Graceful Lobsters Obtained by Component Moving of Diameter Four Trees

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Abstract

We observe that a lobster with diameter at least five has a unique path $x_0, x_1, \ldots, x_m$ (called the central path) such that x_0 and x_m are adjacent to the centers of at least one $K_{1,s}$, $s > 0$, and besides adjacencies in the central path each x_i, $1 \leq i \leq m-1$, is at most adjacent to the centers of some $K_{1,s}$, $s \geq 0$. In this paper we give graceful labelings to some new classes of lobsters with diameter at least five, in which the degree of the vertex x_m is odd and the degree of each of the remaining vertices on the central path is even. The main idea used to obtain these graceful lobsters to form a diameter four tree $T(L)$ from a lobster L of certain type, give a graceful labeling to $T(L)$ and finally get a graceful labeling of L by applying component moving and inverse transformations.

Keywords: graceful labeling, lobster, odd and even branches, inverse transformation, component moving transformation.

AMS classification: 05C78

1 Introduction

Recall that a *graceful labeling* of a tree T with q edges is a bijection $f : V(T) \to \{0, 1, 2, \ldots, q\}$ such that $\{|f(u) - f(v)| : \{u, v\}$ is an edge of $T\} =$

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$\{1, 2, \ldots, q\}$. A tree which has a graceful labeling is called a *graceful tree*. A *lobster* is a tree having a path (of maximum length) from which every vertex has distance at most two. It is easy to check that the following lemma holds.

Lemma 1.1 If L is a lobster with diameter at least five then there exists a unique path $H = x_0, x_1, x_2, \ldots, x_m$ in L such that

(*i*) x_0 and x_m are adjacent to the centers of at least one star $K_{1,s}$ with $s \geq 1$.

(*ii*) besides the adjacency in H, each x_i, $1 \leq i \leq m-1$, is at most adjacent to the centers of some stars $K_{1,s}$, $s \geq 0$.

Notation 1.2 We shall say the unique path $H = x_0, x_1, x_2, \ldots, x_m$ in Lemma 1.1 is the *central path* of the lobster. Throughout the paper we use H to denote the central path of a lobster with diameter at least five. Take $x_i \in V(H)$. If x_i is adjacent to the center of $K_{1,s}, s > 0$ then $K_{1,s}$ will be called an *even branch* if s is even or an *odd branch* if s is odd. If x_i is adjacent to a pendant vertex v (i.e. $K_{1,0}$) then v will be called a *pendant branch*. If x_i is adjacent to the center of some branch, then we say the branch is *incident on* x_i or x_i *is attached to* the branch. In case of a diameter four tree T, if the center vertex a_0 is adjacent to the center of some $K_{1,s}$ with $s \geq 0$, then these $K_{1,s}$ will be called the even, odd and pendant branches as above. Furthermore, whenever we say x_i, for some $0 \leq i \leq m$, is attached to even number of branches we mean "non zero" even number of branches unless otherwise stated.

In 1979, Bermond [1] conjectured that all lobsters are graceful. This conjecture is open and very few classes of lobsters are known to be graceful so far. By the survey paper of Gallian [3] the known graceful lobsters are due to Ng [6], Wang et al. [7], Chen et al. [2] and Morgan [5]. Figures 1 and 2 show the graceful lobsters due to Ng and Chen et al., respectively. Morgan has proved that a lobster is graceful if it has a perfect matching. Wang et al. have given a graceful labeling to the lobsters with diameter at least five in which the degree of x_m is odd, the degree of the rest of the vertices x_i, $0 \leq i \leq m-1$, are even, and the branches incident on the vertices of the central path are either all odd or all even (see Figure 3).

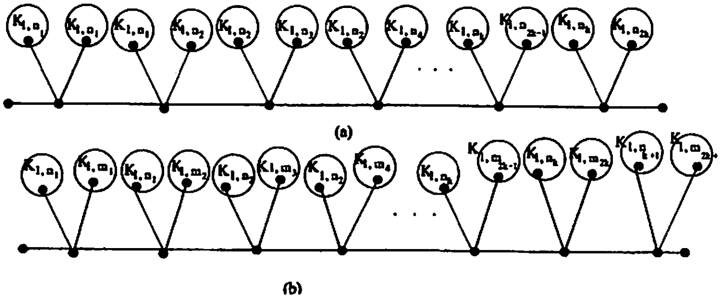

Figure 1: The graceful lobsters given by Ng.

