Some results on mod (integral) sum graphs

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Abstract

Let $N(Z)$ denote the set of all positive integers (integers). The sum graph $G^+(S)$ of a finite subset $S \subset N(Z)$ is the graph (S, E) with $uv \in E$ if and only if $u + v \in S$. A graph G is said to be an (integral) sum graph if it is isomorphic to the sum graph of some $S \subset N(Z)$. The (integral) sum number $\sigma(G)(\zeta(G))$ of G is the smallest number of isolated vertices which when added to G result in an (integral) sum graph. A mod (integral) sum graph is a sum graph with $S \subset Z_m \backslash \{0\}$ $(S \subset Z_m)$ and all arithmetic performed modulo m where $m \geq |S| + 1$ $(m \geq |S|)$. The mod (integral) sum number $\rho(G)$ $(\psi(G))$ of G is the least number ρ (ψ) of isolated vertices ρK_1 (ψK_1) such that $G \cup \rho K_1$ $(G \cup \psi K_1)$ is a mod (integral) sum graph. In this paper, the mod (integral) sum numbers of $K_{r,s}$ and $K_n - E(K_r)$ are investigated and bounded, and n-spoked wheel W_n is shown to be a mod integral sum graph.

1 Introduction

The concept of the (integral) sum graph was introduced by Harary [3]([4]). Let $N(Z)$ denote the set of all positive integers (integers). The sum graph $G^+(S)$ of a finite subset $S \subset N(Z)$ is the graph (S, E) with $uv \in E$ if and only if $u + v \in S$. A graph G is said to be an (integral) sum graph if it is isomorphic to the sum graph of some $S \subset N(Z)$. We say that S give an (integral) sum labelling for G. The (integral) sum number $\sigma(G)(\zeta(G))$ is

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the smallest number of isolated vertices which when added to G result in an (integral) sum graph.

The concept of mod sum graph was introduced by Boland et al. [1] in 1990. A mod sum graph is a sum graph with S$\subset Z_m\backslash\{0\}$ and all arithmetic performed modulo m where $m \geq |S|+1$. The mod sum number $\rho(G)$ of G is the least number ρ of isolated vertices ρK_1 such that $G \cup \rho K_1$ is a mod sum graph. It is obvious that $\rho(G) \leq \sigma(G)$ for any graph G. Similarly we can define the concepts of mod integral sum graph and mod integral sum number. A mod integral sum graph is a sum graph with S$\subset Z_m$ and all arithmetic performed modulo m where $m \geq |S|$. The mod integral sum number $\psi(G)$ of graph G is the least number ψ of isolated vertices ψK_1 such that $G \cup \psi K_1$ is a mod integral sum graph. It is obvious that $\psi(G) \leq \rho(G)$ for any graph G. In this paper, the mod (integral) sum numbers of the complete bipartite graphs and the graph $K_n - E(K_r)$ are investigated and bounded, and n-spoked wheel W_n is shown to be a mod integral sum graph.

To simplify notations, throughout this paper we may assume that the vertices of G are identified with their labels. In addition all arithmetic are performed modulo m if it is not pointed out specially.

2 Complete bipartite graph $K_{r,s}(s \geq r)$

Let $V(K_{r,s}) = (A, B)$ be the bipartition of $K_{r,s}$ with $A = \{a_1, \cdots, a_r\}$, $B = \{b_1, \cdots, b_s\}$ and $S = A \cup B$. We have known that $K_{1,s}(s \geq 2)$ and $K_{2,s}$ are mod sum graphs. We can find the corresponding results in [1]. Since $K_{1,1} = K_2$ and $\rho(K_2) = 1$ we only need to consider the case of $s \geq r \geq 3$.

Lemma 2.1 *$K_{r,s}$ is a mod sum graph for $s > 3r - 4(r \geq 3)$.*

Proof. Let $a_i = (r-2+i)N$, $i = 1, \cdots, r$, $b_j = (j-1)N+1$, $j = 1, \cdots, s$, and take the modulus $m = sN$, where $N \geq 3$ is an integer. It is easy to verify that the following assertions are true.

(1) $A \cap B = \emptyset, A \cup B \subset Z_m\backslash\{0\}$.

(2) $a_i + a_j \notin S$ for any $a_i, a_j \in A(i \neq j)$.

(3) $b_i + b_j \notin S$ for any $b_i, b_j \in B(i \neq j)$.

