension* or simply the *converse dimension* of H and is denoted by $\overleftarrow{dim}(H)$.

The *converse* of a digraph H is the digraph obtained from H by reversing all its arcs, and it is denoted by $\overleftarrow{H}$.

Remark: A set S is a resolving set for H if and only if S is a converse resolving set for $\overleftarrow{H}$. Thus $dim(H) = \overleftarrow{dim}(\overleftarrow{H})$. So if $\overleftarrow{H}$ is isomorphic to H, then $dim(H) = \overleftarrow{dim}(H)$. Since every Cayley digraph is isomorphic to its converse, its metric dimension and converse metric dimension are equal. This need not be the case for all digraphs. We now describe a digraph D_n for which the metric dimension and converse metric dimension are not equal. For some positive integer n let $V(D) = \{v_i, u_i | 0 \leq i \leq n\} \cup \{w_i, r_i, s_i | 1 \leq i \leq n\}$. To describe the arcs of D_n we begin by constructing the directed path $u_0 u_1 \ldots u_n$. Now join v_i to u_0 by the arc (v_i, u_0) for all $i, 1 \leq i \leq n$ and add the arcs (v_i, u_i), (u_i, w_i), (w_i, r_i), (w_i, s_i) and (r_i, w_i) for $1 \leq i \leq n$. Finally join every vertex x from the set $\{r_i, s_i | 1 \leq i \leq n\}$ to every v_j $(0 \leq j \leq n)$ by the arc (x, v_j). Figure 1 shows D_3. Since $N^+(r_i) = N^+(s_i) \cup \{w_i\}$ for $(1 \leq i \leq n)$, every resolving set for D_n must contain one of r_i, s_i or w_i for $1 \leq i \leq n$. Hence $dim(D_n) \geq n$. Since $\{s_i | 1 \leq i \leq n\}$ is a resolving set for D_n it now follows that $dim(D_n) = n$. Since every two vertices in $\{v_i | 0 \leq i \leq n\}$ have the same in-neighbourhood every converse resolving set for D_n must contain at least n of these vertices. Similarly since r_i and s_i have the same in-neighbourhood for $1 \leq i \leq n$ every converse resolving set for D_n must contain at least one vertex from each pair $\{r_i, s_i\}$ $(1 \leq i \leq n)$. Hence $\overleftarrow{dim}(D_n) \geq 2n$. Since $\{r_i, v_i | 1 \leq i \leq n\}$ is a converse resolving set it follows that $\overleftarrow{dim}(D_n) = 2n$.

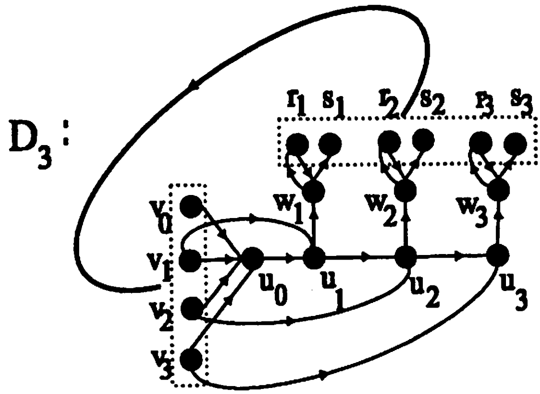

Figure 1: Digraph with unequal dimension and converse dimension

If $W = \{w_1, w_2, \ldots, w_n\}$ is a converse resolving set for a digraph D that has been assigned the given order we define the i^{th} *converse co-ordinate of vertex* v with respect to W to be $\overleftarrow{d}_i(v) = \overleftarrow{d}_D(v, w_i) = d_D(w_i, v)$ for $1 \leq i \leq n$. The sum of the converse co-ordinates of a vertex v, $\overleftarrow{W}(v) = \sum_{i=1}^{n} \overleftarrow{d}_i(v)$, is called the *converse co-ordinate sum* of v with respect to W. So any two vertices having distinct converse co-ordinate sums are conversely resolved by W. However, the converse need not be true.

3 The Metric Dimension of Cayley Digraphs of Direct Products of Three Cyclic Groups

In this section we show that the upper bound of Theorem 1.2 can be improved for the direct product of three cyclic groups.

Theorem 3.1 *If D is the Cayley digraph of $\Gamma = \mathbb{Z}_m \bigoplus \mathbb{Z}_n \bigoplus \mathbb{Z}_k$ where $2 \leq m < n \leq k$ with set of generators $\Delta = \{(1,0,0),(0,1,0),(0,0,1)\}$, then $dim(D) = n$.*

Proof: By Theorem 1.2 we know that $dim(D) \geq n$. We now show that $dim(D) \leq n$ by showing that the converse metric dimension of D is at most n.

To describe the vertices of a converse resolving set $W = \{w_1, w_2, \ldots, w_n\}$ we first describe an $n \times 3$ matrix X as follows: The $(j,3)$ entry is 0 and the $(j,2)$ entry is $j-1$ for $1 \leq j \leq n$. The $(j,1)$ entry is 0 for $1 \leq j \leq n-m+1$ and equals $j-(n-m+1)$ for $n-m+1 < j \leq n$. Let w_i be the vertex whose co-ordinates in the Cayley digraph D are given by the i^{th} row of X, $1 \leq i \leq n$. The matrix X is shown below.

