On Super Edge-Magic Graphs

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Abstract

A graph $G = (V, E)$ is said to be super edge-magic if there exists a one-to-one correspondence λ from $V \cup E$ onto $\{1, 2, 3, \ldots, |V| + |E|\}$ such that $\lambda(V) = \{1, 2, \ldots, |V|\}$ and $\lambda(x) + \lambda(xy) + \lambda(y)$ is constant for every edge xy. In this paper, given a positive integer k ($k \geq 6$) we use the partitions of k having three distinct parts to construct infinitely many super edge-magic graphs without isolated vertices with edge magic number k. Especially we use this method to find graphs with the maximum number of edges among the super edge-magic graphs with ν vertices. In addition, we investigate whether or not some interesting families of graphs are super edge-magic.
Key words. Edge-magic labeling, Super edge-magic graphs, Magic number

1 Introduction

Throughout this paper, we assume that all graphs are finite, simple and undirected. A graph G has vertex set $V(G)$ and edge set $E(G)$ and we let $|V(G)| = \nu(G)$ and $|E(G)| = \varepsilon(G)$. A general reference for graph theoretic notions is West [4].

Given a graph G, let $V = V(G)$, $E = E(G)$, $\nu(G) = \nu$, and $\epsilon(G) = \epsilon$. A one-to-one correspondence λ from $V \cup E$ onto the integers $\{1, 2, \ldots, \nu + \varepsilon\}$ is an *edge-magic labeling* if there is a constant k so that for any edge xy,

$$\lambda(x) + \lambda(xy) + \lambda(y) = k.$$

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The constant k is called the *edge magic number* for λ. An edge-magic labeling λ is called *super edge-magic* if $\lambda(V) = \{1, 2, \ldots, \nu\}$ and $\lambda(E) = \{\nu+1, \nu+2, \ldots, \nu+\varepsilon\}$. A graph G is called *edge-magic* (resp. *super edge-magic*) if there exists an edge-magic (resp. super edge-magic) labeling of G.

In this paper, given a positive integer k ($k \geq 6$) we use the partitions of k having three distinct parts to construct infinitely many super edge-magic graphs with edge magic number k. Then we give an upper bound for the number of edges of a super edge-magic graph with edge magic number k in terms of k. We show that this upper bound is sharp in infinitely many cases by constructing a super edge-magic graph with $2n-3$ edges and edge magic number $3n$ for every $n \geq 2$. Also, we investigate whether several noteworthy families of graphs are super edge-magic or not. Especially, we focus on an (n,t)-kite consisting of a cycle of length n with a t-edge path attached to one vertex and $K_2 \cup C_n$ motivated by Wallis [3] questions to characterize edge-magic graphs among these graphs. In fact, we completely characterize super edge-magic $K_2 \cup C_n$. In addition, we show that a graph derived from a star by adding a pendant edge to each vertex of degree 1 is super edge-magic.

2 Constructing super edge-magic graphs without isolated vertices by using a partition method

We observe that given a super edge-magic graph G without isolated vertices and its super edge-magic labeling λ with the edge magic number k, $\{\{\lambda(x), \lambda(xy), \lambda(y)\} : xy \in E(G)\}$ is a set of the partitions of k with three distinct parts satisfying the following properties:

1. Their union forms $\{1, 2, \ldots, m\}$ for some positive integer m;
2. The maximum number of each partition belongs to only that partition;
3. The maximum numbers from each partition are consecutive.

Conversely, for a positive integer k ($k \geq 6$), if there is a set of ϵ partitions of k with three distinct parts satisfying the above three properties we can construct a super edge-magic graph G without isolated vertices with edge magic number k as follows: Then their union forms $\{1, 2, \ldots, m\}$ for some positive integer m by Property 1. Then $m-\epsilon+1$, $m-\epsilon+2$, $\ldots$, m are the maximum numbers from each partition by properties 2 and 3. Let

$$V(G) = \{1, 2, \ldots, m-\epsilon\}$$

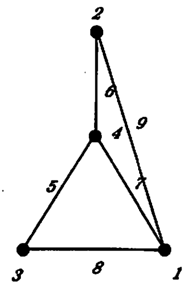

Figure 1: A super edge-magic graph constructed from parts $\{1,2,9\}$, $\{1,3,8\}$, $\{1,4,7\}$, $\{3,4,5\}$, $\{2,4,6\}$.

and define the edge set by vertices i and j being adjacent if i and j are in the same partition.