(4) $a_i + b_j \in S$ for any $a_i \in A$ and $b_j \in B$.

Thus the above labelling is a mod sum labelling of $K_{r,s}$ for $s > 3r-4(r \geq 3)$. □

Lemma 2.2 *For $s > 2r - 1$ and s even, $K_{r,s}$ is a mod sum graph.*

Proof. Let $a_i = (2i-1)N$, $i = 1, \cdots, r$, $b_j = (j-1)N + 1$, $j = 1, \cdots, s$, and take the modulus $m = sN$, where $N \geq 3$ is an integer. It is easy to verify that the following assertions are true.

(1) $A \cap B = \emptyset, A \cup B \subset Z_m \backslash \{0\}$.

(2) $a_i + a_j \notin S$ for any $a_i, a_j \in A (i \neq j)$.

(3) $b_i + b_j \notin S$ for any $b_i, b_j \in B (i \neq j)$.

(4) $a_i + b_j \in S$ for any $a_i \in A$ and $b_j \in B$.

Thus the above labelling is a mod sum labelling of $K_{r,s}$ for $s > 2r - 1$ and s even. □

Lemma 2.3 *$K_{r,s}$ is a mod sum graph for $\frac{5}{2}r \leq s \leq 3r - 4$, s odd and dividable by* 5.

Proof. Let $a_{1i} = [5(i-1)+2]N$, $i = 1, \cdots, \lceil \frac{r}{2} \rceil$, $a_{2j} = [5(j-1)+3]N$, $i = 1, \cdots, \lfloor \frac{r}{2} \rfloor$, $b_k = (k-1)N + 1$, $k = 1, \cdots, s$, and take the modulus $m = sN$, where $N \geq 3$ is an integer. Let $A = \{a_{1i} | \ i = 1, \cdots, \lceil \frac{r}{2} \rceil\} \cup \{a_{2j} | \ j = 1, \cdots, \lfloor \frac{r}{2} \rfloor\}$, $B = \{b_k | \ k = 1, \cdots, s\}$ and $S = A \cup B$. It is easy to verify that the following assertions are true.

(1) $A \cap B = \emptyset, A \cup B \subset Z_m \backslash \{0\}$.

(2) $a_{1i} + a_{1j} \notin S$ for any $a_{1i}, a_{1j} \in A$ $(i \neq j)$.

(3) $a_{2i} + a_{2j} \notin S$ for any $a_{2i}, a_{2j} \in A$ $(i \neq j)$.

(4) $a_{1i} + a_{2j} \notin S$ for any $a_{1i}, a_{2j} \in A$.

(5) $b_i + b_j \notin S$ for any $b_i, b_j \in B$ $(i \neq j)$.

(6) $a_{1i} + b_j \in S$ for any $a_{1i} \in A$ and $b_j \in B$.

(7) $a_{2i} + b_j \in S$ for any $a_{2i} \in A$ and $b_j \in B$.

Thus the above labelling is a mod sum labelling of $K_{r,s}$ for $\frac{5}{2}r \leq s \leq 3r - 4$, s odd and $5 \mid s$. □

Let $\rho = \rho(K_{r,s})(s \geq r \geq 3)$. We will give some properties of the mod sum graph $K_{r,s} \cup \rho K_1$ for $s \geq r \geq 3$. Let $V(K_{r,s}) = (A, B)$ be the bipartition of $K_{r,s}$, and

$A = \{a_1, a_2, \cdots, a_r\}$, where $a_1 < a_2 < \cdots < a_r$.
$B = \{b_1, b_2, \cdots, b_s\}$, where $b_1 < b_2 < \cdots < b_s$.
$S = V(K_{r,s} \cup \rho K_1)$ and the modulus be m.

Lemma 2.4 *If $u \in S$ and there exists a_i $(1 \le i \le r) \ne u$ (or b_j $(1 \le j \le s) \ne u$) such that $u + a_i \in S$ (or $u + b_j \in S$) then $u \in B$ (or $u \in A$).*

This lemma is obvious.

Lemma 2.5 *$a_i + b_j \notin A$ for any $a_i \in A$ and any $b_j \in B$.*

Proof. By contradiction. If there exist $a_p \in A$ and $b_q \in B$ such that $a_p + b_q \in A$ we will prove that $a_p + b_j \in A$ for any $b_j \in B$.

