# Partitioning A Bipartite Graph into Vertex-Disjoint Paths*

Jianping LI
Yunnan University, Kunming, China

George STEINER
McMaster University, Hamilton, Ontario, Canada
e-mails: jianping@ynu.edu.cn and steiner@mcmaster.ca

**Abstract**

Deciding whether a graph can be partitioned into $k$ vertex-disjoint paths is a well-known NP-complete problem. In this paper, we give new sufficient conditions for a bipartite graph to be partitionable into $k$ vertex-disjoint paths. We prove the following results for a simple bipartite graph $G = (V_1, V_2; E)$ of order $n$ : (i) For any positive integer $k$, if $||V_1| - |V_2|| \leq k$ and $d_G(x) + d_G(y) \geq \frac{n-k+1}{2}$ for every pair $x \in V_1$ and $y \in V_2$ of nonadjacent vertices of $G$, then $G$ can be partitioned into $k$ vertex-disjoint paths, unless $k = 1$, $|V_1| = |V_2| = \frac{n}{2}$ and $G = K_{s,s} \cup K_{\frac{n}{2}-s,\frac{n}{2}-s}$, where $1 \leq s \leq \frac{n}{2} - 1$; (ii) For any two positive integers $p_1$ and $p_2$ satisfying $n = p_1 + p_2$, if $G$ does not belong to some easily recognizable classes of exceptional graphs, $||V_1| - |V_2|| \leq 2$ and $d_G(x) + d_G(y) \geq \frac{n-1}{2}$ for every pair $x \in V_1$ and $y \in V_2$ of nonadjacent vertices of $G$, then $G$ can be partitioned into two vertex-disjoint paths $P_1$ and $P_2$ of order $p_1$ and $p_2$, respectively. These results also lead to new sufficient conditions for the existence of a Hamilton path in a bipartite graph.

*Keywords*: bipartite graphs, vertex-disjoint paths and Hamilton paths.

*AMS Classifications (2000)*: 05c38, 05c45, 05c35.

*This research was supported by the Natural Sciences and Engineering Research Council of Canada, under Grant No. 1798-99 and the National Natural Science Foundation of China (No.10271103). Most of this research was done while the first author was a postdoctoral fellow at McMaster University.

# 1 Introduction

We consider only undirected and simple graphs. For notation and terminology not defined here, we refer the reader to [2]. Let $G = (V, E)$ be a graph and $P$ a path of $G$. $P$ is called a *Hamilton path* if $P$ contains all vertices of $G$, and $G$ is called *traceable* in this case. $G$ is called *hamiltonian* if it has a *Hamilton cycle,* i.e., a cycle containing all vertices of $G$. The *order* of a graph $G = (V, E)$ is $|V|$. For a positive integer $k$, if $k$ paths $P_1, P_2, \cdots, P_k$ satisfy $V(P_i) \cap V(P_j) = \emptyset$ for any $i \neq j$, then the paths are called *vertex-disjoint* or *independent.* $G$ is said to be *covered* by the paths $P_1, P_2, \cdots, P_{k'}$ if $V(G) = V(P_1) \cup V(P_2) \cup \cdots \cup V(P_{k'})$. If $k$ vertex-disjoint paths $P_1, P_2, \cdots, P_k$ cover the graph $G$, we say that $G$ can be *partitioned* into $k$ vertex-disjoint paths. Let $\mathcal{N}$ be the set of all *positive* integers. For any $s, t \in \mathcal{N}$, $K_{s,t}$ represents a complete bipartite graph with $s$ vertices in one part and $t$ vertices in the other. For any two vertex-disjoint graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, the *union* of $G_1$ and $G_2$ is the graph $G_1 \cup G_2$ with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

The question whether a given graph can be partitioned into $k$ vertex-disjoint paths arises in many applications in operations research, logistics and VLSI design. For example, in a vehicle routing problem we want to decide whether the routing network can be partitioned into $k$ delivery paths. It is also known as the *hamiltonian completion problem,* since if $k$ is the minimum number of paths a graph can be partitioned into, then $k$ is also the minimum number of edges we need to add to the graph to make it hamiltonian. Unfortunately, the problem is NP-complete even for fixed $k$ [6]. It is solvable in polynomial time, however, for some special classes of graphs, see e.g. [1, 3, 4, 7, 9, 11].

In the past, the $k = 1$ case, i.e., determining whether a graph has a Hamilton path, has attracted the most attention. In 1952, Dirac gave the first sufficient conditions for a graph to have a Hamilton cycle or Hamilton path.

**Theorem 1** [5] *Let $G$ be a graph of order $n \geq 3$. If the minimum degree $\delta(G) \geq \frac{n}{2}$, then $G$ is hamiltonian. Similarly, if the minimum degree $\delta(G) \geq \frac{n-1}{2}$, then $G$ contains a Hamilton path.*

Ore generalized this result in 1963 by considering pairs of nonadjacent vertices.

**Theorem 2** [10] *Let $G$ be a graph of order $n \geq 3$. If $d_G(x)+d_G(y) \geq n$ for each pair of nonadjacent vertices $x$, $y$, then $G$ is hamiltonian. Similarly, if $d_G(x) + d_G(y) \geq n - 1$ for each pair of nonadjacent vertices $x$, $y$, then $G$ contains a Hamilton path.*

For bipartite graphs, Moon and Moser [8] obtained the following result.