We illustrate the above method by examples: Take $k = 12$. Then the partitions of 12 having three distinct parts are as follows:

$$\begin{aligned}12 &= 1+2+9\\ &= 1+3+8\\ &= 1+4+7\\ &= 1+5+6\\ &= 2+3+7\\ &= 2+4+6\\ &= 3+4+5.\end{aligned}$$

We may take parts $\{1,2,9\}$, $\{1,3,8\}$, $\{1,4,7\}$, $\{1,5,6\}$ that satisfy the above properties. From these, we construct a super edge-magic graph as follows: the vertex set is $\{1,2,3,4,5\}$. The edge set is $\{\{1,2\},\{1,3\},\{1,4\},\{1,5\}\}$. Thus, we obtain the graph star $K_{1,4}$ with central vertex labeled 1. Parts $\{1,2,9\}$, $\{1,3,8\}$, $\{1,4,7\}$, $\{2,4,6\}$, $\{3,4,5\}$ also satisfy the above properties and we obtain the super edge-magic graph given in Figure 1. We note that $7 = 1+2+4$ is the only partition with three distinct parts and therefore there is no super edge-magic graph without isolated vertices with edge magic number 7. In fact, we may show that for every integer $k \geq 6$ other than 7, there is a super edge-magic graph without isolated vertices with edge magic number k. For, we may construct a set of partitions of k satisfying the above three conditions: For k even ($k \geq 6$),

$$\begin{aligned} k &= 1+2+(k-3) \\ &= 1+3+(k-4) \\ &\vdots \\ &= 1+(i-1)+(k-i) \\ &\vdots \\ &= 1+(k/2-1)+k/2 \end{aligned}$$

For k odd ($k \geq 9$),

$$\begin{aligned} k &= 1+2+(k-3) \\ &= 1+3+(k-4) \\ &\vdots \\ &= 1+(i-1)+(k-i) \\ &\vdots \\ &= 1+(k-3)/2+(k+1)/2 \\ &= 2+(k-3)/2+(k-1)/2 \end{aligned}$$

As mentioned before, Enomoto *et al.* [1] showed that the following:

Lemma 1 (Enomoto *et al.* [1]) *If a nontrivial graph G is super edge-magic, then $\varepsilon \leq 2\nu - 3$.*

Their lemma is significant in the sense that it eliminates huge number of graphs from being super edge-magic graphs. It is interesting to find families of super edge-magic graphs that satisfy $\varepsilon = 2\nu - 3$ as the super edge-magic graphs satisfying this equality have the maximum number of edges among the super edge-magic graphs with ν vertices.

The following proposition gives an upper bound for the number of edges of a super edge-magic graph with edge magic number k in terms of k.

Proposition 2 *Given k, let G be a super edge-magic graph with edge magic number k. Then*

$$\epsilon \leq \frac{2}{3}k - 3.$$

proof. Since $k-3$ in partition $1+2+(k-3)$ is the largest possible part, it is true that $k-3 \geq \nu + \epsilon$ or $\nu \leq k - \epsilon - 3$. From this inequality and the inequality in Lemma 1, we can derive the inequality in the proposition. □

The bound given in Proposition 2 is also sharp. The graph given in Figure 1 has edge magic number 12 and 5 edges, which satisfies both equalities in

Lemma 1 and Proposition 2. In fact, we may show that the both bounds are sharp if an edge magic number is a multiple of 3 (it should be a multiple of 3 in order for the second inequality to be sharp). For $k = 3n$, define the following partitions:

$$
\begin{aligned}
k &= 1+2+(3n-3)\\
&= 1+3+(3n-4)\\
&= 2+3+(3n-5)\\
&= 2+4+(3n-6)\\
&= \vdots\\
&= (3j-1)/2+(3j+1)/2+(3n-3j)\\
&= (3j-1)/2+(3j+3)/2+(3n-3j-1)\\
&= (3j+1)/2+(3j+3)/2+(3n-3j-2)\\
&= (3j+1)/2+(3j+5)/2+(3n-3j-3)\\
&= (3j+3)/2+(3j+5)/2+(3n-3j-4)\\
&= (3j+3)/2+(3j+7)/2+(3n-3j-5)\\
&= \vdots\\
&= (n-2)+n+(n+2)\\
&= (n-1)+n+(n+1)
\end{aligned}
$$

where j is an odd number satisfying $1 \leq j \leq (2n-1)/3$.
Then it can easily be checked that this collection of partitions satisfies properties 1, 2, 3 and therefore it determines the following super edge-magic graph denoted by BT_n with edge magic number k:

$$V(BT_n) = \{v_1, v_2, \ldots, v_n\};$$

$$E(BT_n) = \{v_i v_{i+1} \,|\, 1 \leq i \leq n-1\} \cup \{v_{2i-1}v_{2i+1} \,|\, 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor\}$$
$$\cup\ \{v_{2i}v_{2i+2} \,|\, 1 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor\}.$$

