Chirality group and chirality index of Coxeter chiral maps

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Abstract

In this paper we compute the chirality group, the chirality index and the smallest regular coverings of the chiral Coxeter maps, the toroidal orientably regular maps described in Coxeter and Moser monograph [H.S.M.Coxeter and W.O.J.Moser, *Generators and Relations for Discrete Groups* (4th ed.), Springer-Verlag, Berlin, 1984]. We also compute the greatest regular maps covered by chiral Coxeter maps.

1 Introduction

Chirality in chemistry is unquestionable an old theme. It terms the handedness, or the non-existence of a plane of symmetry, in molecular structures. These so called chiral molecules come in pairs known as right- and left-handed enantiomers. In mathematics the combinatorial chirality phenomenon is associated to quasi regular surface structures (maps) having no reflection (orientation preserving automorphisms) [5, 6, 7, 8]. A map $\mathcal{M}$ is a 2-cell decomposition of an orientable surface. It is orientably regular if its automorphism group acts transitively on darts of $\mathcal{M}$ (edges endowed with one of the two possible orientations). An orientably regular map may, or may not, admit an automorphism reversing the global orientation of the surface. In the first case we call it regular (reflexible in Coxeter, Moser terminology), while an orientably regular map which does not admit an orientation reversing automorphism will be called chiral.

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Supported in part by UIMA of University of Aveiro, through Program POCTI of FCT cofinanced by the European Community fund FEDER, and in part by Slovak Academy of Science, Grant VEGA, and by Grant APVT.

Measuring chirality is certainly new. In [3] two chirality measures involving orientably regular maps and hypermaps were introduced: a qualitative measure called *chirality group* and a quantitative measure called *chirality index*. Being the later a combinatorial measure it is not related to the familiar mathematical continuous measure introduced by Zabrodsky, Peleg and Avnir [15] in chemistry. The definition of the chirality group is based on the observation that each chiral map $\mathcal{M}$ is covered by a regular (reflexible) map $\mathcal{M}_\Delta$. Moreover, one can prove that if the group of orientation preserving automorphisms is transitive on darts of the map (which is the case of orientably regular maps) then the least regular cover $\mathcal{M}_\Delta \to \mathcal{M}$ is unique. Roughly speaking the chirality group gives a qualitative measure of the regularity deviation of an orientably regular hypermap $\mathcal{M}$ by expressing this regularity deviation through a quotient group, the group of covering transformations of $\mathcal{M}_\Delta \to \mathcal{M}$, which express how $\mathcal{M}_\Delta$ covers $\mathcal{M}$ (or equivalently, how $\mathcal{M}$ covers the biggest regular map covered by $\mathcal{M}$). The chirality index, being the size of the chirality group, gives thus a quantitative measure. While all orientably regular maps on the sphere are mirror symmetric (these are the five Platonic solids, cycles with their duals and semistars), there are three infinite families of chiral maps on the torus: the maps $\{4,4\}_{b,c}$, $\{3,6\}_{b,c}$, $\{6,3\}_{b,c}$ with $bc(b-c) \neq 0$ (following the notation introduced in Coxeter and Moser [5]). The first examples of non-toroidal chiral maps seams to be given by Sherk [10] in 1962 with an infinite family of chiral maps of type $\{6,6\}$, the smallest non-toroidal being on genus 7. Later in 1969, with a map of type $\{9,6\}$, Garbe [6] gave another example of a chiral map of genus 7. Maps are essentially polytopes of rank three. Higher rank abstract chiral polytopes were extensively studied by Schulte and Weiss in [11, 12, 13].

The aim of this paper is to calculate the chirality index and the chirality group for each toroidal chiral map $\mathcal{M}$ in the above families. Also, for each such map $\mathcal{M}$ we identify the smallest regular (reflexible) cover of $\mathcal{M}$ as well as the least regular (reflexible) map covered by $\mathcal{M}$. This paper follows [2] (see [1] as well) which classifies the chiral hypermaps up to genus 4. Henceforth we carry the same notation into this paper and dispense lengthy introduction on the subject.