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & n-m & 0 \\ 1 & n-m+1 & 0 \\ 2 & n-m+2 & 0 \\ \vdots & \vdots & \vdots \\ i-1 & n-m+i-1 & 0 \\ \vdots & \vdots & \vdots \\ m-2 & n-2 & 0 \\ m-1 & n-1 & 0 \end{pmatrix}$$

Figure 2 shows the set W for the group $\mathbb{Z}_4 \bigoplus \mathbb{Z}_6 \bigoplus Z_7$. (Note that the figure shows neither all the vertices nor all the arcs of the Cayley digraph, but it shows the 2-dimensional hyper-plane in which the resolving vertices lie and a part of all the remaining hyper-planes that run 'parallel' to this hyper-plane.)

Let $v = (v_1, v_2, v_3)$ be a vertex of D. Then the converse co-ordinates of v with respect to W are given by:

$\overleftarrow{d}_1(v) = v_1 + (v_2 - 0) \bmod n + v_3$
$\overleftarrow{d}_2(v) = v_1 + (v_2 - 1) \bmod n + v_3$
$\overleftarrow{d}_3(v) = v_1 + (v_2 - 2) \bmod n + v_3$

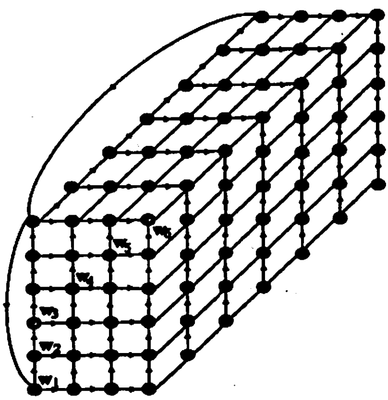

Figure 2: A converse resolving set for a Cayley digraph of $\mathbb{Z}_4 \bigoplus \mathbb{Z}_6 \bigoplus \mathbb{Z}_7$

$$\vdots$$

$\overleftarrow{d}_{n-m+1}(v) = v_1 + (v_2 - (n-m)) \bmod n + v_3$
$\overleftarrow{d}_{n-m+2}(v) = (v_1 + (m-1)) \bmod m + (v_2 + (m-1)) \bmod n + v_3$

$$\vdots$$

$\overleftarrow{d}_{n-1}(v) = (v_1 + 2) \bmod m + (v_2 + 2) \bmod n + v_3$
$\overleftarrow{d}_{n}(v) = (v_1 + 1) \bmod m + (v_2 + 1) \bmod n + v_3$

Hence the *converse co-ordinate sum* of v with respect to W is given by:

$\overleftarrow{W}(v) = (n-m)v_1 + nv_3 + \frac{1}{2}m(m-1) + \frac{1}{2}n(n-1)$

Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be two distinct vertices of D. Suppose that $w = (0,0,0)$ does not conversely resolve a and b and let $y = (a_1 + a_2 + a_3) - (b_1 + b_2 + b_3) = 0$. Suppose also that $\overleftarrow{W}(a) = \overleftarrow{W}(b)$ for otherwise a and b are conversely resolved by some point in W.

Claim 3.2 $a_1 \neq b_1$

Proof: Suppose $a_1 = b_1$. Then, since $\overleftarrow{W}(a) = \overleftarrow{W}(b)$, $(n-m)a_1 + na_3 = (n-m)b_1 + nb_3$ so that $a_3 = b_3$. But $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$ so that $a_2 = b_2$ from which it follows that $a = b$, a contradiction. □

We assume that $a_1 > b_1$ and thus $w' = (b_1 + 1, n - m + b_1 + 1, 0) \in W$.

Claim 3.3 $w' = (b_1 + 1, n - m + b_1 + 1, 0)$ *conversely resolves* a *and* b.

Proof: Consider $\overleftarrow{d}(a, w') - \overleftarrow{d}(b, w') = a_1 - (b_1 + 1) + (a_2 - (n - m + b_1 + 1)) \bmod n + a_3 - [m - 1 + (b_2 - (n - m + b_1 + 1)) \bmod n + b_3]$. We have the following cases.

Case 1 $a_2 - (n-m+b_1+1) \geq 0$ and $b_2 - (n-m+b_1+1) \geq 0$ or $a_2 - (n-m+b_1+1) < 0$ and $b_2 - (n-m+b_1+1) < 0$.

In this case, $\overleftarrow{d}(a,w') - \overleftarrow{d}(b,w') = y - m = -m \neq 0$ so a and b are conversely resolved by w'.
Case 2 $a_2 - (n - m + b_1 + 1) \geq 0$ and $b_2 - (n - m + b_1 + 1) < 0$.
Then $\overleftarrow{d}(a,w') - \overleftarrow{d}(b,w') = -m - n + y = -m - n \neq 0$ so again w' conversely resolves a and b.
Case 3 $a_2 - (n - m + b_1 + 1) < 0$ and $b_2 - (n - m + b_1 + 1) \geq 0$.
Then $\overleftarrow{d}(a,w') - \overleftarrow{d}(b,w') = y + n - m = n - m \neq 0$.