Since the number of edges of BT_n is $2n-3$, the upper bound given in Proposition 2 is achieved for BT_n. (The graph given in Figure 1 is BT_4.) The labeling $\lambda : V \cup E \to \{1, 2, \ldots, 3n-3\}$ determined by the above collection of partitions is as follows:

$$\lambda(v_i) = i, \text{ for } 1 \leq i \leq n;$$
$$\lambda(v_i v_{i+1}) = 3n - (2i+1), \text{ for } 1 \leq i \leq n-1;$$

$$\lambda(v_{2i-1}v_{2i+1}) = 3n - 4i, \text{ for } 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor;$$
$$\lambda(v_{2i}v_{2i+2}) = 3n - (4i+2), \text{ for } 1 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor.$$

Now we have the following proposition:

Proposition 3 *For every $n \geq 2$, the graph BT_n is super edge-magic with the maximum number of edges among the graphs with n vertices.*

3 Labeling some interesting families of super edge-magic graphs

In Section 2, we suggested a way to construct infinitely many super edge-magic graphs. In this section, we take some interesting families of graphs to see whether or not they are super edge-magic.

3.1 (n, t)-kite

An (n, t)-*kite* consists of a cycle of length n with a t-edge path (the tail) attached to one vertex. Wallis [3] posed a problem to investigate the edge-magic properties of (n, t)-kites for general t. We give a necessary condition for an (n, t)-kite being super edge-magic as follows:

Theorem 4 *Suppose that the graph (n, t)-kite is super edge-magic with an edge magic number k and a super edge-magic labeling λ. Let v be the vertex of degree 3 and w be the pendant vertex. Then the following are true:*

1. *n and t must have the same parity;*
2. *The edge magic number is either $k = \frac{5(n+t)}{2} + 1$ and $\lambda(v) - \lambda(w) = -\frac{n+t}{2}$ or $k = \frac{5(n+t)}{2} + 2$ and $\lambda(v) - \lambda(w) = \frac{n+t}{2}$.*

Proof. Let G be a graph of (n, t)-kite and let $v, v_1, \cdots, v_{n-1}, v$ be a vertex sequence of C_n. Let $\lambda(v) = \alpha$ and $\lambda(w) = \beta$. Then

$$\begin{aligned} k(n+t) &= \sum_{xy \in E(G)} [\lambda(x) + \lambda(xy) + \lambda(y)] \\ &= 2 \sum_{x \in V(G)} \lambda(x) + \lambda(v) - \lambda(w) + \sum_{xy \in E(G)} \lambda(xy) \\ &= \frac{2(n+t)(n+t+1)}{2} + \alpha - \beta + \frac{(n+t)[(n+t+1) + (2n+2t)]}{2} \\ &= \frac{(n+t)[5(n+t)+3]}{2} + \alpha - \beta. \end{aligned}$$

This implies that $k = \frac{5(n+t)+3}{2} + \frac{\alpha - \beta}{n+t}$ is an integer. Since $0 < |\frac{\alpha-\beta}{n+t}| < 1$, it is true that $|\frac{\alpha-\beta}{n+t}| = \frac{1}{2}$ and that n and t must have the same parity.

Furthermore, $k = \frac{5(n+t)}{2} + 1$ if $\lambda(v) - \lambda(w) = -\frac{n+t}{2}$, and $k = \frac{5(n+t)}{2} + 2$ if $\lambda(v) - \lambda(w) = \frac{n+t}{2}$. □

It has been shown (Theorem 2.23 in [3]) that an $(n, 1)$-kite is edge-magic. We show that it is also super edge-magic if n is odd and that the converse is also true.

Theorem 5 *An $(n, 1)$-kite is super edge-magic if and only if n is odd.*

Proof. The 'only if' part immediately follows from Theorem 4. To show the 'if' part, let $n = 2m + 1$ for a nonnegative integer m. We define a labeling $\lambda : V \cup E \to \{1, 2, 3, \ldots, 4m + 4\}$ as follows:

$$\begin{aligned}
\lambda(v_i) &= \begin{cases} i/2 + 1 & \text{if } i \text{ is even;} \\ m + 2 + (i + 1)/2 & \text{if } i \text{ is odd,} \end{cases} \\
\lambda(v) &= 1; \\
\lambda(w) &= m + 2; \\
\lambda(vw) &= 4m + 3; \\
\lambda(vv_{n-1}) &= 4m + 4; \\
\lambda(vv_1) &= 4m + 2; \\
\lambda(v_i v_{i+1}) &= 4m + 2 - i \text{ for } 1 \le i \le n - 2.
\end{aligned}$$

It is easily seen that λ is a super edge-magic labeling of an $(n, 1)$-kite with the edge magic number $5m + 6$. Hence, an $(n, 1)$-kite is super edge-magic graph. □

Park *et al.* [2] showed that $(n, 2)$-kite is super edge-magic for every positive even number n. The following theorem shows that $(n, 3)$-kite might not be super edge-magic even if n is odd.