1.1 Maps and hypermaps

A *topological map* is a 2-cell decomposition of an orientable surface (without border). Cells of dimension 0, 1 and 2 are called vertices, edges and faces, respectively. *Darts* of the map are edges of the underlying graph $\mathcal{G}$ endowed with one of the two possible orientations. Fixing an orientation of the surface, counter- or clockwise orientation, denote by R the permutation of the darts that for each vertex v permutes cyclically (following the chosen orientation) all the darts incident to v. Denote by L the involutory permutation interchanging oppositely directed darts sharing the same edge. Then to each topological map there is associated a triple $(D; R, L)$ composed of a set D of darts, two permutations R, L acting on D, with $L^2 = 1$ and $\langle R, L\rangle$ acting transitively on D. Such a triple $\mathcal{M} = (D; R, L)$ will be called an (*oriented*) *map*. The map $\mathcal{M}^* = (D; R^{-1}L, L)$ is called the *dual* of $\mathcal{M}$. One can see that it reflects the standard notion of duality of topological maps.

An *oriented hypermap* $\mathcal{H}$ is a triple $\mathcal{H} = (D; R, L)$, where D is a set of abstract darts (preferably finite), and R, L are permutations of D such that the *monodromy group* $Mon(\mathcal{H}) = \langle R, L\rangle$ is transitive on D. Thus maps are hypermaps satisfying $L^2 = 1$. As for maps, one can define a topological counterpart of a hypermap (see for instance [14]). The *type* of $\mathcal{H}$ is the triple (l, m, n) of integers, where $l = |R|$, $m = |L|$ and $n = |RL|$. Given two hypermaps $\mathcal{H}_1 = (D_1; R_1, L_1)$ and $\mathcal{H}_2 = (D_2; R_2, L_2)$ a *covering* $\mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a function $\phi : D_1 \longrightarrow D_2$ such that $R_1\phi = \phi R_2$ and $L_1\phi = \phi L_2$. Any covering is necessarily onto due to the connectivity of $\mathcal{G}$. If ϕ is injective the covering is an *isomorphism* of hypermaps. An *automorphism* of a hypermap $\mathcal{H} = (D; R, L)$ (also called a *symmetry*) is an isomorphism of $\mathcal{H}$ into itself; in other words, an automorphism of $\mathcal{H}$ is a permutation of the dart set D of $\mathcal{H}$ that commutes with R and L. The automorphism group of $\mathcal{H}$ acts semiregularly on D while the monodromy group acts transitively on D. Hence we have

$$|Aut(\mathcal{H})| \leq |D| \leq |Mon(\mathcal{H})|$$

If one of the equalities holds, that is, if $Aut(\mathcal{H})$ acts transitively on D or $Mon(\mathcal{H})$ acts regularly on D, then the other equality holds as well, and $\mathcal{H}$ is said to be *orientably regular*. If in addition $\mathcal{H}$ has an orientation inverting automorphism, i.e., if there is a permutation ψ of D making $R\psi = \psi R^{-1}$ and $L\psi = \psi L^{-1}$, then $\mathcal{H}$ is said to be *regular*. If $\mathcal{H}$ is orientably regular but not regular then $\mathcal{H}$ will be called *chiral*.

Let Δ denote the free product

$$\Delta = \langle r_0; r_1; r_2 | r_0^2 = r_1^2 = r_2^2 = 1\rangle.$$

and let $\Delta^+ = \langle r_1r_2; r_2r_0\rangle$ be its even word subgroup. The canonical generators of Δ^+ will be denoted by $\rho = r_1r_2$ and $\lambda = r_2r_0$. Observe that the triple $U = (\Delta^+; \rho, \lambda)$, with ρ and λ acting on Δ^+ by the left multiplication, is a hypermap (clearly, an infinite one) which we will call the universal hypermap. For any hypermap $\mathcal{H} = (D; R; L)$, finite or infinite, there is an epimorphism $\Phi : \Delta^+ \to Mon(\mathcal{H})$ sending ρ to R and λ to L. Consequently, $\mathcal{H}$ can be identified with the hypermap $(\Delta^+/H; \bar{\rho}, \bar{\lambda})$ whose darts are the left cosets of the subgroup $H \leq \Delta^+$, H being the kernel of the epimorphism $\Delta^+ \to Mon(\mathcal{H})$ and $\bar{\rho}$, $\bar{\lambda}$ the images of ρ and λ, respectively . Thus the monodromy group of any oriented hypermap is a quotient Δ^+/H. In this context H is called a *hypermap subgroup* for $\mathcal{H}$.