It follows that W is a converse resolving set for D. □

We now turn our attention to Cayley digraphs of abelian groups that are the direct product of three (or more) cyclic groups where some of these cyclic groups have the same order. It follows from Theorem 1.2 that if Γ is the direct product of r copies of $\mathbb{Z}_2$ and one copy of $\mathbb{Z}_k$ where $k \geq 2$ and if $\Delta = \{e_i^{(r+1)} | 1 \leq i \leq r+1\}$, then the Cayley digraph $Cay(\Delta : \Gamma)$ has metric dimension at most $r+1$. Our next result shows that in the case where $r = 2$ this upper bound is also a lower bound.

Theorem 3.4 *If D is the Cayley digraph of $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2 \bigoplus \mathbb{Z}_k$, $k \geq 2$, with set of generators $\Delta = \{(1,0,0),(0,1,0),(0,0,1)\}$, then the metric dimension of D is 3.*

Proof: Let S be a smallest resolving set. Without loss of generality, $(0,0,0)$ is an element of S.

No two of the vertices in the set $\{(0,1,0),(1,0,0),(0,0,k-1)\}$ are resolved by any vertex in the k-cycle which contains $(0,0,0)$ *except* $(0,0,k-1)$ (i.e. vertices of the form $(0,0,i)$, where $0 \leq i \leq k-2$). If $(0,0,k-1)$ is an element of S, then S must contain at least one more vertex to resolve $(0,1,0)$ and $(1,0,0)$. So in this case $|S| \geq 3$.

Suppose now that $(0,0,k-1)$ is not an element of S. If S also contains a vertex of the form $(0,0,i)$ with $1 \leq i \leq k-2$ (i.e. a vertex other than $(0,0,0)$ or $(0,0,k-1)$ which is on the same k-cycle as these), then S must contain at least one more vertex to resolve the pair of points $(1,0,i)$ and $(0,1,i)$. So $|S| \geq 3$.

Suppose that S contains no point of the form $(0,0,i)$ except the origin. Since the vertices $(0,1,0)$ and $(1,0,0)$ are not resolved by any vertex of the form $(1,1,i)$, the set S must contain a vertex of the form $(0,1,i)$ or $(1,0,i)$. Suppose S contains a vertex of the form $(0,1,i)$. The only vertex of the form $(0,1,i)$ that resolves every pair of vertices in the set $\{(0,1,0),(1,0,0),(0,0,k-1)\}$ is $(0,1,k-1)$. Since neither $(0,0,0)$ nor $(0,1,k-1)$ resolves the pair $(1,1,k-1)$, $(0,1,k-2)$, there must be at least one more vertex in S. Once again, $|S| \geq 3$.

Thus, the metric dimension of D is at least 3. This together with the comment prior to the theorem says that $dim(D) = 3$. □

Theorem 3.5 *If D is the Cayley digraph of $\mathbb{Z}_n \bigoplus \mathbb{Z}_n \bigoplus \mathbb{Z}_k$ where $3 \leq n < k$ with set of generators $\Delta = \{(1,0,0),(0,1,0),(0,0,1)\}$, then the metric dimension of D is at most k.*

Proof: To show that $dim(D) \leq k$ we show that D has a converse resolving set $W = \{w_1, w_2, ..., w_k\}$. To describe W we first describe a $k \times 3$ matrix X as follows:

The $(j,1)$ entry of X is 0 and the $(j,3)$ entry of X is $j-1$ for $1 \le j \le k$. The $(j,2)$ entry is 0 for $1 \le j \le k-n+1$ and equals $j-(k-n+1)$ for $k-n+2 \le j \le k$. Let w_i be the vertex whose co-ordinates in the Cayely digraph D are given by the i^{th} row of X, $1 \le i \le k$.

Then W conversely resolves G. This follows as in the proof of Theorem 3.1 □

Note that the upper bound k represents an improvement over the upper bound $m+n-1 = 2n-1$ given by Theorem 1.2 in the case when $k < m+n-1 = 2n-1$. In other words, an upper bound on the metric dimension in the case where $3 \le m = n < k$ is $min(k, 2n-1)$.

Theorem 3.6 *If D is the Cayley digraph of $\Gamma = \mathbb{Z}_n \bigoplus \mathbb{Z}_n \bigoplus \mathbb{Z}_n$ with set of generators $\Delta = \{(1,0,0),(0,1,0),(0,0,1)\}$, where $n > 3$, then the metric dimension of D is at most $\frac{3n}{2}$ if n is even and $\frac{3n-1}{2}$ if n is odd.*

Proof:
Case 1 (n even): We first define three sets $A_1 = \{(0,i,0)|0 \le i \le \frac{n-2}{2}\}$, $A_2 = \{(i, i+\frac{n}{2}, 0)|0 \le i \le \frac{n-2}{2}\}$, and $A_3 = \{(i,i,0)|\frac{n}{2} \le i \le n-1\}$, and then we show that the union of these sets conversely resolves G. (For example, if $n = 6$ we have $A_1 = \{(0,0,0),(0,1,0),(0,2,0)\}$, $A_2 = \{(0,3,0),(1,4,0),(2,5,0)\}$, and $A_3 = \{(3,3,0),(4,4,0),(5,5,0)\}$. Figure 3 shows the case $n = 6$ where again we show all the vertices in the hyperplane containing these vertices, but we do not show all the vertices in the hyperplanes parallel to this one.)