**Theorem 3** [8] *Let $G = (V_1, V_2; E)$ be a bipartite graph. If $|V_1| = |V_2| = m$ and $d_G(x) + d_G(y) \geq m + 1$ for all vertices $x \in V_1$ and $y \in V_2$, then $G$ is hamiltonian.*

This means that the lower bound on the degree sum sufficient to make a balanced bipartite graph hamiltonian is only about half of the lower bound for general graphs.

In this paper, motivated by the preceding theorems, we consider the problem of partitioning a bipartite graph into vertex-disjoint paths and obtain the following results.

**Theorem 4** *Let $G = (V_1, V_2; E)$ be a simple balanced bipartite graph of $n$ vertices. If $G$ is connected and $d_G(x) + d_G(y) \geq \frac{n}{2}$ for every pair $x \in V_1$ and $y \in V_2$ of nonadjacent vertices, then $G$ contains a Hamilton path.*

**Theorem 5** *Let $G = (V_1, V_2; E)$ be a simple bipartite graph of order $n$ satisfying $||V_1| - |V_2|| \leq k$, where $k \leq n$ is a positive integer. If $d_G(x) + d_G(y) \geq \frac{n-k+1}{2}$ for every pair $x \in V_1$ and $y \in V_2$ of nonadjacent vertices, then $G$ can be partitioned into $k$ vertex-disjoint paths $P_1, P_2, \cdots, P_k$, unless $k = 1$, $|V_1| = |V_2| = \frac{n}{2}$ and $G \in \{K_{s,s} \cup K_{\frac{n}{2}-s,\frac{n}{2}-s} | \; 1 \leq s \leq \frac{n}{2} - 1\}$.*

We shall postpone the proofs to the next section. Here we note that the conditions in Theorem 5 are sharp. The graph $K_{m,m+s+1} - e$ ($s \geq k$) shows that $||V_1| - |V_2|| \leq k$ cannot be relaxed. For $1 \leq s \leq \frac{n-k}{2} - 1$, the graph $K_{s,s} \cup K_{\frac{n-k}{2}-s,\frac{n+k}{2}-s}$ cannot be partitioned into $k$ vertex-disjoint paths, in fact, it can be partitioned only into at least $k+1$ vertex-disjoint paths. Thus it is clear that the right hand side of the condition $d_G(x) + d_G(y) \geq \frac{n-k+1}{2}$ cannot be lowered by $\frac{1}{2}$ either.

From Theorem 5, we can easily obtain the following corollaries, which represent extensions of Theorem 3 and 4, respectively, for the Hamilton path problem.

**Corollary 6** *Let $G = (V_1, V_2; E)$ be a simple bipartite graph of order $n$ satisfying $||V_1| - |V_2|| \leq 1$. If $d_G(x) + d_G(y) \geq \frac{n}{2}$ for every pair $x \in V_1$ and $y \in V_2$ of nonadjacent vertices, then $G$ contains a Hamilton path, unless $G \in \{K_{s,s} \cup K_{\frac{n}{2}-s,\frac{n}{2}-s} | \; 1 \leq s \leq \frac{n}{2} - 1\}$.*

**Corollary 7** *Let $G = (V_1, V_2; E)$ be a simple bipartite graph of order $n$ satisfying $||V_1| - |V_2|| \leq 1$. If $d_G(x) + d_G(y) \geq \frac{n}{2}$ for every pair of vertices $x \in V_1$ and $y \in V_2$, then $G$ contains a Hamilton path, unless $G = K_{\frac{n}{4},\frac{n}{4}} \cup K_{\frac{n}{4},\frac{n}{4}}$.*

By Theorem 5, $G$ can be partitioned into two vertex-disjoint paths if

$d_G(x) + d_G(y) \geq \frac{n-1}{2}$ for every pair $x \in V_1$ and $y \in V_2$ of nonadjacent vertices and $||V_1| - |V_2|| \leq 2$, but we have no information about the lengths of these two paths. The next result answers this question for the case $k = 2$.

**Theorem 8** *Let $G = (V_1, V_2; E)$ be a simple bipartite graph of order $n \geq 2$ satisfying $||V_1| - |V_2|| \leq 2$. Let $p_1$ and $p_2$ be two positive integers with $n = p_1 + p_2$. If $G \neq G^*$ and $G$ does not belong to any of the four families $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ of exceptional graphs, and $d_G(x) + d_G(y) \geq \frac{n-1}{2}$ for every pair $x \in V_1$ and $y \in V_2$ of nonadjacent vertices, then $G$ can be partitioned into two vertex-disjoint paths $P_1$, $P_2$ of order $p_1$, $p_2$.*

The exceptional graph $G^*$ is obtained from $K_{1,3}$ by replacing every edge in it by a path of length two. ($G^*$ is a tree with seven vertices and six edges.) The four families $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ of exceptional graphs in Theorem 8 are defined by

$$
\begin{aligned}
\mathcal{G}_1 &= \{K_{s,s} \cup K_{\lfloor \frac{n}{2} \rfloor - s, \lceil \frac{n}{2} \rceil - s} | 1 \leq s \leq \lfloor \tfrac{n}{2} \rfloor - 1 \text{ for } 2s \neq p_1, p_2\} \\
\mathcal{G}_2 &= \{K_{s,s+1} \cup K_{\lfloor \frac{n-1}{2} \rfloor - s, \lceil \frac{n-1}{2} \rceil - s} | 1 \leq s \leq \lfloor \tfrac{n-1}{2} \rfloor - 1 \\
&\qquad \text{for } 2s+1 \neq p_1, p_2\} \\
\mathcal{G}_3 &= \{G | G \subseteq K_{\frac{n-2}{2}, \frac{n+2}{2}} \text{ for } p_1, p_2 \text{ even } \} \\
\mathcal{G}_4 &= \{G | G \text{ is obtained from } K_{m,m+1} \cup K_{m',m'} \text{ by adding} \\
&\qquad \text{at least one edge between the part of } m \text{ vertices} \\
&\qquad \text{and the opposing part of } m' \text{ vertices} \}.
\end{aligned}
$$

In fact, Theorem 8 is a direct consequence of the following stronger result which we will also prove in the next section.