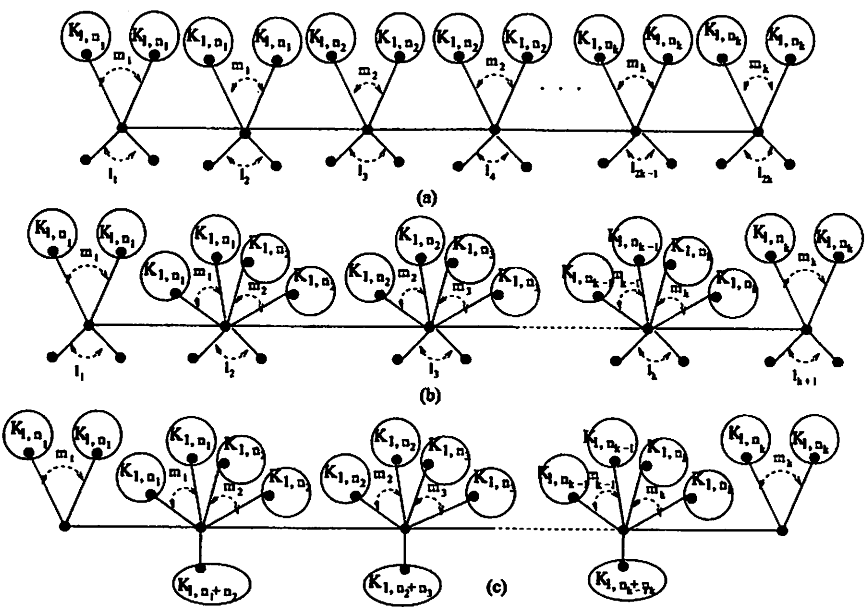

Figure 2: The graceful lobsters given by Chen et al.

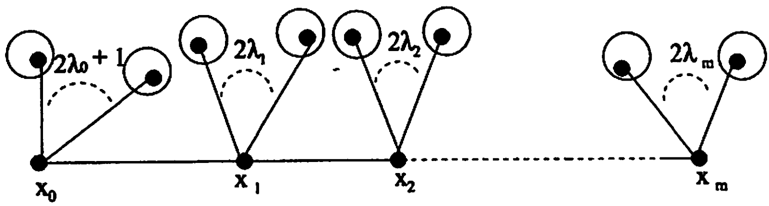

Figure 3: The graceful lobsters given by Wang et al. ($m \geq 1$, $\lambda_0 \geq 0$, $\lambda_i > 0$, $i = 1, 2, \ldots, m$, and the circles enclosing the vertices adjacent to the central path are either all odd or all even branches.)

Motivated by [7] and the graceful labeling of diameter four trees in [4] we show in this paper that the ideas of Wang et al. are not just limited to giving a graceful labeling to the lobsters shown in Figure 3. Here we give a graceful labeling to the lobsters in which each x_i, $i = 1, 2, 3, \ldots, m$, is attached to an even number of branches of the same type or two different types of certain combinations. The vertex x_0 is attached to an odd number of branches and these branches may be of the same type or two or three different types of certain combinations. Of course the lobsters in which x_0 is attached to only one type of branches are the same lobsters that appear in [7] so the graceful lobsters that appear in [7] follow from our result.

To prove our results we need some definitions and terminologies which are described below. Although inverse transformation and component moving transformation are explained in many papers, for example [7] and [4], to make the paper self contained we include them here also.

Definition 1.3 If f is a labeling of a tree T with n edges then the labeling f_n defined as $f_n(v) = n - f(v)$, for all $v \in V(T)$, is called the inverse transformation of f.

Definition 1.4 For an edge $e = \{u, v\}$ of a tree T, we define $u(T)$ as that connected component of $T - e$ which contains the vertex u. Here we say $u(T)$ is a component incident on the vertex v. If a and b are vertices of a tree T, $u(T)$ is a component incident on a, and the component $u(T)$ does not contain the vertex b then deleting the edge $\{a, u\}$ from T and making b and u adjacent is called the component moving transformation.

Here we say the component $u(T)$ has been transformed from a to b. This is illustrated in Figure 4.

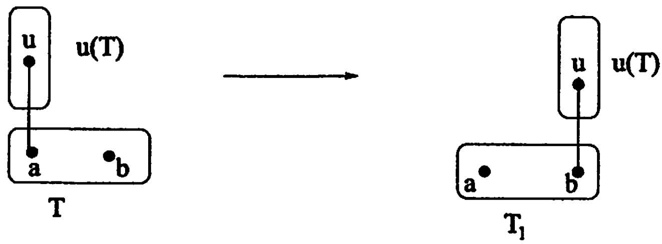

Figure 4: The tree T_1 obtained from the tree T by moving the component $u(T)$ from the vertex a to the vertex b.

Throughout the paper we write "the component u" instead of writing "the component $u(T)$". Thereby whenever we wish to refer to u as a vertex, we write "the vertex u". Moreover we shall not distinguish between a vertex and its label. Lemma 1.5 and Lemma 1.6 below may be found in [7].

Lemma 1.5 If f is a graceful labeling of a tree T then the inverse transformation of f is also a graceful labeling of T.

Lemma 1.6 Let f be a graceful labeling of a tree T; let a and b be two vertices of T; let $u(T)$ and $v(T)$ be two components incident on a where $u(T) \cup v(T)$ does not contain the vertex b. Then the following hold:

(*i*) if $f(u)+f(v) = f(a)+f(b)$ then the tree T^* obtained from T by moving the components $u(T)$ and $v(T)$ from a to b is also graceful.