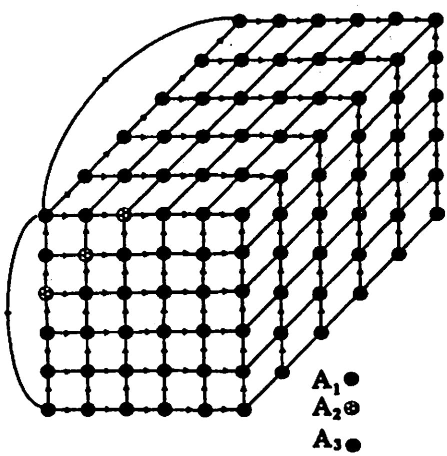

Figure 3: A converse resolving set for a Cayley digraph of $\mathbb{Z}_6 \bigoplus \mathbb{Z}_6 \bigoplus \mathbb{Z}_6$

Let $S = A_1 \cup A_2 \cup A_3$. Note that $|S| = \frac{3n}{2}$. Let $S_{12} = A_1 \cup A_2$. Then $\overleftarrow{S_{12}}(v)$ denotes the converse co-ordinate sum of v with respect to S_{12}. If $v = (v_1, v_2, v_3)$ and $\gamma_v = |\{k|v_1 - k < 0 \text{ and } 1 \le k \le \frac{n-2}{2}\}|$, then

$$\overleftarrow{S_{12}}(v) = \frac{(n+2)v_1}{2} + \sum_{k=1}^{\frac{n-2}{2}} (v_1 - k) \bmod n + \frac{n(n-1)}{2} + nv_3$$

$$= \frac{(n+2)v_1}{2} + (\frac{(n-2)v_1}{2} + \gamma_v n - \sum_{k=1}^{\frac{n-2}{2}} k) + \frac{n(n-1)}{2} + nv_3$$

$$= nv_1 + nv_3 + \gamma_v n - \sum_{k=1}^{\frac{n-2}{2}} k + \frac{n(n-1)}{2}.$$

Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be elements of Γ with $a \neq b$. If $\overleftarrow{S_{12}}(a) \neq \overleftarrow{S_{12}}(b)$ then S is a converse resolving set for D for then at least one element in $A_1 \cup A_2$ conversely resolves the pair a, b. So we assume $\overleftarrow{S_{12}}(a) = \overleftarrow{S_{12}}(b)$. If $(0,0,0)$ conversely resolves the pair a, b, then S is a converse resolving set. We assume this is not the case so that $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$.

Claim 3.7 $a_1 \neq b_1$

Proof: The proof is similar to that of Claim 3.2. If $a_1 = b_1$ then, since $\overleftarrow{S_{12}}(a) = \overleftarrow{S_{12}}(b)$ we have $na_1 + na_3 = nb_1 + nb_3$ and $a_3 = b_3$. But $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$ so that $a_2 = b_2$ and $a = b$, a contradiction. □

So we assume that $a_1 > b_1$.

Let $y = (a_1 + a_2 + a_3) - (b_1 + b_2 + b_3) = 0$. Since $a_1 > b_1$, we have $b_1 + 1 \in \mathbb{Z}_n$ and either $w = (b_1 + 1, b_1 + 1, 0)$ or $w' = (b_1 + 1, b_1 + 1 + n/2, 0)$ belongs to S. If $a_2 \geq b_2$, then that vertex among these two which belongs to S conversely resolves a and b. We now assume $a_2 < b_2$. We consider two subcases.

Subcase 1.1: $a_2 < b_2$ and $0 \leq a_2 + 1 \leq n/2 - 1$. In this case, $w'' = (0, a_2 + 1, 0) \in S$ conversely resolves a and b since $d(w'', a) - d(w'', b) = y + n = n \neq 0$.

Subcase 1.2: $a_2 < b_2$ and $n/2 \leq a_2 + 1 \leq n - 1$. Suppose $(a_2 + 1, a_2 + 1, 0) \in S$ does not conversely resolve a and b. Then $0 = (a_1 - (a_2 + 1)) \bmod n + n - 1 + a_3 - ((b_1 - (a_2 + 1)) \bmod n + (b_2 - (a_2 + 1)) + b_3)$. So $a_1 - (a_2 + 1) \geq 0$ and $(b_1 - (a_2 + 1)) < 0$. Since $n/2 \leq a_2 + 1 \leq n - 1$, we have $0 \leq a_2 + 1 - n/2 \leq n/2 - 1$. Thus $w'' = (a_2 + 1 - n/2, a_2 + 1, 0) \in S$ and w'' resolves a and b unless $b_1 - (a_2 + 1 - n/2) < 0$ and $a_1 - (a_2 + 1 - n/2) \geq 0$. In this case, $(a_2, a_2, 0) \in S$ since $n/2 - 1 \leq a_2 \leq n - 2$ and if $a_2 = n/2 - 1$, then $b_1 < 0$, a contradiction. The point $(a_2, a_2, 0)$ resolves a and b since $(a_1 - a_2) \bmod n + a_3 - ((b_1 - a_2) \bmod n + (b_2 - a_2) \bmod n + b_3) = y - n = -n \neq 0$.