**Theorem 9** *Let $G = (V_1, V_2; E)$ be a simple bipartite graph of order $n$ satisfying $||V_1| - |V_2|| \leq 2$. If $d_G(x) + d_G(y) \geq \frac{n-1}{2}$ for every pair $x \in V_1$ and $y \in V_2$ of nonadjacent vertices, then $G$ contains a Hamilton path or $G = G^*$ or $G$ belongs to one of the four families $\mathcal{G}'_1$, $\mathcal{G}'_2$, $\mathcal{G}'_3$ and $\mathcal{G}'_4$ defined below*

$$
\begin{aligned}
\mathcal{G}'_1 &= \{K_{s,s} \cup K_{\lfloor \frac{n}{2} \rfloor - s, \lceil \frac{n}{2} \rceil - s} | \ 1 \leq s \leq \lfloor \tfrac{n}{2} \rfloor - 1 \ \} \\
\mathcal{G}'_2 &= \{K_{s,s+1} \cup K_{\lfloor \frac{n-1}{2} \rfloor - s, \lceil \frac{n-1}{2} \rceil - s} | \ 1 \leq s \leq \lfloor \tfrac{n-1}{2} \rfloor - 1 \ \} \\
\mathcal{G}'_3 &= \{G | G \subseteq K_{\frac{n-2}{2}, \frac{n+2}{2}}\} \\
\mathcal{G}'_4 &= \{G | G \text{ is obtained from } K_{m,m+1} \cup K_{m',m'} \text{ by adding} \\
&\qquad \text{at least one edge between the part of } m \text{ vertices} \\
&\qquad \text{and the opposing part of } m' \text{ vertices} \}.
\end{aligned}
$$

## 2 The proofs

First, we introduce some notation and terminology. We consider only bipartite graphs $G = (V_1, V_2; E)$ in this section, and for convenience, we call the vertices $x \in V_1$ *black* and the vertices $y \in V_2$ *white*. For any vertex $x \in V(G)$, we denote its neighborhood by $N(x) = \{y \in V(G) | xy \in E(G)\}$. Let $S \subseteq V(G)$, the set of neighbors of $x$ *in* $S$ is defined by $N_S(x) = \{y \in S | xy \in E(G)\}$. We also define the degree function *in* $S$ by $d_S(x) = |N_S(x)|$. If $P[u_0, u_p] := u_0 u_1 \cdots u_p$ is a path or a cycle (if $u_0 = u_p$ ) of the graph $G$, we denote by $\overrightarrow{P}$ the path or cycle with the orientation from $u_0$ to $u_p$, and by $\overleftarrow{P}$ the path or cycle with the reverse orientation. If $0 \le i \le j \le p$, then $u_i \overrightarrow{P} u_j$ denotes the consecutive vertices or the subpath of $P$ from $u_i$ to $u_j$ in the direction specified by $\overrightarrow{P}$. The same vertices or subpath in reverse order are given by $u_j \overleftarrow{P} u_i$. We use $u^+$ to denote the successor of $u$ on $\overrightarrow{P}$ and $u^-$ to denote its predecessor. For any path or cycle $P$ with a given orientation and $S \subseteq V(P)$, let $S^+ = \{x^+ | x \in S\}$ and $S^- = \{x^- | x \in S\}$. Since it is convenient to use $P$ both for a path and the vertices $V(P) = \{u_0, u_1, \cdots, u_p\}$ on it, we will use the short notation $d_P(x)$ for $d_{V(P)}(x)$.

We start with the following lemma.

**Lemma 1** *Let* $G = (V_1, V_2; E)$ *be a bipartite graph and* $P = x_1 x_2 \cdots x_{2m-1} x_{2m}$ *a path in* $G$. *If* $d_P(x_1) + d_P(x_{2m}) \ge m + 1$, *then the subgraph* $G[P]$ *induced by* $V(P)$ *is hamiltonian.*

*Proof.* Suppose $x_1 x_{2m} \notin E$. Give an orientation $\overrightarrow{P}$ to $P$ by directing it from $x_1$ to $x_{2m}$. Then we get

$$\begin{aligned} m+1 &\le d_P(x_1) + d_P(x_{2m}) \\ &= |N_P(x_1)| + |N_P(x_{2m})| \\ &= |N_P(x_1)^-| + |N_P(x_{2m})| \\ &= |N_P(x_1)^- \cup N_P(x_{2m})| + |N_P(x_1)^- \cap N_P(x_{2m})| \\ &\le |\{x_1, x_3, \cdots, x_{2m-3}, x_{2m-1}\}| + |N_P(x_1)^- \cap N_P(x_{2m})| \\ &= m + |N_P(x_1)^- \cap N_P(x_{2m})|, \end{aligned}$$

which leads to $|N_P(x_1)^- \cap N_P(x_{2m})| \ge 1$, *i.e.*, there exists an integer $i \in \{1, 2, \cdots, m\}$ such that $x_{2i-1} x_{2m}, x_{2i} x_1 \in E$. So $G[P]$ has the Hamilton cycle $C = x_1 \overrightarrow{P} x_{2i-1} x_{2m} \overleftarrow{P} x_{2i} x_1$, *i.e.*, $G[P]$ is hamiltonian.