(*ii*) if $2f(u) = f(a) + f(b)$ then the tree T^{**} obtained from T by moving the component $u(T)$ from a to b is also graceful.

The following lemma is due to [4].

Lemma 1.7 Let T be a diameter four tree with q edges. If a_0 is the center vertex and the degree of a_0 is $2k+1$ then there exists a graceful labeling f of T such that

(*a*) $f(a_0) = 0$ and the labelings of the neighbours of a_0 are $1, 2, \ldots, k, q, q-1, \ldots, q-k$.

(b) from the sequence $S = (q, 1, q-1, 2, q-2, 3, \ldots, q-k+1, k, q-k)$ of vertex labels, r_1 elements from the beginning are the labels of the centers of the odd branches then the next r_2 elements are the labels of the centers of the even branches and rest of the elements are the labels of the centers of the pendant branches, where $0 \leq r_1, r_2 \leq 2k+1$, and $r_1 + r_2 \geq 2$.

2 Results

In this section we give a theorem (Theorem 2.3) for general graceful trees and then apply it to diameter four trees to obtain our graceful lobsters. The lemma below plays an important role in proving Theorem 2.3.

Lemma 2.1 Let $S = (t_1, t_2, \ldots, t_{2p})$ be a finite sequence of natural numbers in which the sums of consecutive terms are alternately $l+1$ and l, beginning (and ending) with the sum $l+1$. Then the sums of consecutive terms in the sequence $S_1 = (\phi_{l+1}(t_{2k+2}), \phi_{l+1}(t_{2k+3}), \ldots, \phi_{l+1}(t_{2p-2k_1-1}))$ where $\phi_n(x) = n - x$, $0 \leq k, k_1 \leq p-2$, and $0 \leq k + k_1 \leq p-2$, are alternately $l+2$ and $l+1$, beginning (and ending) with $l+2$.

Proof: We have for $2k+2 \leq i \leq 2p-2k_1-1$

$$\phi_{l+1}(t_i) + \phi_{l+1}(t_{i+1}) = 2(l+1) - (t_i + t_{i+1})$$

$$= \begin{cases} 2(l+1) - (l+1) & \text{if } (t_i + t_{i+1}) = l+1 \\ 2(l+1) - l & \text{if } (t_i + t_{i+1}) = l \end{cases}$$

$$= \begin{cases} l+1 & \text{if } (t_i + t_{i+1}) = l+1 \\ l+2 & \text{if } (t_i + t_{i+1}) = l \end{cases}$$

Therefore, the sums of consecutive terms of the sequence S_1 are $l+1$ and $l+2$ alternately. Moreover, the sum of the first two terms, $\phi_{l+1}(t_{2k+2}) + \phi_{l+1}(t_{2k+3})$, is $l+2$ as $t_{2k+2} + t_{2k+3} = l$. Since the total number of terms in S_1 is even the sum of the last two terms is $l+2$. □

Construction 2.2 Let T be a graceful tree with q edges. Let a_0 be a non pendant vertex of T with degree $2k+1$ such that there exists a graceful labeling f of T in which a_0 gets the label 0 and the labels of the neighbours of a_0 are $1, 2, \ldots, k, q, q-1, q-2, \ldots, q-k$ (see Figure 5).

We consider the sequence $A = (q, 1, q-1, 2, q-2, 3, \ldots, k, q-k)$ of vertices adjacent to a_0 (as we do not distinguish between a vertex and its label).

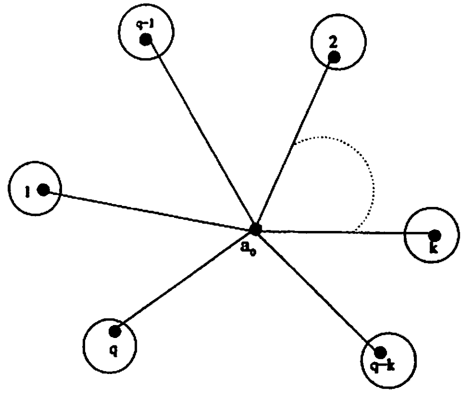

Figure 5: The tree T with vertex a_0 and its neighbours. The circle around the neighbouring vertices of a_0 represent the respective components incident on them.

We construct a tree T_1 (see Figure 6) from T by identifying the vertex y_0 of a path $H' = y_0, y_1, \ldots, y_m$ with a_0 and moving the components (incident on the vertex a_0) in A to y_i in the following way:

(1) At y_0 we retain $2\lambda_0 + 1 (\lambda_0 \geq 0)$ branches. In particular, we retain $2p_0$ branches, $0 \leq p_0 \leq \lambda_0$, whose center vertices are from the beginning of A namely $q, 1, q-1, 2, q-2, 3, \ldots, q-p_0+1, p_0$ and $2\lambda_0 + 1 - 2p_0$ branches whose center vertices are from the end of A namely $q-k, k, q-k+1, k-1, \ldots, k-\lambda_0+p_0+1, q-k+\lambda_0-p_0$. Then we delete these vertices from A which are kept at y_0 and name the remaining sequence as $A^{(1)}$.