Case 2 (n odd): Let $S = A_1 \cup A_2 \cup A_3 \cup A_4$ where $A_1 = \{(0,0,0), (0,1,0), \ldots, (0, \frac{n-3}{2}, 0)\}$, $A_2 = \{(0, \frac{n+1}{2}, 0)\}$, $A_3 = \{(1, \frac{n-1}{2}, 0), (i, i + \frac{n-1}{2}, 0) | 2 \leq i \leq \frac{n-1}{2}\}$, and $A_4 = \{(\frac{n+1}{2}, \frac{n-1}{2}, 0), (i, i, 0) | \frac{n+3}{2} \leq i \leq n - 1\}$. (For example, if $n = 7$ we have $A_1 = \{(0,0,0), (0,1,0), (0,2,0)\}$, $A_2 = \{(0,4,0)\}$, $A_3 = \{(1,3,0), (2,5,0), (3,6,0)\}$, and $A_4 = \{(4,3,0), (5,5,0), (6,6,0)\}$.) Note that in general, $|S| = \frac{3n-1}{2}$. Let

$S_{123} = A_1 \cup A_2 \cup A_3$. Then $\overleftarrow{S_{123}}(v)$ denotes the converse co-ordinate sum of v with respect to S_{123}. Let $\gamma_v = |\{k | v_1 - k < 0, 1 \le k \le \frac{n-1}{2}\}|$. Then we have

$$\begin{aligned}
\overleftarrow{S_{123}}(v) &= \frac{(n+1)v_1}{2} + \sum_{k=1}^{\frac{n-1}{2}} (v_1 - k) + \frac{n(n-1)}{2} + nv_3 \\
&= \frac{(n+1)v_1}{2} + \frac{(n-1)v_1}{2} + \gamma_v n - \sum_{k=1}^{\frac{n-1}{2}} k + \frac{n(n-1)}{2} + nv_3 \\
&= nv_1 + nv_3 + \gamma_v n - \sum_{k=1}^{\frac{n-1}{2}} k + \frac{n(n-1)}{2}.
\end{aligned}$$

Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be distinct vertices in G with $\overleftarrow{S_{123}}(a) = \overleftarrow{S_{123}}(b)$. As in the even case, we may assume that $a_1 > b_1$. (The proof that $a_1 \neq b_1$ is similar to that of Claim 3.7.) Let $y = (a_1+a_2+a_3) - (b_1+b_2+b_3) = 0$.

If $a_2 \ge b_2$ and if $b_1 + 1 \neq 1$ and $b_1 + 1 \neq (n+1)/2$, then one of $w_1 = (b_1+1, b_1+1, 0)$ or $w_2 = (b_1+1, b_1+1+(n-1)/2, 0)$ belongs to S and conversely resolves a and b. If $b_1+1 = 1$ or $b_1+1 = (n+1)/2$, then $w_3 = (b_1+1, (n-1)/2, 0)$ belongs to S and if $a_2 \ge b_2$, then w_3 conversely resolves a and b. We now assume $a_2 < b_2$ and consider several subcases:

Subcase 2.1 $a_2 < b_2$ and either $0 \le a_2 + 1 \le (n-3)/2$ or $a_2 + 1 = (n+1)/2$. Then $w'' = (0, a_2+1, 0) \in S$ and as in the even case, w'' conversely resolves a and b.
Subcase 2.2 $a_2 < b_2$ and $(n+3)/2 < a_2 + 1 \le n - 1$. As in the even case, if $(a_2+1, a_2+1, 0) \in S$ does not resolve a and b, then $a_1 - (a_2+1) \ge 0$ and $b_1 - (a_2+1) < 0$. Also, $w'' = (a_2 + 1 - (n-1)/2, a_2+1), 0) \in S$. If w'' does not conversely resolve a and b, then as in the even case $(a_2, a_2, 0)$ conversely resolves a and b.
Subcase 2.3 $a_2 < b_2$ and $a_2 + 1 = (n-1)/2$, i.e., $a_2 = (n-3)/2$. If $b_2 \neq (n-1)/2$, then since $\frac{n-3}{2} = a_2 < b_2$ we have $b_2 \ge \frac{n+1}{2}$. In this case $w'' = (0, \frac{n+1}{2}, 0)$ conversely resolves a and b since $d(w'', a) - d(w'', b) = y + n = n \neq 0$. If $b_2 = (n-1)/2$, then $(1, \frac{n-1}{2}, 0)$ conversely resolves a and b unless $(b_1 - 1) < 0$ (and $(a_1 - 1) > 0$). So we assume $b_1 = 0$. Suppose that $a_1 \ge (n+1)/2$. Then $w'' = (\frac{n-1}{2}, n-1, 0) \in A_3 \subseteq S$ provided $n > 3$ (since this is the vertex $(i, i + \frac{n-1}{2}, 0)$ with $i = (n-1)/2$). Also w'' conversely resolves a and b since $d(w'', a) - d(w'', b) = y - 2n \neq 0$. Finally, if $a_1 < (n+1)/2$, the vertex $(\frac{n+1}{2}, \frac{n-1}{2}, 0)$ conversely resolves a and b since $d(w'', a) - d(w'', b) = y + n = n \neq 0$.
Subcase 2.4 $a_2 < b_2$ and $a_2+1 = (n+3)/2$. In this case the vertex $(\frac{n+1}{2}, \frac{n-1}{2}, 0)$, which belongs to S, will conversely resolve a and b. □