∎

**The proof of Theorem 4**

Assume, contrary to the theorem, that $G$ has no Hamilton path. Extend $G$ into the balanced bipartite graph $G' = (V_1', V_2'; E')$ by letting $V_1' =$

$V_1 \cup \{x'\}$, $V_2' = V_2 \cup \{y'\}$ and $E' = E \cup \{x'y | y \in V_2\} \cup \{xy' | x \in V_1\}$, where $x', y' \notin V_1 \cup V_2$. It is easy to see that $G'$ satisfies the conditions of Theorem 3, and thus it is hamiltonian. Deleting $x'$ and $y'$ from a Hamilton cycle of $G'$, we obtain a partition of $G$ into two even paths $P_1 = x_1y_1...x_ky_k$ and $P_2 = u_1v_1...u_lv_l$. We can assume without loss of generality that $P_1$ is the longest possible among such partitions and $x_i, u_i \in V_1$. Since $G$ is not traceable by assumption, we have $P_2 \neq \emptyset$.

The induced subgraph $G[P_1]$ is not hamiltonian, since otherwise, using the fact that $G$ is connected, we could easily find a path partition $P_1', P_2'$ where $P_1'$ is longer than $P_1$. Therefore, $x_1y_k \notin E$ and $d_{P_1}(x_1)+d_{P_1}(y_k) \leq k$, by applying Lemma 1 to $G[P_1]$. It also follows from the assumption of maximality for $P_1$ that $d_{P_2}(x_1) = d_{P_2}(y_k) = 0$. Adding the inequality to this, we get $d_G(x_1) + d_G(y_k) = d_{P_1}(x_1) + d_{P_2}(x_1) + d_{P_1}(y_k) + d_{P_2}(y_k) \leq k < \frac{n}{2}$, a contradiction.

■

## The proof of Theorem 5

We distinguish three cases in the proof.

**Case 1.** $n - k$ is even

Assume without loss of generality that $|V_1| = (n-k)/2 + l$ and $|V_2| = (n-k)/2 + (k-l)$ for some $l \in [0, k]$. Extend $G$ into the balanced bipartite graph $G' = (V_1', V_2'; E')$ by adding $k - l$ vertices to $V_1$ and $l$ vertices to $V_2$, and adding every edge from $(V_1' \backslash V_1)$ to $V_2'$ and from $(V_2' \backslash V_2)$ to $V_1'$. Then we have $d_{G'}(x) = d_G(x) + l$ for $x \in V_1$ and $d_{G'}(y) = d_G(y) + k - l$ for $y \in V_2$. Therefore, $d_{G'}(x) + d_{G'}(y) = d_G(x) + d_G(y) + k \geq (n-k+1)/2 + k = (|V_1'| + |V_2'| + 1)/2$, which imply that $d_{G'}(x) + d_{G'}(y) \geq (|V_1'| + |V_2'|)/2 + 1$ since $|V_1'| + |V_2'|$ is even and $d_{G'}(x) + d_{G'}(y)$ is integer. So $G'$ satisfies the conditions of Theorem 3. Thus $G'$ has a Hamiltonian cycle $C$. Deleting the newly added $k$ vertices from $C$ yields a partition of $G$ into $k$ paths.

**Case 2.** $n - k$ is odd and $k > 1$

Assume without loss of generality that $|V_1| = (n-k-1)/2 + l$ and $|V_2| = (n-k-1)/2 + (k-l+1)$ for some $l \in [1, k-1]$. Extend $G$ into the balanced bipartite graph $G' = (V_1', V_2'; E')$ by adding $k - l$ vertices to $V_1$ and $l - 1$ vertices to $V_2$, and adding every edge from $(V_1' \backslash V_1)$ to $V_2'$ and from $(V_2' \backslash V_2)$ to $V_1'$. Then we have $d_{G'}(x) = d_G(x) + l - 1$ for $x \in V_1$ and $d_{G'}(y) = d_G(y) + k - l$ for $y \in V_2$. Thus, $d_{G'}(x) + d_{G'}(y) = d_G(x) + d_G(y) + k - 1 \geq (n-k+1)/2 + k - 1 = (n+k-1)/2 = (|V_1'| + |V_2'|)/2$. As $G'$ is connected by construction, it satisfies the conditions of Theorem 4. Thus $G'$ has a Hamilton path $P$. Deleting the newly added $k - 1$ vertices from $P$ yields a partition of $G$ into $k$ paths.