(2) For $1 \leq i \leq m$, to y_i we move $2\lambda_i$ ($\lambda_i \geq 1$) branches. In particular, we move $2p_i + 1$ branches, $0 \leq p_i < \lambda_i$ whose center vertices are from the beginning of $A^{(i)}$ and $2\lambda_i - 2p_i - 1$ branches whose center vertices are from the end of $A^{(i)}$, where $A^{(i)}$ for $i \geq 2$ is obtained from $A^{(i-1)}$ by deleting the branches which are moved to y_{i-1}. The numbers λ_i, $i = 0, 1, 2, \ldots, m$, are chosen in such a way that

$$\sum_{i=0}^{m} \lambda_i = k$$

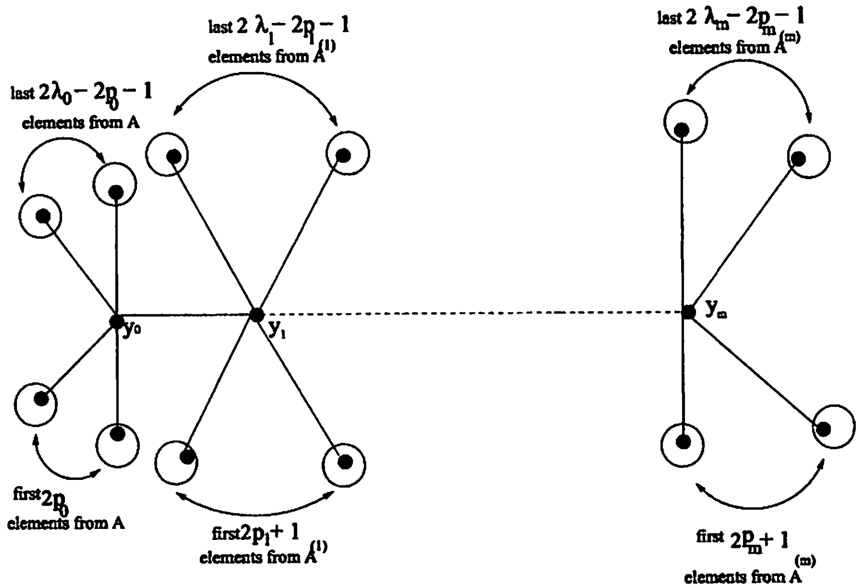

Figure 6: The tree T_1 obtained from T.

In the following theorem, for a graceful tree R with n edges and a graceful labeling g of R we use the notation "$g(R)$" to denote the tree R with the graceful labeling g. Also, for any sequence $F = (a_1, a_2, \dots, a_r)$, $g_n(F)$ is the sequence $(n - a_1, n - a_2, \dots, n - a_r)$.

Theorem 2.3 The tree T_1 in Construction 2.2 is graceful.

Proof: Recall that we denote an edge whose end points are x and y by $\{x, y\}$. We first consider the tree $T \cup \{y_0, y_1\}$ where the vertices a_0 and y_0 are identified. We give the label $q+1$ to y_1. Clearly $T \cup \{y_0, y_1\}$ is graceful with a graceful labeling f' where f' is same as f on T and it gives the label $q+1$ to y_1. Then we move all the components in $A^{(1)}$ to y_1 and let the resultant tree be T'. One can notice that $A^{(1)}$ can be partitioned into pairs of labels whose sum is $q+1$ (consecutive terms). So T' is graceful by Lemma 1.6(i) and f' is a graceful labeling of it.

Next we consider the inverse transformation f'_{q+1} of the graceful labeling f' of T'. So f'_{q+1} is a graceful labeling of T' by Lemma 1.5 and the label of y_1 in $f'_{q+1}(T')$ is 0. Next we make y_2 adjacent to y_1 in $f'_{q+1}(T')$ and give the label $q+2$ to y_2. Obviously $T' \cup \{y_1, y_2\}$ is graceful with a graceful labeling f'' where f'' is same as f'_{q+1} on T' and it gives the label $q+2$ to y_2. We move all the components in $f'_{q+1}(A^{(2)})$ from y_1 to y_2 and let the resultant tree be T''. Observe that the sums of consecutive terms in $A^{(1)}$ are alternately $q+1$ and q (beginning and ending with $q+1$) so by Lemma 2.1 the sums of consecutive terms in $f'_{q+1}(A^{(2)})$ are alternately $q+2$ and $q+1$. One sees that $f'_{q+1}(A^{(2)})$ can be partitioned into pairs of labels whose sum is $q+2$. So from Lemma 1.6(i) T'' is graceful.