2 Chirality Group

The objective of this section is to show how to compute the chirality group of a chiral hypermap from a presentation of its monodromy group. Given an orientably regular hypermap $\mathcal{H}$ with hypermap subgroup $H \triangleleft \Delta^+$ denote by $H^r = H^{r_0} = H^{r_1} = H^{r_2}$ the conjugate of H in Δ. Following [3] define the *chirality group* of $\mathcal{H}$ to be the factor group $X(\mathcal{H}) = H/H_{\Delta} = H^{\Delta}/H$, where $H_{\Delta} = H \cap H^r$ and $H^{\Delta} = HH^r$. It is proved in [3] that $X(\mathcal{H})$ is isomorphic to a normal subgroup of $Mon(\mathcal{H})$. In the following theorem we identify $X(\mathcal{H})$ as a subgroup of $Mon(\mathcal{H})$ in a more detail.

Theorem 1 *Let $\mathcal{H}$ be an orientably regular hypermap with monodromy group $G = Mon(\mathcal{H})$. If G has presentation $\langle x, y \mid R(x,y)\rangle$ then the chirality group $X(\mathcal{H})$ is the normal closure of $\langle R(x^{-1}, y^{-1})\rangle$ in G.*

Proof:
The chirality group is the quotient group $X(\mathcal{H}) = H/H_{\Delta} \cong H^{\Delta}/H$. Since $G = \Delta^+/H$ and

$$G/X(\mathcal{H}) \cong {}^{\Delta^+/H}\!\!\diagup_{H^{\Delta}/H} \cong \Delta^+/H^{\Delta} = \langle x, y \mid R(x,y), R(x^{-1}, y^{-1})\rangle,$$

by von Dyck theorem [9, page 28], $X(\mathcal{H}) = \langle R(x^{-1}, y^{-1})\rangle^{G}$. □

Corollary 2 *If $\mathcal{H}$ is an orientably regular hypermap with Monodromy group $Mon(\mathcal{H}) = \langle x, y \mid x^l = y^m = (xy)^n = R(x,y) = 1\rangle$ then the chirality group $X(\mathcal{H})$ is the normal closure of $\langle R(x^{-1}, y^{-1})\rangle$ in G.*

Proof:
In fact, $x^{-l} = y^{-m} = (yx)^{-n} = 1$ in $Mon(\mathcal{H})$ so we have

$$X(\mathcal{H}) = \langle x^{-l}, y^{-m}, (yx)^{-n}, R(x^{-1}, y^{-1})\rangle^{Mon(\mathcal{H})} = \langle R(x^{-1}, y^{-1})\rangle^{Mon(\mathcal{H})}.$$

□

The *chirality index* of $\mathcal{H}$, denoted by $\kappa(\mathcal{H})$, is the size of its chirality group. As an example of application of Theorem 1, we compute in Table 1 the chirality group of the 16 chiral maps obtained by Conder and Dobcsányi [4] as a result of the classification of the orientably regular maps from genus 7 up to 15.

$\mathcal{M}$	Genus	Type	Size	$X(\mathcal{M})$	$\kappa(\mathcal{M})$
$C7.1$	7	$\{6,9\}$	54	C_3	3
$C7.2$	7	$\{7,7\}$	56	$C_2 \times C_2 \times C_2$	8
$C8.1$	8	$\{6,6\}$	84	C_7	7
$C10.1$	10	$\{3,8\}$	432	$C_3 \times C_3$	9
$C10.2$	10	$\{4,8\}$	144	$C_3 \times C_3$	9
$C10.3$	10	$\{8,8\}$	72	$C_3 \times C_3$	9
$C11.1$	11	$\{4,8\}$	160	C_5	5
$C11.2$	11	$\{4,8\}$	160	C_5	5
$C11.3$	11	$\{4,12\}$	120	C_5	5
$C11.4$	11	$\{8,8\}$	80	C_5	5
$C11.5$	11	$\{8,8\}$	80	C_5	5
$C11.6$	11	$\{12,12\}$	60	C_5	5
$C12.1$	12	$\{5,10\}$	110	C_{11}	11
$C12.2$	12	$\{5,10\}$	110	C_{11}	11
$C14.1$	14	$\{6,6\}$	156	C_{13}	13
$C15.1$	15	$\{3,12\}$	336	C_7	7

Table 1.