**Case 3.** $n - k$ is odd and $k = 1$

In this case, $n$ must be even and thus $G$ is a balanced bipartite graph. If $G$ is connected, then Theorem 4 implies that $G$ has a Hamilton path. If $G$ is not connected, we can partition $G = (V_1, V_2; E)$ into two vertex-disjoint subgraphs $G_1 = (X_1, Y_1; E_1)$ and $G_2 = (X_2, Y_2; E_2)$ with no edge between them and satisfying $V_1 = X_1 \cup X_2$ and $V_2 = Y_1 \cup Y_2$. We have $x_1y_2, x_2y_1 \notin E(G)$ for any $x_1 \in X_1$, $x_2 \in X_2$, $y_1 \in Y_1$, $y_2 \in Y_2$ and thus

$$\frac{n}{2} \leq d_G(x_1) + d_G(y_2) \leq |Y_1| + |X_2|$$

and

$$\frac{n}{2} \leq d_G(x_2) + d_G(y_1) \leq |Y_2| + |X_1|.$$

Combining the two preceding inequalities with $|X_1|+|X_2|+|Y_1|+|Y_2| = n$, we get

$$n \leq (d_G(x_1) + d_G(y_2)) + (d_G(x_2) + d_G(y_1)) \leq |Y_1| + |X_2| + |Y_2| + |X_1| = n.$$

So all inequalities hold as equalities, i.e., $|Y_1| + |X_2| = |Y_2| + |X_1| = \frac{n}{2}$, which implies that $G_1$ and $G_2$ are complete bipartite subgraphs of $G$. Since $G$ is balanced, $|X_1| + |X_2| = |Y_1| + |Y_2| = \frac{n}{2}$, and thus $|X_1| = |Y_1|$ and $|X_2| = |Y_2|$. Hence, $G_1 = (X_1, Y_1; E_1)$ and $G_2 = (X_2, Y_2; E_2)$ are both complete balanced bipartite subgraphs of $G(V_1, V_2; E)$, i.e., $G$ is one of the exceptional graphs described in our theorem.

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## The proof of Theorem 9

We distinguish two main cases in the proof.

**Case 1** $G$ is connected

Assume that $G$ contains no Hamilton path. By Theorem 5, $G$ can be partitioned into two vertex-disjoint paths $P = x_1x_2\cdots x_p$ and $Q = y_1y_2\cdots y_q$, where $p + q = n$. We can assume without loss of generality that $q$ is as large as possible . We direct $P$ from $x_1$ to $x_p$ and denote by it $\overrightarrow{P}$. Similarly, $\overrightarrow{Q}$ is $Q$ oriented from $y_1$ to $y_q$. Let $G[P]$ and $G[Q]$ be the subgraphs of $G$ induced by $V(P)$ and $V(Q)$, respectively. We shall distinguish two subcases.

**Case 1a** $G[Q]$ is hamiltonian

In this case, there is no edge between $G[P]$ and $G[Q]$, since otherwise we could easily construct a partition of $G$ into two paths with a longer $q$, contradicting our assumption. This, however, contradicts the fact that $G$ is connected.

Case 1b $G[Q]$ is nonhamiltonian

(i) Suppose $p$ and $q$ are both even
In this case $G$ must be balanced and $n = p + q$ is even. Thus the degree conditions of the theorem are equivalent to $d_G(x) + d_G(y) \geq \frac{n}{2}$ for every nonadjacent pair $x \in V_1, y \in V_2$. By Theorem 4, however, $G$ would have a Hamilton path, a contradiction.

(ii) Suppose $p$ and $q$ are both odd

Note that $x_1$ and $x_p$ are of the same colour and $y_1$ and $y_q$ are also of the same colour in this case.

(ii-1) If $x_1$ and $y_1$ are of different colour, then $G$ must be balanced, and the same argument as in (i) leads to a contradiction. Thus this case cannot occur.

(ii-2) If $x_1, x_p, y_1$ and $y_q$ are all of the same colour, then, without loss of generality, we may assume that they are all black. So we can obtain the vertex partition $V_1 = \{x_1, x_3, \cdots, x_p, y_1, y_3, \cdots, y_q\}$, $V_2 = \{x_2, x_4, \cdots, x_{p-1}, y_2, y_4, \cdots, y_{q-1}\}$ and $G \subseteq K_{\frac{p+q-2}{2}, \frac{p+q+2}{2}}$. Therefore $G \in \mathcal{G}_3'$.

(iii) Exactly one of $\{p, q\}$ is odd

We may assume without loss of generality that both $x_1$ and $y_1$ are black. We must have $N_P(y_1) = N_P(y_q) = \emptyset$ by the maximality of $Q$.

(iii-1) $p$ is even and $q$ is odd

If there was a vertex $z \in N_Q(x_p) \cap N_Q(y_1)^-$, then $y_q \overleftarrow{Q} z^+ y_1 \overrightarrow{Q} z x_p \overleftarrow{P} x_1$ would yield a Hamilton path in $G$, so $N_Q(x_p) \cap N_Q(y_1)^- = \emptyset$. This implies $d_Q(x_p) \leq \frac{q+1}{2} - d_Q(y_1) - 1$ (the $-1$ for $y_q$). Substituting these and $d_P(x_P) \leq \frac{p}{2}$, we obtain

$$d_G(x_p) + d_G(y_1) = d_P(x_p) + d_Q(x_p) + d_Q(y_1) \leq \frac{p}{2} + \frac{q-1}{2} = \frac{n-1}{2},$$