We repeat the above procedure m times (we can repeat because of Lemma 2.1). The graceful tree that is obtained on the vertex set $V(T) \cup V(H')$ is easily seen to be the tree T_1. □

The lobsters we consider here will be of diameter at least five, so we use the representation of Lemma 1.1. Given a lobster L in which x_0 is attached to odd number of branches and each x_i, $i = 1, 2, \ldots, m$, is attached to even number of branches we construct a diameter four tree, say $T(L)$, from L by successively identifying the vertices x_i, $i = 1, 2, \ldots, m$ with x_0. It is clear that x_0 is the center of $T(L)$ and its degree is odd. By [4], $T(L)$ has a graceful labeling in which x_0 gets the label 0 and the neighbours of x_0 get labels in the sequence A of Construction 2.2.

Next we apply Theorem 2.3 to $T(L)$ and to the central path $H = x_0, x_1, \ldots, x_m$ to get a graceful labeling of L. However, we note that the order in which the centers of the branches attached to x_0 get labels from the sequence A plays an important role. To get back L and a graceful labeling of it we have to follow an appropriate ordering of the branches attached at x_0 in $T(L)$, which will be clear from the proof of the theorem which follows.

Theorem 2.4 The lobsters in the Tables 2.1, 2.2 and 2.3 below are graceful.

Explanation of Tables: The first, second, and third components of the triple (., ., .) signify the number of odd branches, even branches, and pendant branches, respectively. The symbol e stands for an even number of branches, o stands for an odd number of branches and 0 stands for no branches. For example $(e, 0, o)$ means an even number of odd branches, no even branches and an odd number of pendant branches. If a branch is listed under x_i it means that these branches are attached to x_i.

Table 2.1

Lobsters↓	x_0	x_i, $1 \leq i \leq t_1$	x_i, $t_1+1 \leq i \leq t_2$	x_i, $t_2+1 \leq i \leq m$
(a)	$(e,0,o)$	$(o,0,o)$	$(o,o,0)$	$(e,0,0)$ or $(0,e,0)$
(b)	same as (a)	same as (a)	$(0,o,o)$	$(0,e,0)$

Table 2.2

Lobsters↓	x_0	x_i, $1 \leq i \leq t$	$x_i, t+1 \leq i \leq m$
(a)	$(e,0,o)$	$(o,o,0)$	$(e,0,0)$ or $(0,e,0)$
(b)	same as (a)	$(0,o,o)$	$(0,e,0)$
(c)	same as (a)	$(o,0,o)$	same as (a)
(d)	same as (a)	same as (c)	$(o,o,0)$
(e)	same as (a)	same as (c)	$(0,o,o)$
(f)	$(0,e,o)$	$(0,o,o)$	same as (b)
(g)	$(e,o,0)$	$(o,o,0)$	same as (a)
(h)	(e,e,o)	same as (f)	same as (b)
(i)	(o,o,o)	same as (f)	same as (b)
(j)	same as (h)	same as (g)	same as (a)
(k)	(e,o,e)	same as (g)	same as (a)

Table 2.3

Lobsters↓	x_0	x_i, $1 \leq i \leq m$	Lobsters↓	x_0	x_i, $1 \leq i \leq m$
(a)	$(e,0,o)$	$(e,0,0)$ or $(0,e,0)$	(i)	$(o,0,0)$ (respectively $(0,o,0)$)	$(e,0,0)$ (respectively $(0,e,0)$)
(b)	same as (a)	$(o,o,0)$	(j)	(o,o,o)	same as (g)
(c)	same as (a)	$(0,o,o)$	(k)	(e,e,o)	same as (c)
(d)	same as (a)	$(o,0,o)$	(l)	same as (j)	same as (c)
(e)	$(0,e,o)$	same as (c)	(m)	same as (k)	same as (a)
(f)	$(e,o,0)$	same as (b)	(n)	(e,o,e)	same as (a)
(g)	same as (e)	$(0,e,0)$	(o)	same as (k)	same as (b)
(h)	same as (f)	same as (a)	(p)	same as (n)	same as (b)

Proof: For every lobster L of this theorem we first construct the diameter four tree $T(L)$ corresponding to L. Then we give a graceful labeling to $T(L)$ using the techniques of [4]. Moreover, in the techniques of [4] the center

x_0 and the centers of the branches attached to x_0 get labels according to Lemma 1.7. Recall that in Lemma 1.7 the center of $T(L)$ gets label zero and the centers of the branches attached to x_0 get labels from the sequence A of Construction 2.2. However to get a graceful labeling of L the manner in which we give a graceful labeling to $T(L)$ (in particular to the center of the branches attached to x_0) is different for the different cases.

First let L be a lobster of type (a) in Table 2.1. Then to $T(L)$ we give the following labeling :

1. The centers of the odd (respectively pendant) branches attached to x_0 in L get labels consecutively from the beginning (respectively end) of A.