Next we determine lower bounds for the metric dimension of Cayley digraphs of abelian groups of multiple copies of $\mathbb{Z}_n$. Let Γ be the group that is the direct product of $k \ge 2$ copies of the cyclic group $\mathbb{Z}_n$, $n \ge 2$. We denote this by $(\mathbb{Z}_n)^k$. Let $\Delta = \{e_i^{(k)} | 1 \le i \le n\}$. Let $D = Cay(\Delta : \Gamma)$. The $(k-1)$-dimensional hyperplane with respect to the i^{th} coordinate whose i^{th} coordinate equals a is denoted by $X_i(a)$. So $X_i(a) = \{(x_1, x_2, \dots, x_k) \in (\mathbb{Z}_n)^k |\ x_i = a\}$. Then

$$X_i(a) \cap X_j(b) = \{(x_1, x_2, \dots, x_k) \in (\mathbb{Z}_n)^k |\ x_i = a \text{ and } x_j = b\}.$$

Moreover, the complement of $X_i(a) \cup X_j(b)$ is the set

$$\overline{X_i(a) \cup X_j(b)} = \{(x_1, x_2, \dots, x_k) \in (\mathbb{Z}_n)^k |\ x_i \neq a \text{ and } x_j \neq b\}.$$

If $k = 2$, then $dim(D) \geq 2$. We now assume $k \geq 3$. Let $A_1 = (x_1, x_2, \dots, x_k)$ and $A_2 = (y_1, y_2, \dots, y_k)$ be vertices of D that agree on all coordinates except on the i^{th} and j^{th} coordinates. So $x_l = y_l$ for $l \neq i, j$. Moreover, we assume that $x_i = a+1$, $x_j = b$ and $y_i = a$, $y_j = b+1$ for some $0 \leq a < n-1$ and $0 \leq b < n-1$ in $\mathbb{Z}_n$. The next result describes the set of vertices that can resolve A_1 and A_2 and will be used to establish a lower bound for the metric dimension of D.

Proposition 3.8 *Let A_1 and A_2 be defined as above. Then a vertex w of D resolves A_1 and A_2 if and only if w does not belong to $(X_i(a) \cap X_j(b)) \cup \overline{X_i(a) \cup X_j(b)}$.*

Proof: Suppose first that $w = (w_1, w_2, \dots, w_k) \in X_i(a) \cap X_j(b)$. Then $w_i = a$ and $w_j = b$ and so $d(A_1, w) = \sum_{l \in \{1,2,\dots,k\} - \{i,j\}} (w_l - x_l) \bmod n \ + \ (w_i - (a+1)) \bmod n \ + \ (w_j - b) \bmod n = \sum_{l \in \{1,2,\dots,k\} - \{i,j\}} (w_l - x_l) \bmod n \ + n - 1$ and $d(A_2, w) = \sum_{l \in \{1,2,\dots,k\} - \{i,j\}} (w_l - y_l) \bmod n \ + \ (w_i - a) \bmod n \ + \ (w_j - (b+1)) \bmod n = \sum_{l \in \{1,2,\dots,k\} - \{i,j\}} (w_l - y_l) \bmod n \ + n - 1$. Since $y_l = x_l$ for $l \neq i, j$ it follows that $d(A_1, w) = d(A_2, w)$. Thus A_1 and A_2 are not resolved by any vertex in $X_i(a) \cap X_j(b)$.

Suppose now that $w = (w_1, w_2, \dots, w_n) \in \overline{X_i(a) \cup X_j(b)}$. Then $w_i \neq a$ and $w_j \neq b$. We consider three cases:

Case 1 $a + 1 \leq w_i \leq n - 1$ and $b + 1 \leq w_j \leq n - 1$. Then $d(A_1, w) = \sum_{l \in \{1,2,\dots,k\} - \{i,j\}} (w_l - x_l) \bmod n \ + w_i - (a+1) + w_j - b$ and $d(A_2, w) = \sum_{l \in \{1,2,\dots,k\} - \{i,j\}} (w_l - y_l) \bmod n \ + w_i - a + w_j - (b+1)$. Hence $d(A_1, w) = d(A_2, w)$.

Case 2 $0 \leq w_i < a$ and $0 \leq w_j < b$. Then $d(A_1, w) = \sum_{l \in \{1,2,\dots,k\} - \{i,j\}} (w_l - x_l) \bmod n \ + n + w_i - (a+1) + n + w_j - b$ and $d(A_2, w) = \sum_{l \in \{1,2,\dots,k\} - \{i,j\}} (w_l - y_l) \bmod n \ + n + w_i - a + n + w_j - (b+1)$. Once again $d(A_1, w) = d(A_2, w)$.

Case 3 $0 \leq w_i < a$ and $b+1 \leq w_j \leq n-1$ or $a+1 \leq w_i \leq n-1$ and $0 \leq w_j < b$. We prove this only for the first of these two subcases as the second one follows by an identical argument. In that case $d(A_1, w) = \sum_{l \in \{1,2,\dots,k\} - \{i,j\}} (w_l - x_l) \bmod n \ + n + w_i - (a+1) + w_j - b$ and $d(A_2, w) = \sum_{l \in \{1,2,\dots,k\} - \{i,j\}} (w_l - y_l) \bmod n \ + n + w_i - a + w_j - (b+1)$. Once again $d(A_1, w) = d(A_2, w)$.