which by the assumptions of the theorem means that equality must hold everywhere, i.e, $d_P(x_p) = p/2$ and $d_Q(x_p) + d_Q(y_1) = \frac{q-1}{2}$. This implies that $x_p$ is connected to every black vertex on $P$, in particular $x_1 x_p \in E$. If $p > 2$, then this means that $G[P]$ is hamiltonian. We show next that this is not possible: Suppose $x_1 x_p \in E$, then if we had a $z \in N_P(y_2)$ this would imply the existence of the path $y_q \overleftarrow{Q} y_2 z \overrightarrow{P} x_p x_1 \overrightarrow{P} z^-$, which is longer than $Q$, contradicting the definition of $Q$. Similarly, if we had a $z \in N_Q(x_1) \cap N_Q(y_2)^-$, then we again could find a path longer than $Q$, so $N_Q(x_1) \cap N_Q(y_2)^- = \emptyset$. This implies $d_Q(x_1) + d_Q(y_2) \leq \frac{q-1}{2}$ and

$$d_G(x_1) + d_G(y_2) = d_P(x_1) + d_Q(x_1) + d_Q(y_2) \leq \frac{p}{2} + \frac{q-1}{2} = \frac{n-1}{2},$$

which using the assumptions of the theorem means that $d_P(x_1) = p/2$ and $d_Q(x_1) + d_Q(y_2) = \frac{q-1}{2}$. Thus from every consecutive pair $(y_{2i}, y_{2i+1})$ $(1 \le i \le \frac{q-1}{2})$ exactly one must be a neighbor of $x_1$ or $y_2$. In particular, since $x_1 y_{q-1} \in E$ would again yield a path longer than $Q$, we must have $y_2 y_q \in E$. Now suppose that there is a $z \in N_Q(x_1)$, then $x_p \overleftarrow{P} x_1 z \overrightarrow{Q} y_q y_2 \overrightarrow{Q} z^-$ would be a longer path than $Q$, so $N_Q(x_1) = \emptyset$. As $G[P]$ is assumed to be hamiltonian, the path on $G[P]$ could start in any black vertex of $P$ and we could repeat the above argument for this vertex. Therefore, no black vertex of $P$ can have a neighbor on $Q$. Similarly, if we had a black vertex $z \in N_Q(x_p)$, then $x_1 \overrightarrow{P} x_p z \overrightarrow{Q} y_q y_2 \overrightarrow{Q} z^-$ would be a longer path than $Q$, so $x_p$ does not have a neighbor on $Q$ either. Using again the hamiltonicity of $G[P]$, this applies to every white vertex on $P$. This, however, contradicts the assumption that $G$ is connected, so we cannot have $x_1 x_p \in E$ if $p > 2$. Since $d_P(x_1) = p/2$ implies $x_1 x_p \in E$, we must have $p = 2$. Furthermore, if there was a vertex $z \in N_Q(x_2) \cap N_Q(y_q)^+$, then $x_1 x_2 z \overrightarrow{Q} y_q z^- \overleftarrow{Q} y_1$ would be a Hamilton path in $G$, so $N_Q(x_2) \cap N_Q(y_q)^+ = \emptyset$. This implies $d_Q(x_2) \le \frac{q+1}{2} - d_Q(y_q) - 1$ (the $-1$ for $y_1$) and from the degree conditions of the theorem, $d_Q(x_2) + d_Q(y_q) = \frac{q-1}{2}$. This again means that from every consecutive pair $(y_{2i}, y_{2i+1})$ $(1 \le i \le \frac{q-1}{2})$ of vertices exactly one must be a neighbor of $x_2$ or $y_q$.

Suppose that $x_2$ has a neighbor on $Q$ and let $y_{2j+1} (1 \le j \le \frac{q-3}{2})$ the first of these. Then it follows from the above observations that $y_{2j} y_q \notin E, y_1 y_{2j+2} \notin E$. Furthermore, if $j > 1$ then, by the definition of $j$, $x_2 y_{2j-1} \notin E$ and thus $y_{2j-2} y_q \in E$ and $y_1 y_{2j} \in E$. This, however, yields the Hamilton path $x_1 x_2 y_{2j+1} \overrightarrow{Q} y_q y_{2j-2} \overleftarrow{Q} y_1 y_{2j} y_{2j-1}$, a contradiction. Therefore $j = 1$, i.e., $x_2 y_3 \in E, y_2 y_q \notin E, y_1 y_4 \notin E$. We cannot have $x_1 y_2 \in E$, since this would yield the path $x_2 x_1 y_2 \overrightarrow{Q} y_q$, which is longer than $Q$, contradicting its maximality. We have $x_1 y_4 \notin E$, since otherwise $y_1 y_2 y_3 x_2 x_1 y_4 \overrightarrow{Q} y_q$ would be a Hamilton path in $G$. Suppose that $d_Q(x_1) = 0$. Then since $y_2 y_q \notin E$,

$$\frac{n-1}{2} \le d_G(x_1) + d_G(y_2) = 1 + d_Q(y_2) \le 1 + \frac{q+1}{2} - 1 = \frac{n-1}{2},$$

implying that $y_2 y_5 \in E$, unless $q = 5$. Similarly, since $y_1 y_4 \notin E$,

$$\frac{n-1}{2} \le d_G(x_1) + d_G(y_4) = 1 + d_Q(y_4) \le 1 + \frac{q+1}{2} - 1 = \frac{n-1}{2},$$

implying that $y_4 y_q \in E$. However, this yields the path $x_1 x_2 y_3 y_2 y_5 \overrightarrow{Q} y_q y_4$, which is longer than $Q$, a contradiction. So we must have $d_Q(x_1) > 0$. Let $k$ be the smallest index for which $x_1 y_{2k} \in E$. Note that $k > 2$. We have $x_2 y_{2k-1} \notin E$, otherwise $y_1 \overrightarrow{Q} y_{2k-1} x_2 x_1 y_{2k} \overrightarrow{Q} y_q$ would be a Hamilton path.