2. For $i = 1, 2, \ldots, t_1$, the centers of the odd branches (respectively pendant branches) attached to x_i in L get labels consecutively from the beginning (respectively end) of $A^{(i)}$, where $A^{(i)}$ are as defined in Construction 2.2.

3. For $i = t_1 + 1, t_1 + 2, \ldots, t_2$, the centers of the odd branches (respectively even branches) attached to x_i in L get labels consecutively from the beginning (respectively end) of the sequence $A^{(i)}$, where $A^{(i)}$ are as defined in Construction 2.2.

4. For $i = t_2 + 1, \ldots, m$, among the odd (or even) branches attached to x_i in L, the centers of any odd number of these branches get labels consecutively from the beginning of $A^{(i)}$ and the centers of rest of these branches get labels consecutively from the end of $A^{(i)}$, where $A^{(i)}$ are as defined in Construction 2.2.

Finally, we apply Theorem 2.3 to $T(L)$ with this graceful labeling, as well as to the path $H = x_0, x_1, \ldots, x_m$, so as to get a graceful labeling of L (see example below). This approach will be common for all the lobsters of this theorem so for the remaining cases we mention the changes in steps 1 to 4 only.

Example: Consider the lobster L presented in Figure 7 which is of the type given in Table 2.1(a). We construct the graceful diameter four tree $T(L)$

shown in Figure 8 by following the steps mentioned in the proof above. The total number of edges in $T(L)$ is $q = 85$ and the degree of the center is $2k+1 = 31$. So $k = 15$ and the sequence $A = \{85, 1, 84, 2, \ldots, 15, 70\}$. From [4] we obtain a graceful labeling of $T(L)$ given in Figure 8. Then in Figure 9 we make x_1 adjacent with x_0, give label 86 to x_1 and move all the components in $A^{(1)}$ to x_1. The tree in Figure 10 is obtained by applying the inverse transformation to the lobster found in Figure 9, making x_2 adjacent to x_1, giving label 87 to x_2, and moving all the components in $f'_{86}(A^{(2)})$ to x_2. Continuing in this manner, we finally get the graceful labeling of L presented in Figure 11.

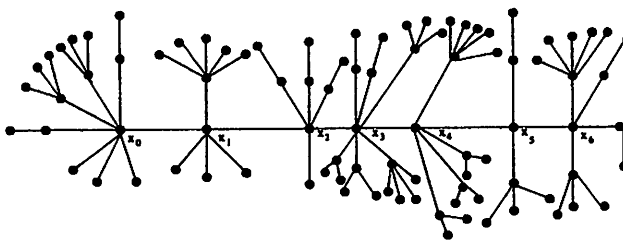

Figure 7: The lobster L of type (a) in Table 2.1. Here $m = 6$, $t_1 = 2$, $t_2 = 4$ and each x_i, $i = t_2+1, \ldots, m$, is attached to even number of odd branches.

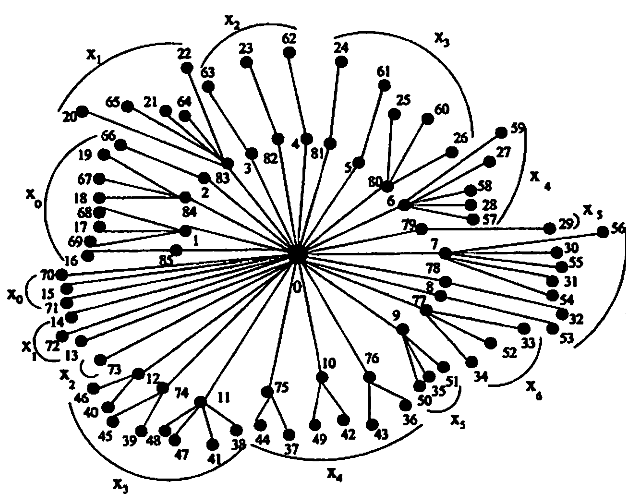

Figure 8: The tree $T(L)$ corresponding to the lobster in Figure 7.

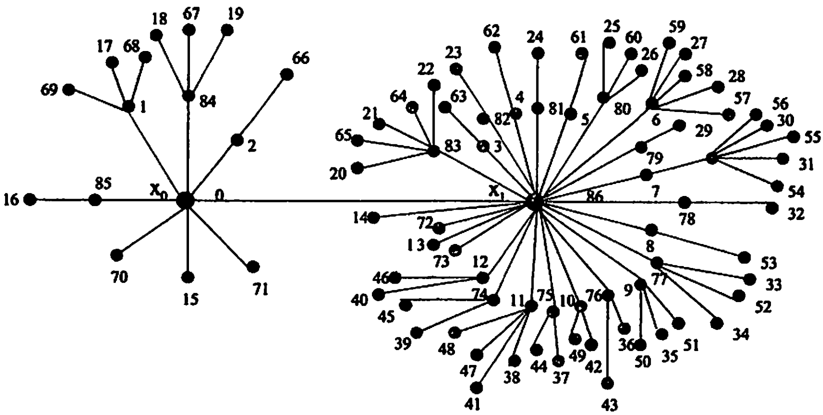

Figure 9: The graceful lobster obtained by making x_1 adjacent to x_0, giving x_1 the label 86, and moving all of the branches in $A^{(1)}$ to x_1.