For the converse suppose $w = (w_1, w_2, \dots, w_k) \in X_i(a) - X_j(b)$. (The case where $w \in X_j(b) - X_i(a)$ can be argued similarly.) Then $w_i = a$ and $w_j \neq b$. Thus $d(A_1, w) = \sum_{l \in \{1,2,\dots,k\} - \{i,j\}} (w_l - x_l) \bmod n + (a - (a+1)) \bmod n + (w_j - b) \bmod n$ and $d(A_2, w) = \sum_{l \in \{1,2,\dots,k\} - \{i,j\}} (w_l - y_l) \bmod n \ + (w_j - (b+1)) \bmod n$. If $w_j \geq b+1$, then $d(A_1, w) = \sum_{l \in \{1,2,\dots,k\} - \{i,j\}} (w_l - x_l) \bmod n \ + n - 1 + w_j - b$ and $d(A_2, w) = \sum_{l \in \{1,2,\dots,k\} - \{i,j\}} (w_l - y_l) \bmod n \ + w_j - b - 1$; otherwise, $w_j < b$ and $d(A_1, w) = \sum_{l \in \{1,2,\dots,k\} - \{i,j\}} (w_l - x_l) \bmod n \ + n - 1 + n + w_j - b$ and

$d(A_2, w) = \sum_{l \in \{1,2,\ldots,k\}-\{i,j\}}(w_l - y_l) \bmod n \; + n + w_j - b - 1$. In either case $d(A_1, w) \neq d(A_2, w)$. □

Remark: By Proposition 3.8 we know that any two out-neighbours of $\mathbf{0} = (0, 0, \ldots, 0)$ are resolved precisely by any vertex in $(X_i(0) - X_j(0)) \cup (X_j(0) - X_i(0))$ where i and j are the co-ordinates in which the two out-neighbours differ from those in $\mathbf{0}$. By symmetry it follows that if A_1 and A_2 are any two distinct out-neighbours of a vertex $B = (b_1, b_2, \ldots, b_k)$ in D, say A_1 and A_2 differ from B in precisely coordinates i and j, respectively, then A_1 and A_2 are resolved precisely by any vertex in $(X_i(b_i) - X_j(b_j)) \cup (X_j(b_j) - X_i(b_i))$. Note that this holds even if b_i or b_j is $n-1$. We now establish a lower bound on the metric dimension of the Cayley digraph D described prior to Proposition 3.8

Proposition 3.9 *Let D be the Cayley digraph of Proposition 3.8. If W is a resolving set for D, then $|W| \geq 1 + log_2 k$.*

Proof: By symmetry we may assume that $w_0 = (0, 0, \ldots, 0) \in W$. Then w_0 does not resolve any pair of out-neighbours of w_0. So $W - \{w_0\}$ is non empty and contains some vertex w_1 that resolves some pair of out-neighbours of w_0. Let $S_1(0)$ be the collection of co-ordinates of w_1 that equal 0 and $\overline{S}_1(0)$ the collection of all co-ordinates of w_1 that are not 0. (Then w_1 resolves $|S_1(0)| \cdot |\overline{S}_1(0)|/2$ pairs of out-neighbours of w_0, namely those that have a 1 in any co-ordinate in $S_1(0)$ and those having a 1 in any co-ordinate in $\overline{S}_1(0)$.) One of $S_1(0)$ and $\overline{S}_1(0)$ contains at least $k/2$ elements. Let T_1 be that one of these two sets that has at least $k/2$ elements. If $k = 3$, then $|W| \geq 1 + \lceil log_2 k \rceil = 3$. Suppose now that $k \geq 4$. Then $|T_1| \geq 2$. Thus w_1 does not resolve any two distinct out-neighbours of w_0 that have a 1 in a co-ordinate in T_1. So $W - \{w_0, w_1\}$ must contain a vertex w_2 that resolves some pair of out-neighbours of w_0 each with a 1 in a co-ordinate in T_1. Let $S_2(0)$ be the collection of co-ordinates of w_2 that are equal to 0 and $\overline{S}_2(0) = T_1 - S_2(0)$. (Then w_2 resolves $|S_2(0)| \cdot |\overline{S}_2(0)|/2$ pairs of out-neighbours of w_0 that were not resolved by either w_0 nor w_1.) Again at least one of $S_2(0)$ or $\overline{S}_2(0)$ contains at least $|T_1|/2$ elements. Let T_2 be the one of these two sets that has at least $|T_1|/2$ elements. If T_2 has only one element we stop; otherwise, we continue in this manner constructing a strictly decreasing chain of sets $T_1 \supset T_2 \supset \ldots$ and a sequence of vertices $w_0, w_1, w_2, \ldots$ of elements of W such that no vertex in $\{w_0, w_1, \ldots, w_i\}$ resolves any pair of out-neighbours of w_0 with a 1 in a co-ordinate in T_{i+1} for $i = 0, 1, \ldots$. By the choice of the T_i's this chain must contain at least $log_2 k$ elements. The result now follows. □

Corollary 3.10 *If D is the Cayley digraph of Proposition 3.8, then $dim(D) \geq max(n, 1 + \lceil log_2 k \rceil)$.*

Proof: This follows immediately from Theorem 1.2 and Proposition 3.9. □

4 The Metric Dimension of Cayley Digraphs of Abelian Groups

The following result was established in [6].