Since $x_2y_{2k-1} \notin E$, we must have $y_{2k-2}y_q \in E$, but then we have the path $x_1y_{2k}\overrightarrow{Q}y_qy_{2k-2}\overleftarrow{Q}y_1$, contradicting that $Q$ is longest possible. In summary, $x_2$ cannot have a neighbor on $Q$ if $q > 5$. But this then implies

$$\frac{n-1}{2} \le d_G(x_2) + d_G(y_{2i+1}) = d_P(x_2) + d_Q(y_{2i+1}) \le 1 + \frac{q-1}{2} = \frac{n-1}{2},$$

for $0 \le i \le \frac{q-1}{2}$. Thus $G[Q]$ is isomorphic to $K_{\frac{q-1}{2},\frac{q+1}{2}}$, i.e., $G \in \mathcal{G}_4'$.

If $q \le 5$ and $d_G(x_2) > 0$, then it is easy to see that $q = 3$ would yield a Hamilton path, so we must have $q = 5$. In this case, $x_2y_3 \in E$ is the only edge between $x_2$ and $Q$. Any edge from $x_1$ to $Q$ would yield a contradiction with the maximality of $Q$, so $G$ must be isomorphic to the exceptional graph $G^*$ of the theorem.

(iii-2) $p$ is odd and $q$ is even

Since we have assumed that $x_1$ is *black*, we have more *black* vertices on $P$ than *white* ones, so $d_P(x_1) \le \frac{p-1}{2}$. If we had a vertex $z \in N_Q(x_1) \cap N_Q(y_q)^+$, then $x_p\overleftarrow{P}x_1z\overrightarrow{Q}y_qz^-\overleftarrow{Q}y_1$ would yield a Hamilton path in $G$, so $N_Q(x_1) \cap N_Q(y_q)^+ = \emptyset$. This implies $d_Q(x_1) \le \frac{q}{2} - d_Q(y_q)$. Substituting this, we obtain

$$d_G(x_1) + d_G(y_q) = d_P(x_1) + d_Q(x_1) + d_Q(y_q) \le \frac{p-1}{2} + \frac{q}{2} = \frac{n-1}{2}.$$

By the degree conditions of the theorem, we must have equality everywhere, i.e., $d_Q(x_1) + d_Q(y_q) = \frac{q}{2}$. Thus every pair of consecutive positions $y_{2j-1}, y_{2j}$ on $Q$ $(1 \le j \le q/2)$ must contain a neighbor of $x_1$ or a neighbor of $y_q$, but not both. This also implies that $x_1y_2 \in E$, since $y_1y_q \notin E$. By the maximality of $Q$, then we must have $p = 1$. Also note that we can substitute $y_1$ for $x_1$ in the argument, since $x_1y_2 \in E$, so every pair of consecutive positions $y_{2j-1}, y_{2j}$ on $Q$ $(1 \le j \le q/2)$ also contains exactly one neighbor of $y_1$ or one neighbor of $y_q$. It also follows that from every such pair of consecutive positions, $x_1$ and $y_1$ have exactly the same neighbors. Let $s$ be the largest index for which $x_1y_{2s} \in E$. We also have $y_1y_{2s} \in E$ and $y_qy_{2s-1} \notin E(G)$. Note that $y_3y_q \notin E$, since otherwise we would have the Hamilton path $x_1y_2y_1y_{2s}\overrightarrow{Q}y_qy_3\overrightarrow{Q}y_{2s-1}$. By our above observation about consecutive positions, this implies $x_1y_4 \in E$ and $y_1y_4 \in E$. Similarly, $y_5y_q \notin E$, since otherwise we would have the Hamilton path $x_1y_2\overrightarrow{Q}y_4y_1y_{2s}\overrightarrow{Q}y_qy_5\overrightarrow{Q}y_{2s-1}$. Again, this implies $x_1y_6 \in E$ and $y_1y_6 \in E$. Continuing with this argument in a similar fashion, we obtain that $y_{2j-1}y_q \notin E$ for $1 \le j \le s$ and $y_{2j}$ is a neighbor of $x_1$ and $y_1$. Thus, $|N_G(x_1)| = |N_G(y_1)| = s$. Since $d_G(y_1) + d_G(y_q) = \frac{q}{2}$, it follows that $d_Q(y_q) = \frac{q}{2} - s$. Therefore, $N_G(y_q) = \{y_{2s+1}, y_{2s+3}, \ldots, y_{q-1}\}$. If we had an edge $y_{2i+1}y_{2j}$ for some $1 \le i \le s-1$ and $s+1 \le j < \frac{q}{2}$, then $y_{2i+1}y_{2j}\overrightarrow{Q}y_qy_{2j-1}\overleftarrow{Q}y_{2i+2}y_1\overrightarrow{Q}y_{2i+1}$ would be a Hamilton cycle in $G[Q]$, so none of these edges exists. Hence for any $w \in \{x_1, y_1, y_3, ..., y_{2s-1}\}$ and $z \in \{y_{2s+2}, y_{2s+4}, ..., y_q\}$ we have $wz \notin E$. Applying the theorem's degree conditions for any such $w$, we get $\frac{n-1}{2} = \frac{q}{2} \le d_G(w)+d_G(y_q) \le s+\frac{q}{2}-s = \frac{q}{2}$, implying $d_G(w) = s$. We can similarly get $d_G(z) = \frac{q}{2} - s$.