Theorem 6 *A $(3, 3)$-kite is not super edge-magic.*

Proof. Suppose that a $(3, 3)$-kite denoted by G is a super edge-magic graph with an edge magic number k. Then there is a labelling λ from $V(G) \cup E(G)$ to $\{1, 2, \ldots, 12\}$. Let v be the vertex of degree 3 and w be the pendant. In addition, let $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, $\lambda(v) = \alpha$, and $\lambda(w) = \beta$. Then, by Theorem 4 either $k = 16$ and $\alpha - \beta = -3$ or $k = 17$ and $\alpha - \beta = 3$. Suppose that $k = 16$ and $\alpha - \beta = -3$. Then for each edge label, the possible labels of its endpoints are as follows:
From Table 1, we know that the labels of the endpoints of e_6 are 1 and 3. For convenience, we denote by $\Lambda(e_i)$ the set of labels for endpoints of edge e_i. There are exactly three cases to consider: (i) $\alpha = 1$ and $\beta = 4$; (ii) $\alpha = 2$ and $\beta = 5$; (iii) $\alpha = 3$ and $\beta = 6$. If $\alpha = 1$ and $\beta = 4$, then $\Lambda(e_5) \neq \{1, 4\}$ since v and w are not adjacent. Thus $\Lambda(e_5) = \{2, 3\}$. Since

Edge	e_1	e_2	e_3	e_4	e_5	e_6
Edge label	7	8	9	10	11	12
Possible sets of labels of endpoints	$\{3,6\}$ $\{4,5\}$	$\{2,6\}$ $\{3,5\}$	$\{1,6\}$ $\{2,5\}$ $\{3,4\}$	$\{1,5\}$ $\{2,4\}$	$\{1,4\}$ $\{2,3\}$	$\{1,3\}$

Table 1: Possible labels for $k = 16$

3 is used twice for a vertex label, $\Lambda(e_1) = \{4,5\}$ and $\Lambda(e_2) = \{2,6\}$. Since 2 is used twice, $\Lambda(e_3) = \{1,6\}$ and $\Lambda(e_4) = \{1,5\}$. From the fact that $\Lambda(e_1) = \{4,5\}$ and $\Lambda(e_4) = \{1,5\}$ have a common element, we know that edges e_1 and e_4 are adjacent and that the vertex with label 1 and that with label 4 are at distance two. Now we reach a contradiction. If $\alpha = 2$ and $\beta = 5$, then $\Lambda(e_3) \neq \{2,5\}$ since v and w are not adjacent. Since 2 is a label for v of degree 3, 2 must be used three times. Thus $\Lambda(e_2) = \{2,6\}$, $\Lambda(e_4) = \{2,4\}$, and $\Lambda(e_5) = \{2,3\}$. Since 3 is used twice, $\Lambda(e_1) = \{4,5\}$. Then edges e_1 and e_4 are adjacent, and so the vertex with label 2 and that with label 5 are at distance two. This is an contradiction again. If $\alpha = 3$ and $\beta = 6$, then $\Lambda(e_1) \neq \{3,6\}$ and so $\Lambda(e_1) = \{4,5\}$. Since label 6 must be used once, either $\Lambda(e_2) = \{2,6\}$ or $\Lambda(e_3) = \{1,6\}$. If $\Lambda(e_3) = \{1,6\}$, then e_3 and e_6 are adjacent, and so the vertex with label 3 and that with label 6 are at distance two, which is a contradiction. If $\Lambda(e_2) = \{2,6\}$, then $\Lambda(e_3) = \{3,4\}$ and $\Lambda(e_5) = \{2,3\}$ since 3 must be used three times. Then edges e_2 and e_5 are adjacent. Thus the vertex with label 3 and that with label 6 are at distance two and we reach a contradiction. Thus it is impossible that $k = 16$ and $\alpha - \beta = -3$.