3 Chirality group and index of the toroidal chiral maps

3.1 The toroidal maps

As classified and described by Coxeter and Moser in [5] the orientably regular maps on the Torus are the maps $\{4,4\}_{b,c}$, $\{3,6\}_{b,c}$ and $\{6,3\}_{b,c}$, where b and c are non-negative integers. The map $\{4,4\}_{b,c}$ has $n = b^2 + c^2$ vertices, $2n$ edges, n 4-gonal faces and $4n$ darts. It arises by directly identifying the opposite sides of a square with corners having coordinates $(0;0)$, $(b;c)$, $(b-c;b+c)$ and $(-c;b)$ in a rectangular grid defined by perpendicular unit vectors in the euclidean plane. Write $x = (RL)^{-1}$ and $y = L$. The monodromy group of $\{4,4\}_{b,c}$ has presentation

$$\langle x, y \mid x^4 = y^2 = (xy)^4 = (xyx)^b (x^2 y)^c = 1 \rangle .$$

The map $\{3,6\}_{b,c}$ with $n = b^2+bc+c^2$ vertices, $3n$ edges, $2n$ triangular faces and $6n$ darts arises by directly identifying a square with corners $(0;0)$, $(b;c)$, $(b-c;b+2c)$ and $(-c;b+c)$ in the triangular grid generated by unit vectors making an angle of 60 degrees. Its monodromy group has presentation

$$\langle x, y \mid x^3 = y^2 = (xy)^6 = (x^{-1}yxy)^b (xyx^{-1}y)^c = 1 \rangle ,$$

where, as above, $x = (RL)^{-1}$ and $y = L$. The last map $\{6,3\}_{b,c}$ is just the dual of $\{3,6\}_{b,c}$.

In all cases the maps are regular if and only if $bc(b-c) = 0$ (see [5, pages 104,107]). Consequently the toroidal chiral maps are the orientably regular maps $\{4,4\}_{b,c}$, $\{3,6\}_{b,c}$, $\{6,3\}_{b,c}$ together with their mirror images $\{4,4\}_{c,b}$, $\{3,6\}_{c,b}$, $\{6,3\}_{c,b}$, where $b > c > 0$. Since $\{6,3\}_{b,c}$ is the dual of $\{3,6\}_{b,c}$ both these two maps have the same chirality group.

3.2 The map $\{4,4\}_{b,c}$

The map $\{4,4\}_{b,c}$ has monodromy group $G = \langle x, y \mid x^4 = y^2 = (xy)^4 = (xyx)^b(x^2y)^c = 1\rangle$ so by Corollary 2,

$$X(\{4,4\}_{b,c}) = \langle (x^{-1}y^{-1}x^{-1})^b(x^{-2}y^{-1})^c\rangle^G = \langle (xyx)^{-b}(yx^2)^{-c}\rangle^G = \langle (yx^2)^c(xy$$

Theorem 3 *The chirality group of $\{4,4\}_{b,c}$ is cyclic generated by some vertical translation $(xyx)^{2d}$, where $d = (b,c)$ is the greatest common divisor of b and c. The chirality index is given by*

$$\kappa = \frac{n}{(n, 2d^2)}$$

where $n = b^2 + c^2$.

Proof:
First we show that the chirality group is cyclic. Inside G we have the equalities $(yx^2)^c = (xyx)^b$ and, conjugating by x^{-1}, $(xyx)^c = (x^2y)^b$. Then $X(\{4,4\}_{b,c}) = \langle (xyx)^{2b}\rangle^G = \langle u^{2b}\rangle^G$, by setting $u = xyx$ and using the first equality. Let $v = u^x = yx^2$. Then $v^c = u^b$ and $u^c = v^{-b}$ (second equality). Let us take the conjugates u^θ, v^θ of u and v under the generators of G:

θ	x	x^2	x^{-1}	y
u^θ	v	u^{-1}	v^{-1}	u^{-1}
v^θ	u^{-1}	v^{-1}	u	v^{-1}

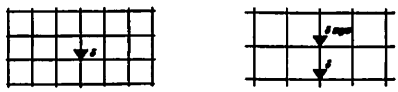
This table shows that the set $\{u, u^{-1}, v, v^{-1}\}$ is a complete system of conjugates (i.e. it is closed under conjugation). Then

$$\begin{aligned}
X(\{4,4\}_{b,c}) &= \langle u^{2b}\rangle^G \\
&= \langle (u^\theta)^{2b}, (v^\theta)^{2b}\rangle_{\theta=1,x,x^2,x^{-1},y} \\
&= \langle u^{2b}, v^{2b}\rangle \\
&= \langle u^{2b}, u^{2c}\rangle \\
&= \langle u^{2(b,c)}\rangle \\
&= \langle (xyx)^{2(b,c)}\rangle
\end{aligned}$$

So $X(\{4,4\}_{b,c})$ is a cyclic group of order

$$|(xyx)^{2(b,c)}| = \frac{|xyx|}{(|xyx|, 2(b,c))}.$$

Now we calculate the order of xyx. Fix the counterclockwise orientation so that $\{4,4\}_{b,c}$ is obtained by identifying the opposite sides of the dark large square shown in Fig. 2. Fix also a dart δ as shown in Fig. 1 (a).