Theorem 4.1 *Let Γ be a group and let $\Delta = \{g_1, g_2, \ldots, g_k\}$ be a generating set for Γ. Let $H = Cay(\Delta : \Gamma)$. Then $\Delta' = \{(g_1, 0), (g_2, 0) \ldots, (g_k, 0), (e_\Gamma, 1)\}$ is a generating set for the group $\Gamma' = \Gamma \bigoplus \mathbb{Z}_m$, where $m \geq 2$ and e_Γ is the identity element of Γ. Then for $H' = Cay(\Delta' : \Gamma')$,*

$$dim(H) \leq dim(H') \leq dim(H) + m - 1.$$

This result together with those given in the previous section can be used to improve the upper bound of Theorem 1.2.

Theorem 4.2 *Let $k, n_1, n_2, \ldots, n_k$ be positive integers, each of which is at least 2. Suppose $\Gamma = \mathbb{Z}_{n_1} \bigoplus \mathbb{Z}_{n_2} \bigoplus \ldots \bigoplus \mathbb{Z}_{n_k}$ and*
$\Delta = \{e_i^{(k)} | 1 \leq i \leq k\}$. Let $G = Cay(\Delta : \Gamma)$

1. *If $n_1 \leq n_2 \leq \ldots \leq n_{k-2} < n_{k-1} \leq n_k$, then*

$$dim(G) \leq n_{k-1} + \sum_{i=1}^{k-3} (n_i - 1).$$

2. *If $n_1 \leq n_2 \leq \ldots \leq n_{k-2} = n_{k-1} \leq n_k$, then*

$$dim(G) \leq n_k + \sum_{i=1}^{k-3} (n_i - 1).$$

3. *If $n_1 \leq n_2 \leq \ldots \leq n_{k-2} = n_{k-1} = n_k$, then*

$$dim(G) \leq \frac{3}{2} n_k + \sum_{i=1}^{k-3} (n_i - 1).$$

Proof: This result follows from Theorems 4.1, 3.1, 3.5 and 3.6. □

5 Closing Remarks

Let $k, n_1, n_2, \ldots, n_k$ be positive integers at least 2 and $D = Cay(\Delta : \Gamma)$ where $\Gamma = \mathbb{Z}_{n_1} \bigoplus \mathbb{Z}_{n_2} \bigoplus \ldots \bigoplus \mathbb{Z}_{n_k}$ and $\Delta = \{e_i^{(k)} | 1 \leq i \leq k\}$. Let W be a basis for D and $d = dim(D)$. The ideas used in [6] to prove the lower bound of Theorem 4.1 can be used to show that if $W = \{w_i = (w_{1i}, w_{2i}, \ldots, w_{ki}) | 1 \leq i \leq d\}$ is a resolving set for D, then $W' = \{w'_i = (w_{1i}, w_{2i}, \ldots, w_{(k-1)i}) | 1 \leq i \leq d\}$ is a resolving set for the Cayley digraph $D' = Cay(\Delta' : \Gamma')$ where $\Gamma' = \mathbb{Z}_{n_1} \bigoplus \mathbb{Z}_{n_2} \bigoplus \ldots \bigoplus \mathbb{Z}_{n_{k-1}}$ and $\Delta' = \{e_i^{(k-1)} | 1 \leq i \leq k - 1\}$.

In view of the fact that the asymptotically exact value for the metric dimension of the n-cube is $2n/log_2 n$ it would be interesting to determine if a similar result holds for Cayley digraphs of $(\mathbb{Z}_n)^k$ $(n \geq 3)$ with a canonical set of generators.

It is our belief that the Cayley digraph $D = Cay(\Delta : \Gamma)$ where Γ and Δ are as above has a minimum resolving set that belongs to some $(k-1)$-dimensional hyperplane with respect to some co-ordinate. We now show that a converse resolving set for D (and hence a resolving set) cannot lie in a $(k-2)$-dimensional hyperplane with respect to two co-ordinates. Suppose $W = \{w_i = (w_{1i}, w_{2i}, \ldots, w_{ki}) | 1 \leq i \leq d\}$ is a converse resolving set for D that lies in some $(k-2)$-dimensional hyperplane with respect to co-ordinates l and m. We assume $n_l \leq n_m$. We may also assume that co-ordinates l and m for every vertex in W equal 0. Let x be the vertex whose l^{th} co-ordinate is 1 and all of whose other co-ordinates are 0. Let y be the vertex of D whose m^{th} co-ordinate is $n_m - n_l + 1$ and all of whose other co-ordinates are 0. Then $d(w_i, x) = w_{1i} + w_{2i} + \ldots + w_{(l-1)i} + (n_l - 1) + w_{(l+1)i} + \ldots + w_{(m-1)i} + 0 + w_{(m+1)i} + \cdots + w_{n_k}$ and $d(w_i, y) = w_{1i} + w_{2i} + \ldots + w_{(l-1)i} + 0 + w_{(l+1)i} + \ldots + w_{(m-1)i} + (n_m - (n_m - n_l + 1)) + w_{(m+1)i} + \ldots w_{n_k}$. So $d(w_i, x) = d(w_i, y)$ for all $w_i \in W$. So W does not conversely resolve D. Hence D has no resolving set that is contained in a $(k-2)$- dimensional hyperplane with respect to any two of its co-ordinates.

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