We suppose that $k = 17$ and $\alpha - \beta = 3$. Then for each edge label, the possible labels of its endpoints are as follows:

Edge	e_1	e_2	e_3	e_4	e_5	e_6
Edge label	7	8	9	10	11	12
$\Lambda(e_i)$	$\{4,6\}$	$\{3,6\}$ $\{4,5\}$	$\{2,6\}$ $\{3,5\}$	$\{1,6\}$ $\{2,5\}$ $\{3,4\}$	$\{1,5\}$ $\{2,4\}$	$\{1,4\}$ $\{2,3\}$

Table 2: Possible labels for $k = 17$

From Table 2, we know that $\Lambda(e_1) = \{4,6\}$. There are exactly three cases to consider: (i) $\alpha = 4$ and $\beta = 1$; (ii) $\alpha = 5$ and $\beta = 2$; (iii) $\alpha = 6$ and $\beta = 3$. If $\alpha = 4$ and $\beta = 1$, then $\Lambda(e_6) \neq \{1,4\}$ and so $\Lambda(e_6) = \{2,3\}$. Since 1 must be used once, either $\Lambda(e_4) = \{1,6\}$ or $\Lambda(e_5) = \{1,5\}$. If $\Lambda(e_4) = \{1,6\}$, then

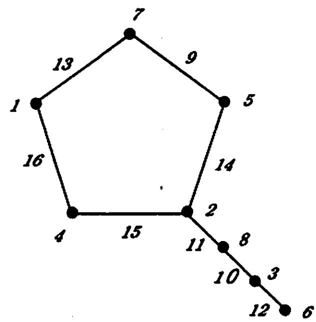

Figure 2: A super edge-magic labeling of $(5,3)$-kite.

edges e_1 and e_4 are adjacent, and so the vertex with label 1 and that with label 4 are at distance two. This is a contradiction. Thus $\Lambda(e_5) = \{1,5\}$. Since 4 is used three times, $\Lambda(e_2) = \{4,5\}$, $\Lambda(e_4) = \{3,4\}$. Then edges e_2 and e_5 are adjacent, and so so the vertex with label 1 and that with label 4 are at distance two. Thus we reach a contradiction. If $\alpha = 5$ and $\beta = 2$, then $\Lambda(e_4) \neq \{2,5\}$. Since 5 must be used three times, $\Lambda(e_2) = \{4,5\}$, $\Lambda(e_3) = \{3,5\}$, and $\Lambda(e_5) = \{1,5\}$. Since 4 is used twice, $\Lambda(e_6) = \{2,3\}$. Then edges e_3 and e_6 are adjacent, and so vertex with label 2 and that with label 5 are at distance two. Thus we reach a contradiction. If $\alpha = 6$ and $\beta = 3$, then $\Lambda(e_2) \neq \{3,6\}$ and so $\Lambda(e_2) = \{4,5\}$. Since 6 must be used three times, $\Lambda(e_3) = \{2,6\}$ and $\Lambda(e_4) = \{1,6\}$. Since 4 is used twice, $\Lambda(e_6) = \{2,3\}$. Then edges e_3 and e_6 are adjacent, and so the vertex with label 3 and that with label 6 are at distance two. Thus we reach a contradiction. Hence it is impossible that $k = 17$ and $\alpha - \beta = 3$. This completes the proof. □

Except the $(3,3)$-kite, an $(n,3)$-kite is super edge-magic as long as n is odd:

Theorem 7 *An $(n,3)$-kite is super edge-magic if and only if n is an odd integer greater than or equal to* 5.

Proof. The 'only if' part follows from Theorems 4 and 6. To show the 'if' part, let $v_0, v_1, \ldots, v_{n-1}, v_0$ be a vertex sequence of C_n with the tail yxw attached to v_0. Since n is an odd integer greater than or equal to 5, either $n = 4m+1$ or $4m+3$ for a positive integer m. First suppose that $n = 4m+1$ for a positive integer m. A labeling of $(5,3)$-kite is given in Figure 2.

For $m \geq 2$, we define a labeling $\lambda : V \cup E \to \{1,2,3,\ldots,8m+8\}$ as follows:

$$\lambda(v_i) = \begin{cases} i/2+1 & \text{if } i = 0, 2, \ldots, 2m+2; \\ 2m+6+(i-1)/2 & \text{if } i = 1, 3, \ldots, 2m-3; \\ m+3 & \text{if } i = 2m-1; \\ 3m+5 & \text{if } i = 2m+1; \\ (i-1)/2+3 & \text{if } i = 2m+3, 2m+5, \ldots, 4m-1; \\ i/2+2m+4 & \text{if } i = 2m+4, 2m+6, \ldots, 4m; \end{cases}$$

$$\begin{aligned} \lambda(y) &= 2m+4; \\ \lambda(x) &= 2m+5; \\ \lambda(w) &= 2m+3; \end{aligned}$$

$$\lambda(v_i v_{i+1}) = \begin{cases} 8m+4-i & \text{if } 0 \le i \le 2m-3 \\ 10m+6-i & \text{if } i = 2m-2, 2m-1; \\ 8m+5-i & \text{if } i = 2m, 2m+1; \\ 8m+5 & \text{if } i = 2m+2; \\ 8m+4-i & \text{if } 2m+3 \le i \le 4m-1; \end{cases}$$

$$\begin{aligned} \lambda(v_{4m} v_0) &= 6m+6; \\ \lambda(v_0 y) &= 8m+6; \\ \lambda(yx) &= 6m+2; \\ \lambda(xw) &= 6m+3. \end{aligned}$$

It is easily seen that λ is a super edge-magic labeling of $(n,3)$-kite with the edge magic number $10m+11$.