(a) (b)

Figure 1

Relatively to this dart the word xyx is a translation one step along a vertical line (Fig. 2 (b)). Take one step as unit of measure and consider the system of coordinates XOY whose axles XX and YY make an angle $\frac{\pi}{2}$. Here we consider two square grides: the smallest or finer gride, will be referred simply as the *square gride*; the larger square gride where the shaded square belongs, will be referred as the *large square gride* (Figure 2).

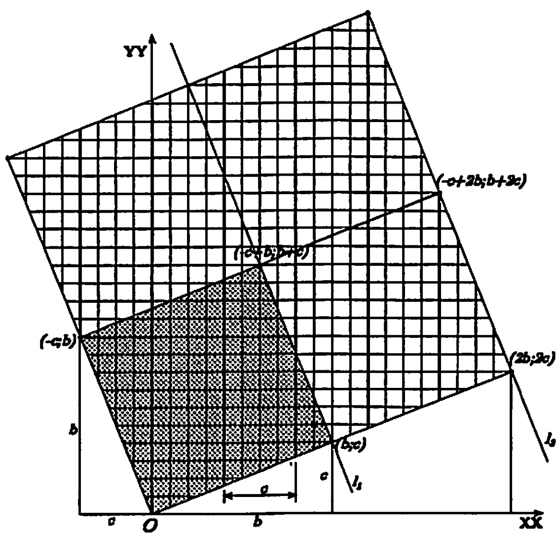


Figure 2

Let l_n be the line passing through the points with coordinates $(nb;nc)$ and $(-c+nb;b+nc)$, whose equation is $Y-nc=(X-nb)\frac{b}{-c}$. Choose n to be the smallest positive integer such that l_n intersects the YY axle in a vertex of the large square gride. We note that for such n the positive integer Y is the order of xyx. Let V_n be the set of vertices of the large square gride that belong to l_n, that is,

$$V_n = \{(-mc + nb\,;\, mb + nc) \mid m \in \mathbb{N}\}.$$

Then n is the smallest positive integer such that

$$\begin{aligned}(0;Y) \in V_n \quad &\Leftrightarrow \quad nc + n\tfrac{b^2}{c} = mb + nc \\ &\Leftrightarrow \quad n\tfrac{b}{c} = m \in \mathbb{N},\end{aligned}$$

that is, n is the smallest positive integer such that $n\frac{b}{c}$ is an integer. Hence

$$n = \frac{c}{(b,c)}$$

where (,) stands for the greatest common divisor. Then

$$|xyx| = Y = \frac{b^2 + c^2}{(b,c)}$$

and so

$$|(xyx)^{2(b,c)}| = \frac{b^2 + c^2}{(b^2 + c^2\,,\, 2(b,c)^2)}$$

is the chirality index of $\{4,4\}_{b,c}$. □

3.3 The map $\{3,6\}_{b,c}$

The map $\{3,6\}_{b,c}$ has the monodromy group $G = \langle x, y \mid x^3 = y^2 = (xy)^6 = (x^{-1}yxy)^b(xyx^{-1}y)^c = 1\rangle$ so by Corollary 2,

$$X(\{3,6\}_{b,c}) = \langle (xyx^{-1}y)^b(x^{-1}yxy)^c \rangle^G.$$

Theorem 4 *The chirality group of $\{3,6\}_{b,c}$ is cyclic generated by some "horizontal" translation $(xyxyx)^{b-c}$. The chirality index is given by*

$$\kappa = \frac{n}{(n,(b-c)d)}$$

where $n = b^2 + bc + c^2$ and $d = (b,c)$.

Proof:
Putting $u = xyx^{-1}y$, $v = x^{-1}yxy$ and $w = xyxyx$, we can write

$$X(\{3,6\}_{b,c}) = \langle u^b v^c \rangle^G.$$

In G we have $(x^{-1}yxy)^b = (yxyx^{-1})^c \Leftrightarrow v^b = u^{-c}$. The following table gives the conjugates of u, v and w by x, x^{-1} and y.