It is obvious that $a_p + b_j \in S$ for any $b_j \in B$. Since $a_p + b_q \in A$ and $b_j \in B$ we have that $(a_p + b_q) + b_j = (a_p + b_j) + b_q \in S$. If $a_p + b_j \ne b_q$ then $a_p + b_j \in A$. Suppose that $a_p + b_j = b_q$. We have that $b_j = b_q - a_p$ and there exists at most one such j. So $a_p + b_k \ne b_q$ for any $b_k \in B$ and $b_k \ne b_j$. Thus $a_p + b_k \in A$. Since $b_q - a_p \in B$ and $a_p + b_q \in A$ we have that $(a_p + b_k) + (b_q - a_p) = 2b_q \in S$. In a similar way we obtain that $a_p + 2b_q \in S$ by $b_q \in B$ and $a_p + b_q \in A$. We consider the following two cases, respectively.

Case 1: $a_p = 2b_q$.

We have that $b_j = b_q - a_p = -b_q \in B$ and $a_p + b_q = 3b_q \in A$. Let $l \ge 4$ be the smallest integer such that $lb_q \notin A$. If such an integer l exists then $(l-1)b_q \in A$, $lb_q = (l-1)b_q + b_q \in S$. Notice that $lb_q + (-b_q) = (l-1)b_q$. So $lb_q = -b_q$ i.e., $(l+1)b_q = 0$. If $l \ge 5$ then $a_p + (a_p + b_q) = 2b_q + 3b_q = 5b_q \in S$ and $a_p \ne a_p + b_q$, contradicting the fact that a_p, $a_p + b_q \in A$. So $l = 4$, $5b_q = 0$. We can obtain $r = |A| \ge s$ by $\{a_p,\ a_p + b_1,\ a_p + b_2,\ \cdots,\ a_p + b_{j-1},\ a_p + b_{j+1},\ \cdots,\ a_p + b_s\} \subset A$. Thus $s = r \ge 3$. Therefore there exists k ($1 \le k \le s$, $k \ne j, q$) such that $a_p + b_k \in A$. Notice that $(a_p + b_q) + (a_p + b_k) = 3b_q + (2b_q + b_k) = b_k$. So $a_p + b_q$ is adjacent to $a_p + b_k$, contradicting the fact that $a_p + b_q$, $a_p + b_k \in A$. If such an integer l does not exist then $ib_q \in A$ for $i = 2, 3, \cdots$, contradicting the finiteness of A.

Case 2: $a_p \ne 2b_q$.

Since $2b_q \in S$ and $a_p + 2b_q \in S$ we have that $2b_q \in B$. By $a_p + b_q \in A$ we obtain that $(a_p + b_q) + 2b_q = (a_p + 2b_q) + b_q \in S$ and $a_p + 2b_q \ne b_q$. So $a_p + 2b_q \in A$. Since $b_j = b_q - a_p \in B$ we know that $(a_p + 2b_q) + (b_q - a_p) = 3b_q \in S$. Thus b_q is adjacent to $2b_q$, contradicting b_q, $2b_q \in B$.

From the above we have that $a_p + b_j \in A$ for any $b_j \in B$. Notice that $a_p + b_1, a_p + b_2, \cdots, a_p + b_s, a_p$ are distinct. Therefore, $r = |A| \ge s + 1 > r$, which is a contradiction. □

Lemma 2.6 *If there exist $a_p \in A$ and $b_q \in B$ such that $a_p + b_q \in B$, then $a_i + b_q \in B$ for any $a_i \in A$.*

Proof. It is obvious that $a_i + b_q \in S$ for any $a_i \in A$. Since $a_p + b_q \in B$ and $a_i \in A$ we have that $a_i + (a_p + b_q) = a_p + (a_i + b_q) \in S$. Thus by Lemma 2.5 we can obtain that $a_i + b_q \in B$. □

Lemma 2.7 *$\rho(K_{r,s}) \geq r$ for $r \leq s \leq 2r-1 (r \geq 3)$ or $s = 2r+1 (r \geq 5)$.*

Proof. If there exist $a_p \in A$ and $b_q \in B$ such that $a_p + b_q \notin B$. By Lemmas 2.5 and 2.6 we have that $a_i + b_q \in S - (A \cup B)$ for any $a_i \in A$, i.e., $\{a_1 + b_q, \cdots, a_r + b_q\} \subset S - (A \cup B)$. Therefore $\rho(K_{r,s}) \geq r$. Hence we only need to prove that there exist $a_p \in A$ and $b_q \in B$ such that $a_p + b_q \notin B$.