Now suppose that $n = 4m+3$. We define a labeling $\lambda : V \cup E \to \{1, 2, 3, \ldots, 8m+12\}$ as follows:

$$\lambda(v_i) = \begin{cases} i/2+m & \text{if } i = 0, 2, \ldots, 2m+2; \\ 3m+6+(i-1)/2 & \text{if } i = 1, 3, \ldots, 2m+1; \\ 2m+2 & \text{if } i = 2m+3; \\ i/2+m+1 & \text{if } i = 2m+4, 2m+6, \ldots, 4m+2; \\ (i-3)/2-m & \text{if } i = 2m+5, 2m+7, \ldots, 4m+1; \end{cases}$$

$$\begin{aligned} \lambda(y) &= 3m+4; \\ \lambda(x) &= 3m+5; \\ \lambda(w) &= 3m+3; \end{aligned}$$

$$\lambda(v_i v_{i+1}) = \begin{cases} 6m+10-i & \text{if } 0 \le i \le 2m+1 \\ 6m+13 & \text{if } i = 2m+2; \\ 6m+11 & \text{if } i = 2m+3; \\ 10m+16-i & \text{if } 2m+4 \le i \le 4m+1; \end{cases}$$

$$\begin{aligned}
\lambda(v_{4m+2}v_0) &= 6m+14;\\
\lambda(v_0y) &= 6m+12;\\
\lambda(yx) &= 4m+7;\\
\lambda(xw) &= 4m+8.
\end{aligned}$$

It is easily seen that λ is a super edge-magic labeling of $(n,3)$-kite with the edge magic number $10m+16$. □

3.2 $K_2 \cup C_n$

Wallis [3] proved that $K_2 \cup C_3$ is not edge-magic, but $K_2 \cup C_4$ is edge-magic. Then Wallis [3] proposed the following problem: For which values of n, is $K_2 \cup C_n$ edge-magic?

Park *et al.* [2] showed that $K_2 \cup C_n$ is super edge-magic for an even integer $n \neq 10$. They left the case $n = 10$ open as this case does not fit into the formula that they found. Here we present a super edge-magic labeling of $K_2 \cup C_n$ to complete the case where n is even. We could find this labeling by an exhaustive search based on the following observation:

Proposition 8 *If the graph $K_2 \cup C_n$ has a super edge-magic labeling λ for a positive even integer, then the edge magic number for λ is $\frac{1}{2}(5n+12)$.*

Proof. Let G denote $K_2 \cup C_n$ and let $v_0, v_1, \cdots, v_{n-1}, v_0$ be a vertex sequence of C_n and let u and w be the vertices of K_2. Since n is even, $n = 2l$ for a positive integer l. Let k be the edge magic number for λ. In addition, let $\lambda(u) = \alpha$ and $\lambda(w) = \beta$. Since $\nu = 2l+2$ and $\epsilon = 2l+1$,

$$\begin{aligned}
k(2l+1) &= \sum_{xy \in E(G)} [\lambda(x) + \lambda(xy) + \lambda(y)]\\
&= 2\sum_{x \in V(G)} \lambda(x) - [\lambda(u) + \lambda(w)] + \sum_{xy \in E(G)} \lambda(xy)\\
&= \frac{2(2l+2)(2l+3)}{2} - (\alpha+\beta) + \frac{(2l+1)[(2l+3)+(4l+3)]}{2}\\
&= (10l^2 + 19l + 9) - (\alpha+\beta).
\end{aligned}$$

Since $3 \leq \alpha+\beta \leq 4l+3$, it holds that $10l^2+15l+6 \leq k(2l+1) \leq 10l^2+19l+6$. Since k is an integer, $k = 5l+6 = \frac{1}{2}(5n+12)$. □

Proposition 9 *The graph $K_2 \cup C_{10}$ is super edge-magic.*

Proof. A super edge-magic labeling is given in Figure 3 when an edge magic number is 31. □

The following theorem completely characterizes super edge-magic $K_2 \cup C_n$:

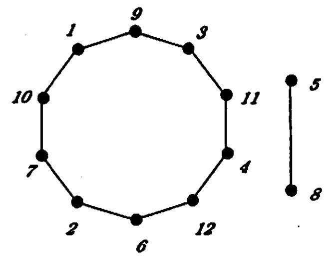

Figure 3: An super edge-magic labeling of $K_2 \cup C_{10}$.