θ	x	x^{-1}	y
u^θ	v^{-1}	w^{-1}	u^{-1}
v^θ	w	u^{-1}	v^{-1}
w^θ	u^{-1}	v	w^{-1}

This shows that $\{u, u^{-1}, v, v^{-1}, w, w^{-1}\}$ is a complete system of conjugates. Conjugating $v^b = u^{-c}$ by x and x^{-1} we have $w^b = v^c$ and $u^b = w^{-c}$. Then

$$\begin{aligned} X(\{3,6\}_{b,c}) &= \langle u^b v^c \rangle^G \\ &= \langle w^{b-c} \rangle^G \\ &= \langle w^{b-c}, (w^x)^{b-c}, (w^{x^{-1}})^{b-c}, (w^y)^{b-c} \rangle \\ &= \langle w^{b-c}, u^{b-c}, v^{b-c} \rangle \\ &= \langle w^{b-c}, v^{b-c} \rangle \end{aligned}$$

since $wv = u = vw$, the chirality group $X(\{3,6\}_{b,c})$ is abelian. So we have $u^{b-c} = v^{b-c}w^{b-c}$.

Taking the clockwise orientation and starting from dart ϵ as shown below

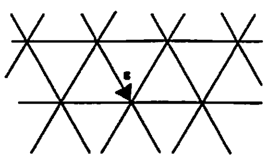


Figure 3

the relation $(x^{-1}yxy)^b(xyx^{-1}y)^c = 1$ determines the opposite sides identification of the dark square shown in Fig. 5. Consider the system of coordinates XOY with axles making an angle $\frac{\pi}{6}$ and with unit vectors e_x and e_y. Relatively to the fixed dart δ (Fig. 4) the elements w, u and v act as translations one unit along $< e_y >$, $< e_x >$ and $< e_x - e_y >$.

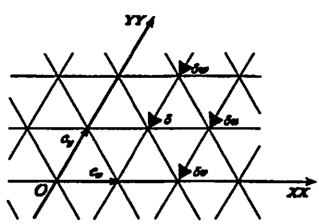


Figure 4

The map $\{3,6\}_{b,c}$ is obtained by identifying opposite sides of the shaded square in Figure 5). As w, u and v are conjugate to each other, they all have the same order.

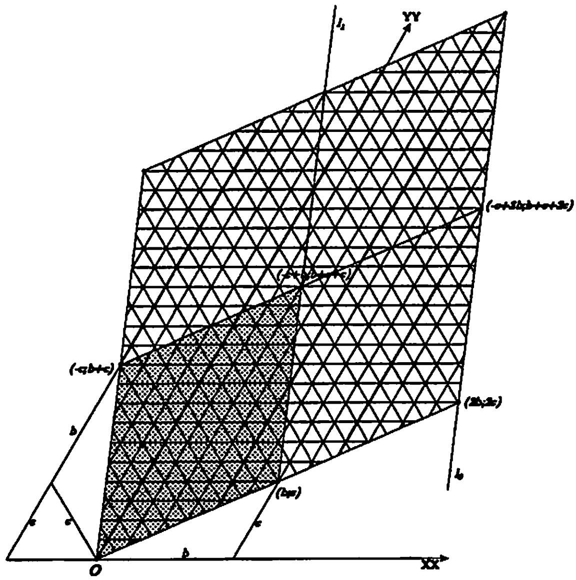


Figure 5

The generators w^{b-c} and v^{b-c} are translations by $b-c$ steps along $< e_y >$ and $< e_x - e_y >$, respectively. The line l_n passing through the points with coordinates $(nb; nc)$ and $(-c+bn; b+c+nc)$ has equation $Y - nc = (X - nb)\frac{b+c}{-c}$. Let n be the smallest positive integer such that l_n intersects the line $X = 0$ in a vertex of the square tessellation. l_n intersects this line in $Y = n\frac{b^2+bc+c^2}{c}$. Denoting by $V_n = \{(-kc+nb; k(b+c)+nc) \mid k \in \mathbb{N}\}$ the set of vertices of the square tessellation then n is the smallest positive integer such that

$$(0;Y) \in V_n \Leftrightarrow n\frac{b^2+bc+c^2}{c} = k(b+c)+nc \Leftrightarrow \frac{nb}{c} = k \in \mathbb{N}\,.$$