In summary, we have proved that $G[\{x_1, y_1, y_3, ..., y_{2s-1}\} \cup \{y_2, y_4, ..., y_{2s}\}] = K_{s+1,s}$ and $G[\{y_{2s+1}, \ldots, y_{q-1}\} \cup \{y_{2s+2}, \ldots, y_q\}] = K_{q/2-s,q/2-s}$. Thus, $G$ is the union of $K_{s+1,s}$ and $K_{q/2-s,q/2-s}$ with the additional edge $y_{2s}y_{2s+1}$, i.e., $G \in \mathcal{G}_4'$.

**Case 2** $G$ is not connected

We note that it is clear that $G$ cannot have a Hamilton path in this case, but we have to deal with this case in order to obtain the claimed characterization of the extremal graphs in the theorem. Let $G_1 = (X_1, Y_1; E_1)$ and $G_2 = (X_2, Y_2; E_2)$ be the partition of $G$ into two disjoint subgraphs with $V_1 = X_1 \cup X_2$ and $V_2 = Y_1 \cup Y_2$. Since $||V_1| - |V_2|| \le 2$, we assume, without loss of generality, that $|V_1| \le |V_2| \le |V_1| + 2$.

For any vertices $x \in X_1$, $y \in Y_1$, $s \in X_2$ and $t \in Y_2$, we have $xt$, $ys \notin E$ and thus,

$$|Y_1| + |X_2| \ge d_G(x) + d_G(t) \ge \frac{n-1}{2} \quad (1)$$

and

$$|X_1| + |Y_2| \ge d_G(y) + d_G(s) \ge \frac{n-1}{2} \quad (2)$$

For even $n$, this implies

$$|Y_1| + |X_2| \ge d_G(x) + d_G(t) \ge \frac{n}{2}, \qquad |X_1| + |Y_2| \ge d_G(y) + d_G(s) \ge \frac{n}{2},$$

from which $n = |X_1| + |Y_1| + |X_2| + |Y_2| \ge n$. So all inequalities hold as equalities, i.e., $G_1$ and $G_2$ are two complete bipartite graphs, and $|X_1| + |Y_2| = |Y_1| + |X_2| = \frac{n}{2}$.

Since $|V_1| \le |V_2| \le |V_1|+2$, we get either $|V_1| = |V_2| = \frac{n}{2}$ or $|V_1| = \frac{n}{2} - 1$ and $|V_2| = \frac{n}{2} + 1$. When $|V_1| = |V_2| = \frac{n}{2}$, we have

$$|X_1| + |Y_2| = |X_1| + |X_2| = |Y_1| + |X_2| = |Y_1| + |Y_2| = \frac{n}{2},$$

implying that $|X_1| = |Y_1|$ and $|X_2| = |Y_2|$. So, $G = K_{m,m} \cup K_{\lfloor \frac{n}{2} \rfloor - m, \lceil \frac{n}{2} \rceil - m}$, where $1 \le m \le \lfloor \frac{n}{2} \rfloor - 1$, i.e., $G \in \mathcal{G}_1'$.

When $|V_1| = \frac{n}{2} - 1$ and $|V_2| = \frac{n}{2} + 1$, i.e., $|V_2| = |V_1| + 2$, we get

$$|X_1|+|Y_2|=\frac{n}{2}, |X_1|+|X_2|=\frac{n}{2}-1, |Y_1|+|X_2|=\frac{n}{2}, |Y_1|+|Y_2|=\frac{n}{2}-1,$$

which imply that $|Y_1|=|X_1|+1$ and $|Y_2|=|X_2|+1$. So, $G=K_{m,m+1}\cup K_{\lfloor\frac{n-1}{2}\rfloor-m,\lceil\frac{n-1}{2}\rceil-m}$, where $1\le m\le\lfloor\frac{n-1}{2}\rfloor-1$, i.e., $G\in\mathcal{G}_2'$.

For odd $n$, we cannot have $|V_1|=|V_2|$ or $|V_2|=|V_1|+2$, so we get $|V_1|=\frac{n-1}{2}$ and $|V_2|=\frac{n+1}{2}$. Using (1) and (2), we can get

$$|X_1|+|Y_2|\ge\frac{n-1}{2}, \qquad |X_1|+|X_2|=\frac{n-1}{2}$$

and

$$|Y_1|+|X_2|\ge\frac{n-1}{2}, \qquad |Y_1|+|Y_2|=\frac{n+1}{2},$$

these imply that $|X_2|\le|Y_2|\le|X_2|+1$. When $|X_2|=|Y_2|$, we get $G=K_{m,m}\cup K_{\lfloor\frac{n}{2}\rfloor-m,\lceil\frac{n}{2}\rceil-m}$, where $1\le m\le\lfloor\frac{n}{2}\rfloor-1$, i.e., $G\in\mathcal{G}_1'$; and when $|Y_2|=|X_2|+1$, we get $G=K_{m,m+1}\cup K_{\lfloor\frac{n-1}{2}\rfloor-m,\lceil\frac{n-1}{2}\rceil-m}$, where $1\le m\le\lfloor\frac{n-1}{2}\rfloor-1$, i.e., $G\in\mathcal{G}_2'$.

This completes the proof of Theorem 9.

■

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