Theorem 10 *The graph $K_2 \cup C_n$ is super edge-magic if and only if n is even.*

Proof. The 'if' part immediately follows from the result of Park *et al.* [2] and Proposition 9.

To show the 'only if' part, let G denote $K_2 \cup C_n$ and let $v_0, v_1, \cdots, v_{n-1}, v_0$ be a vertex sequence of C_n and let u and w be the vertices of K_2. We prove by contradiction. Suppose that n is odd and that G has a super edge-magic labeling λ with edge magic number k. Then $n = 2m + 1$ for a nonnegative integer m. Let $\lambda(u) = a$ and $\lambda(w) = b$. Since $\nu = 2m + 3$ and $\epsilon = 2m + 2$,

$$\begin{aligned} k(2m+2) &= \sum_{xy \in E(G)} [\lambda(x) + \lambda(xy) + \lambda(y)] \\ &= 2 \sum_{x \in V(G)} \lambda(x) - [\lambda(u) + \lambda(w)] + \sum_{xy \in E(G)} \lambda(xy) \\ &= \frac{2(2m+3)(2m+4)}{2} - (a+b) + \frac{(2m+2)\,[(2m+4)+(4m+5)]}{2} \\ &= (10m^2 + 29m + 21) - (a+b). \qquad (1) \end{aligned}$$

Since $3 \leq (a+b) \leq 4m+5$, it follows from the equality (1) that

$$10m^2 + 25m + 16 \leq k(2m+2) \leq 10m^2 + 29m + 18.$$

Since k is an integer, $k = 5m+8$ or $5m+9$. Firstly suppose that $k = 5m+8$. For any distinct vertices x and y, since $2m+4 \leq \lambda(xy) \leq 4m+5$ and $\lambda(x) + \lambda(xy) + \lambda(y) = 5m+8$,

$$m + 3 \leq \lambda(x) + \lambda(y) \leq 3m + 4. \qquad (2)$$

However, it follows from equality (1) that $(5m+8)(2m+2) = (10m^2 + 29m + 21) - (a+b)$ and so $a + b = \lambda(u) + \lambda(w) = 3m + 5$. This contradicts (2).

Now suppose that $k = 5m + 9$. Then for any distinct vertices x and y, by a similar argument to the case $k = 5m + 8$, we can show that

$$m + 4 \le \lambda(x) + \lambda(y) \le 3m + 5. \tag{3}$$

However, if follows from equality (1) that $\lambda(u) + \lambda(w) = m + 3$. This contradicts (3). □

3.3 A graph derived from a star by adding a pendant edge to each vertex of degree 1

It is known that the star $K_{1,n}$ is super edge-magic. In addition, it is known that a graph derived from a star by adding a pendant edge to each vertex of degree 1 is super edge-magic. The following theorem shows that graphs derived from a star by adding a pedant edge to each vertex of degree 1 are super edge-magic. These graphs were studied while we sought a counterexample to the conjecture that every tree is super edge-magic. It took quite an effort to find a super edge-magic labelings of such a graph and this might suggest that the conjecture seems to be rather difficult to answer.

Theorem 11 *A graph G derived from a star by adding a pedant edge to each vertex of degree* 1 *is super edge-magic.*

Proof. Let v_0 be a central vertex of $K_{1,n}$ and $v_1, \ldots, v_n$ be the pendant vertices of $K_{1,n}$. Also, let w_i be the pendant vertex of G adjacent to v_i for $i = 1, \ldots, n$. First we consider the case where n is odd. Then $n = 2m + 1$ for some nonnegative integer m. Define a labeling $\lambda : V \cup E \to \{1, \ldots, \nu + \epsilon\}$ as follows:

$$\lambda(v_0) = m + 2;$$

$$\lambda(v_i) = \begin{cases} i + 1 & \text{if } 1 \le i \le m; \\ i + 2 & \text{if } m + 1 \le i \le 2m; \\ 2m + 3 & \text{if } i = 2m + 1; \end{cases}$$

$$\lambda(w_i) = \begin{cases} 4m - 2i + 5 & \text{if } 1 \le i \le m; \\ 6m - 2i + 4 & \text{if } m + 1 \le i \le 2m; \\ 1 & \text{if } i = 2m + 1; \end{cases}$$

$$\lambda(v_0 v_i) = \begin{cases} 8m - i + 6 & \text{if } 1 \le i \le m; \\ 8m - i + 5 & \text{if } m + 1 \le i \le 2m; \\ 6m + 4 & \text{if } i = 2m + 1; \end{cases}$$

$$\lambda(v_i w_i) = \begin{cases} 5m + i + 3 & \text{if } 1 \le i \le m; \\ 3m + i + 3 & \text{if } m + 1 \le i \le 2m; \\ 7m + 5 & \text{if } i = 2m + 1. \end{cases}$$

It can easily be seen that λ is a super edge-magic labeling of G with the edge magic number $9m+9$.