Hence

$$n = \frac{c}{(b,c)}\,.$$

This integer Y is the order of w (hence of v), then

$$|w| = \frac{b^2+bc+c^2}{(b,c)}$$

and so,

$$|w^{b-c}| = |v^{b-c}| = \frac{b^2+bc+c^2}{(b^2+bc+c^2,(b-c)(b,c))}.$$

The word w^{b-c} moves a fixed dart δ along a line l which is, in the square shaded region, represented by several line segments. The integer (b,c) reflects the minimal distance between two these consecutive line segments (Figure 5). Since (b,c) divides both b and c, w^b and w^c are elements of $\langle w^{(b,c)}\rangle$. On the other hand, $v^c = w^b$ so $v^c \in \langle w^{(b,c)}\rangle$. Powering both sides of the equation $wv = u$ by b we get $v^b = u^b w^{-b} = w^{-(b+c)} \in \langle w^{(b,c)}\rangle$. Hence the chirality group $X(\{3,6\}_{b,c}) = \langle w^{b-c}, v^{b-c}\rangle$ is a subgroup of the cyclic group $\langle w^{(b,c)}\rangle$, so a cyclic group itself. Since a cyclic group C contains only one subgroup of order k, for each divisor k of $|C|$, then $\langle w^{b-c}\rangle = \langle v^{b-c}\rangle$ and hence

$$X(\{3,6\}_{b,c}) = \langle w^{b-c}\rangle.$$

Then the chirality index of $\{3,6\}_{b,c}$ is given by

$$|X(\{3,6\}_{b,c})| = |w^{b-c}| = \frac{b^2+bc+c^2}{(b^2+bc+c^2,(b-c)(b,c))}.$$

□

4 Regular coverings

We have just computed the chirality group and the chirality index of a chiral Coxeter map $\mathcal{H}$. Yet we have said nothing about its smallest regular cover $\mathcal{H}_\Delta$ and about the biggest regular map $\mathcal{H}^\Delta$ covered by $\mathcal{H}$. It follows from [3] that if $\mathcal{H}$ is chiral with chirality index κ then $\mathcal{H}_\Delta$ with hypermap subgroup H_Δ is the smallest regular hypermap that covers $\mathcal{H}$ and the covering $\mathcal{H}_\Delta \longrightarrow \mathcal{H}$ is smooth. On the other hand, the regular hypermap $\mathcal{H}^\Delta$ with hypermap subgroup H^Δ is the largest regular hypermap covered by $\mathcal{H}$. The covering $\mathcal{H} \longrightarrow \mathcal{H}^\Delta$ may be not smooth in general. However, in our case the chirality group $X(\mathcal{H})$ of $\mathcal{H}$ is cyclic generated by some word ω. This ω is a translation along a vertical (Theorem 3) or along a horizontal (Theorem 4) line, so factoring out the chirality group the vertex and face valency (i.e. the type) of $\mathcal{H}$ remains unaltered; in other words, the covering $\mathcal{H} \longrightarrow \mathcal{H}^\Delta$ is smooth. Thus both $\mathcal{H}_\Delta$ and $\mathcal{H}^\Delta$ are toroidal.

4.1 The map $\{4,4\}_{b,c}$

Let $\mathcal{H} = \{4,4\}_{b,c}$, where $b > c > 0$. This map has size $4n$, where $n = b^2+c^2$. As said above, $\mathcal{H}_\Delta$ and $\mathcal{H}^\Delta$ are toroidal. As they are regular, they must be one of $\{4,4\}_{x,0}$ of size $4x^2$ or $\{4,4\}_{y,y}$ of size $8y^2$ for some integers x,

y. Notice that the equation $4x^2 = 8y^2 \Leftrightarrow x^2 = 2y^2$ has no integer solution. This means that if a hypermap $\mathcal{H}$ has size $4x^2$ then $\mathcal{H}$ cannot be $\{4,4\}_{y,y}$ for no y, and vice-versa.

Theorem 5 *Let $\mathcal{H} = \{4,4\}_{b,c}$ and $d = (b,c)$. If $\frac{n}{d^2}$ is odd then $\mathcal{H}_\Delta = \{4,4\}_{d\kappa,0}$ and $\mathcal{H}^\Delta = \{4,4\}_{d,0}$. If $\frac{n}{d^2}$ is even then $\mathcal{H}_\Delta = \{4,4\}_{d\kappa,d\kappa}$ and $\mathcal{H}^\Delta = \{4,4\}_{d,d}$.*