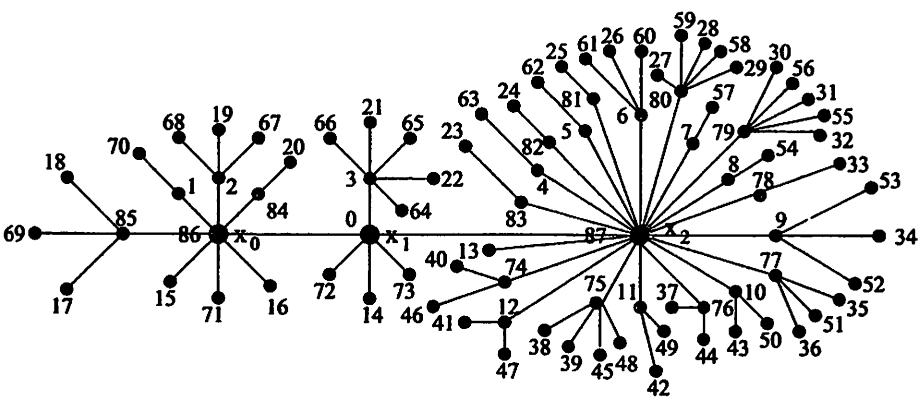

Figure 10: The graceful lobster obtained by applying inverse transformation to the lobster in Figure 9, making x_2 adjacent to x_1, giving the label 87 to x_2, and moving all the components in $f'_{86}(A^{(2)})$ to x_2.

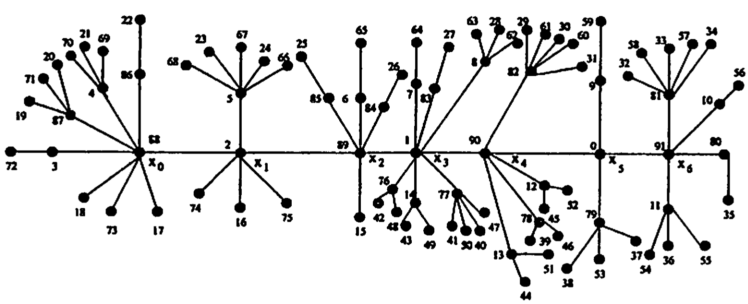

Figure 11: The lobster L with a graceful labeling.

For lobsters of type (b) in Table 2.1, the proof follows if we proceed as in the previous case by replacing the odd branches by even branches and even branches by pendant branches in step 3 and taking only even branches in step 4.

For lobsters of type (a) (respectively (b)) in Table 2.2, the proof follows if we proceed as the proof involving the lobsters described in Table 2.1(a) (respectively 2.1(b)) by repeating steps 1, 3, and 4, by setting $t_1 = 0$ and $t_2 = t$.

For lobsters of type (c), (d) and (e) the proof follows if we proceed as the proof involving the lobsters described in Table 2.1(a) by giving the following labeling to $T(L)$:

1. Same as step 1.
2. Set $t_1 = t$ in step 2.
3. If L is of type (c), then set $t_2 = t$ in step 4. If L is of type (d) (respectively (e)), then set $t_1 = t$ and $t_2 = m$ in step 3 in the proof involving the lobsters described in Table 2.1(a) (respectively 2.1(b)).

For lobsters of type (f) (respectively (g)) in Table 2.2, the proof follows if we proceed as the proof involving the lobsters described in Table 2.2(b) (respectively 2.2(a)) by replacing the odd (respectively pendant) branches by even branches in step 1 only.

For lobsters of type (h) and (i) in Table 2.2, the proof follows if we proceed as the proof involving the lobsters described in Table 2.2(a) by giving the following labeling to $T(L)$:

1. The centers of the odd branches followed by the even branches attached to x_0 get labels consecutively from the beginning of A and the centers of the pendant branches attached to x_0 get labels consecutively from the end of A.
2. Same as steps 2 and 3 in the proof for lobsters in Table 2.2(f).

For lobsters of type (j) and (k) in Table 2.2, the proof follows if we proceed as the proof involving the lobsters described in Table 2.2(a) by giving the following labeling to $T(L)$:

1. The centers of the odd branches attached to x_0 get labels consecutively from the beginning of A and the centers of the pendant branches followed by the even branches attached to x_0 get labels consecutively from the end of A.
2. Same as steps 2 and 3 in the proof for lobsters in Table 2.2(g).