Now we consider the case where n is even. Then $n = 2m$ for some positive integer m. We consider two subcases. First assume that m is even. We define a labeling $\lambda : V \cup E \to \{1, \ldots, \nu + \epsilon\}$ as follows:

$$\lambda(v_0) = m+2;$$

$$\lambda(v_i) = \begin{cases} i+1 & \text{if } 1 \le i \le m; \\ i+3 & \text{if } m+1 \le i \le 2m-1; \\ 2m+4 & \text{if } i = 2m; \end{cases}$$

$$\lambda(w_i) = \begin{cases} 3m+i+2 & \text{if } 1 \le i \le m-1; \\ m+3 & \text{if } i = m; \\ m+i+5 & \text{if } i = m+1, m+3, \ldots, 2m-3; \\ m+i+1 & \text{if } i = m+2, m+4, \ldots, 2m-2; \\ 3m+1 & \text{if } i = 2m-1; \\ 1 & \text{if } i = 2m; \end{cases}$$

$$\lambda(v_0v_i) = \begin{cases} 8m-i+2 & \text{if } 1 \le i \le m; \\ 8m-i & \text{if } m+1 \le i \le 2m-1; \\ 6m-1 & \text{if } i = 2m; \end{cases}$$

$$\lambda(v_iw_i) = \begin{cases} 6m-2i+2 & \text{if } 1 \le i \le m-1; \\ 7m+1 & \text{if } i = m; \\ 8m-2i-3 & \text{if } i = m+1, m+3, \ldots, 2m-3; \\ 8m-2i+1 & \text{if } i = m+2, m+4, \ldots, 2m-2; \\ 4m+2 & \text{if } i = 2m-1; \\ 7m & \text{if } i = 2m. \end{cases}$$

Since m is even, $m+1$, $m+3$, $\ldots$, $2m-3$ and $m+2$, $\ldots$, $2m-2$ both are arithmetic sequences with common difference 2 and so λ is well-defined. Then, λ is a super edge-magic labeling of G with edge magic number $9m+5$ for m even. Finally, we consider the case where m is odd. We define a labeling $\lambda : V \cup E \to \{1, \ldots, \nu + \epsilon\}$ as follows:

$$\lambda(v_0) = m+2;$$

$$\lambda(v_i) = \begin{cases} i+1 & \text{if } 1 \le i \le m; \\ i+3 & \text{if } m+1 \le i \le 2m-1; \\ 2m+4 & \text{if } i = 2m; \end{cases}$$

$$\lambda(w_i) = \begin{cases} 3m+i+2 & \text{if } 1 \le i \le m-1; \\ m+3 & \text{if } i = m; \\ m+i+5 & \text{if } i = m+1, m+3, \ldots, 2m-4; \\ m+i+1 & \text{if } i = m+2, m+4, \ldots, 2m-3; \\ 3m+2 & \text{if } i = 2m-2; \\ 3m & \text{if } i = 2m-1; \\ 1 & \text{if } i = 2m; \end{cases}$$

$$\lambda(v_0v_i) = \begin{cases} 8m-i+2 & \text{if } 1 \le i \le m; \\ 8m-i & \text{if } m+1 \le i \le 2m-1; \\ 6m-1 & \text{if } i = 2m; \end{cases}$$

$$\lambda(v_iw_i) = \begin{cases} 6m-2i+2 & \text{if } 1 \le i \le m-1; \\ 7m+1 & \text{if } i = m; \\ 8m-2i-3 & \text{if } i = m+1, m+3, \ldots, 2m-4; \\ 8m-2i+1 & \text{if } i = m+2, m+4, \ldots, 2m-3; \\ 4m+2 & \text{if } i = 2m-2; \\ 4m+3 & \text{if } i = 2m-1; \\ 7m & \text{if } i = 2m. \end{cases}$$

Since m odd, $m+1$, $m+3$, $\ldots$, $2m-4$ and $m+2$, $\ldots$, $2m-3$ both are arithmetic sequences with common difference 2 and so λ is well-defined. Then, λ is a super edge-magic labeling of G with edge magic number $9m+5$ for m odd. This completes the proof. □

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