Proof:
Notice that d^2 divides n, say $n = d^2x$, and the chirality index of $\mathcal{H}$ is given by Theorem 3

$$\kappa = \frac{n}{(n,2d^2)} = \frac{x}{(x,2)}.$$

If x is odd then $x = \kappa$ and so $|\mathcal{H}_\Delta| = |\mathcal{H}|\kappa = 4xd^2\kappa = 4(d\kappa)^2$ and $|\mathcal{H}^\Delta| = \frac{|\mathcal{H}|}{\kappa} = \frac{4xd^2}{\kappa} = 4d^2$, and hence $\mathcal{H}_\Delta = \{4,4\}_{d\kappa,0}$ and $\mathcal{H}^\Delta = \{4,4\}_{d,0}$. If x is even then $x = 2\kappa$ and so $|\mathcal{H}_\Delta| = 4xd^2\kappa = 8(d\kappa)^2$ and $|\mathcal{H}^\Delta| = \frac{4xd^2}{\kappa} = 8d^2$, and hence $\mathcal{H}_\Delta = \{4,4\}_{d\kappa,d\kappa}$ and $\mathcal{H}^\Delta = \{4,4\}_{d,d}$. □

4.2 The map $\{3,6\}_{b,c}$

Let $\mathcal{H} = \{3,6\}_{b,c}$, where $b > c > 0$. This has size $6n$ where $n = b^2+bc+c^2$ and chirality index

$$\kappa = \frac{n}{(n,(b-c)d)},$$

where $d = (b,c)$. As above, $\mathcal{H}_\Delta$ and $\mathcal{H}^\Delta$ lie on the torus. As they are regular, they must be one of the maps $\{3,6\}_{x,0}$ of size $6x^2$, or $\{3,6\}_{y,y}$ of size $18y^2$. Similarly, since the equation $6x^2 = 18y^2 \Leftrightarrow x^2 = 3y^2$ has no integer solution, if $\mathcal{H}$ has size $6x^2$ then $\mathcal{H}$ cannot be $\{3,6\}_{y,y}$ for none y, and vice-versa.

Theorem 6 *Let $\mathcal{H} = \{3,6\}_{b,c}$ and let $d = (b,c)$. If $\frac{b-c}{d} \not\equiv 0 \bmod 3$ then $\mathcal{H}_\Delta = \{3,6\}_{d\kappa,0}$ and $\mathcal{H}^\Delta = \{3,6\}_{d,0}$. If $\frac{b-c}{d} \equiv 0 \bmod 3$ then $\mathcal{H}_\Delta = \{3,6\}_{d\kappa,d\kappa}$ and $\mathcal{H}^\Delta = \{3,6\}_{d,d}$.*

Proof:
Let $m = b-c$. Then $n = m^2+3cm+3c^2$ and $d = (b,c) = (m+c,c) = (m,c)$. As $d^2 \mid n$, let $t = \frac{n}{d^2}$. Then by Theorem 4,

$$\kappa = \frac{t}{(t,\frac{m}{d})}.$$

Let $\mu = \frac{m}{d}$ and $\gamma = \frac{c}{d}$. Then $(\mu,\gamma) = 1$ (hence $(\mu^2,\gamma) = 1$ as well),

$t = \mu^2 + 3\gamma\mu + 3\gamma^2$ and $(t,\mu) = (3\gamma^2,\mu) = (3,\mu)$. Thus

$$\kappa = \frac{t}{(3,\mu)}\,.$$

We distinguish two cases:

Case (i) $(3,\mu) = 1$. Then $t = \kappa$ and we get $|\mathcal{H}_{\vartriangle}| = |\mathcal{H}|\kappa = 6td^2\kappa = 6(d\kappa)^2$ and $|\mathcal{H}^{\vartriangle}| = \frac{|\mathcal{H}|}{\kappa} = \frac{6td^2}{\kappa} = 6d^2$. Hence $\mathcal{H}_{\vartriangle} = \{3,6\}_{d\kappa,0}$ and $\mathcal{H}^{\vartriangle} = \{3,6\}_{d,0}$.

Case (ii) $(3,\mu) = 3$. Then $t = 3\kappa$ and we have $|\mathcal{H}_{\vartriangle}| = 6td^2\kappa = 18(d\kappa)^2$ and $|\mathcal{H}^{\vartriangle}| = \frac{6td^2}{\kappa} = 18d^2$. Hence $\mathcal{H}_{\vartriangle} = \{3,6\}_{d\kappa,d\kappa}$ and $\mathcal{H}^{\vartriangle} = \{3,6\}_{d,d}$. □

Aknowledgement

We are grateful to the referee for giving us some valuable suggestions related to the final output.

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