For lobsters of type (a) in Table 2.3, the proof follows if we proceed as the proof involving the lobsters described in Table 2.2(a) by repeating steps 1 and 3 by setting $t = 0$. For lobsters of type (b), (c), (d), (e) and (f) in Table 2.3, the proof follows if we proceed as the proof involving the lobsters described in Table 2.2(a), 2.2(b), 2.2(c), 2.2(f) and 2.2(g), respectively, by repeating steps 1 and 2 by setting $t = m$. For lobsters of type (g) and (h) in Table 2.3, the proof follows if we proceed as the proof involving the lobsters described in Table 2.2(f) and 2.2(g), respectively, by repeating steps 1 and 3 by setting $t = 0$.

For lobsters of type (i) in Table 2.3, the proof follows if we proceed as the proof involving the lobsters described in Table 2.3(a) by giving the following labeling to $T(L)$:

1. Among the odd (respectively even) branches attached to x_0 the centers of any even number (may be 0) of these branches get labels consecutively from the beginning of A and the centers of rest of these branches get labels consecutively from the end of A.
2. Same as step 2 in the proof for lobsters in Table 2.3(a).

For lobsters of type (j), (k) and (l) in Table 2.3, the proof follows if we proceed as the proof involving the lobsters described in Table 2.3(a) by giving the following labeling to $T(L)$:

1. Same as step 1 in the proof for lobsters in Table 2.2(h).
2. Same as step 2 in the proof for lobsters in Table 2.3(g) (respectively 2.3(c)) if L is a lobster of type (j) (respectively (k) or (l)).

For lobsters of type (m), (n), (o) and (p) in Table 2.3, the proof follows if we proceed as the proof involving the lobsters described in Table 2.3(a) by giving the following labeling to $T(L)$:

1. Same as step 1 in the proof for lobsters in Table 2.2(j).
2. Same as step 2 in the proof for lobsters in Table 2.3(a) (respectively 2.3(b)) if L is a lobster of type (m) or (n) (respectively (o) or (p)). □

Remark 2.5 The graceful labeling of the lobsters in Table 2.3(i) is the main result of Wang et al. [7] except the case in which the number of branches attached to x_0 is equal to one. In their work the number of branches attached to x_0 is at least three but in our result it may be any odd number (i.e. may be equal to one also).

In all the lobsters of Theorem 2.4 for which the vertex x_m is attached to two different types of branches, both of the types are odd in number. But in Theorem 2.6 we shall show that each of them can be made even in number.

Theorem 2.6 The following lobsters are graceful.

(a) The lobsters of Theorem 2.4 in which x_m is attached to an even number of odd branches, an even number of even branches, and the branches attached to the rest of the x_i's are the same as the lobsters in Table 2.2(d), Table 2.3(b), Table 2.3(f), Table 2.3(o) or Table 2.3(p) of Theorem 2.4.

(b) The lobsters of Theorem 2.4 in which x_m is attached to an even number of odd branches and an even number of pendant branches, and the branches attached to rest of the x_i's are the same as the lobsters in Table 2.3(d) of Theorem 2.4.

(c) The lobsters of Theorem 2.4 in which x_m is attached to an even number of even branches and an even number of pendant branches, and the branches attached to rest of the x_i's are the same as the lobsters in Table 2.2(e), Table 2.3(c), Table 2.3(e), Table 2.3(k) or Table 2.3(l) of Theorem 2.4.

Proof: (a) In the proof of lobsters in Tables 2.2(d), 2.3(b), 2.3(f), 2.3(o) or 2.3(p) of Theorem 2.4 we give a labeling to the centers of an odd number of odd branches attached to x_m from the beginning of $A^{(m)}$ and to the centers of an odd number of even branches attached to x_m from the end of $A^{(m)}$. Instead, we label the centers of an even number of odd branches attached to x_m from the beginning of $A^{(m)}$ and label the centers of an even number of even branches attached to x_m from the end of $A^{(m)}$.

(*b*) The proof follows from the proof involving the lobsters described in Table 2.3(*d*) of Theorem 2.4 with the only change that we label the centers of an even number of odd branches attached to x_m from the beginning of $A^{(m)}$ and label the centers of an even number of pendant branches attached to x_m from the end of $A^{(m)}$.

(*c*) The proof follows from the proof of the respective lobsters in Table 2.2(*e*), 2.3(*c*), 2.3(*f*), 2.3(*o*) or 2.3(*p*) of Theorem 2.4 with the only change that we label the centers of an even number of even branches attached to x_m from the beginning of $A^{(m)}$ and label the centers of an even number of pendant branches attached to x_m from the end of $A^{(m)}$. □

Concluding Remark: In all the lobsters to which we have given a graceful labeling in this paper, the vertex x_m gets the largest label and x_{m-1} gets the label 0. Therefore, we can obtain more graceful lobsters by attaching a caterpillar to the vertex x_m or by attaching a suitable caterpillar (any number of pendant branches or an odd (or even) branch or the combination of both) to the vertex x_{m-1} in any of the lobsters discussed in Theorem 2.4 and Theorem 2.6.

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