Suppose that $a_i + b_j \in B$ for any $a_i \in A$ and $b_j \in B$. Then $\{a_i + b_1, a_i + b_2, \cdots, a_i + b_s\} = B$, $i = 1, 2, \cdots, r$. So $sa_i = 0$, i.e. $m \mid sa_i$, $i = 1, 2, \cdots, r$. We assume that a is the maximum common factor of $a_1, \cdots, a_r$. Then $a \mid a_i$, $i = 1, 2, \cdots, r$ and $m \mid sa$. There exists $l (l \leq s-1)$ such that $la < m$ and $a_r = la$. Let $R = \{a, 2a, 3a, \cdots, la\}$, then $A \subset R$. Now we will aim at proving that $a_r = la \geq (2r-2)a$.

If $1 \leq l \leq 2r-3$ then since $ja + (l-j)a = la$ for $j = 1, 2, \cdots, \lceil \frac{l-1}{2} \rceil$ we have that $|A \cap \{ja,\ (l-j)a\}| \leq 1$. Thus there are at most $\lceil \frac{l-1}{2} \rceil$ elements in $R - \{la\}$ belonging to A. So $r = |A| \leq \lceil \frac{l-1}{2} \rceil + 1 \leq \lceil \frac{2r-4}{2} \rceil + 1 = r - 1$, which is a contradiction. So $a_r \geq (2r-2)a$, $m > (2r-2)a$ and $s = 2r-1$ or $2r+1$. Since $m \mid sa$ we have that $m = sa \leq (3r-4)a$. Firstly we will give three claims.

Claim 1: $|A \cap \{(s-1)a, (s-2)a\}| \leq 1$.

Suppose that both $(s-1)a$ and $(s-2)a$ belong to A, then $a_r = (s-1)a$, $a_{r-1} = (s-2)a$. Therefore, if we have chosen a_i then $a_i + a$, $a_i + 2a$ cannot belong to A, i.e., $a_{i+1} \geq a_i + 3a$, $i = 1,\ 2,\ \cdots,\ r-3$. So $a_{r-2} \geq a_{r-3} + 3a \geq a_{r-4} + 6a \geq \cdots \geq a_1 + 3(r-3)a \geq (3r-8)a$. Since $a_{r-2} + a \notin A$ (otherwise $a_{r-1} = a_{r-2} + a$ and $a_r + a_{r-1} = a_{r-2}$, which is a contradiction.) we have that $a_{r-1} = (s-2)a \geq a_{r-2} + 2a \geq (3r-6)a$. So $sa \geq (3r-4)a$, i.e. $s \geq 3r-4$. Thus $a_1 = a$. However $a_1 + a_{r-1} = a + (s-2)a = (s-1)a = a_r$, which is a contradiction.

Analogously we can obtain the following claims:

Claim 2: $|A \cap \{a, (s-2)a\}| \leq 1$.

Claim 3: $|A \cap \{(s-1)a, 2a\}| \leq 1$.

We will consider the following three cases.

Case 1: $a_r = 2ra$, $s = (2r+1)a$.

We have that $(s-2)a = (2r-1)a \notin A$ (by Claim 1), $ra \notin A$ (otherwise $(r-1)a, (r+1)a \notin A$. Since $ja + (2r-j)a = 2ra$ for $j = 1, 2, \cdots, r-2$ we have that $|A \cap (\{ja\} \cup \{(2r-j)a\})| \leq 1$. Hence there exists exactly one which belongs to A between ja and $(2r-j)a$ ($j = 1, 2, \cdots, r-2$). If $ja \in A$ for $1 \leq j \leq r-2$ and $j \neq \frac{r}{2}$ then we have that $ra + ja \notin A$ by $ra \in A$. Since $ja + (ra - ja) = ra$ and $j \neq \frac{r}{2}$ we have that $ra - ja \notin A$. However $(ra+ja)+(ra-ja) = 2ra$, contradicting the fact that there exists exactly one which belongs to A between ja and $(2r-j)a$ ($j = 1, 2, \cdots, r-1$). So we have that $ja \notin A$ for any $1 \leq j \leq r-2$ and $j \neq \frac{r}{2}$. If r is odd then $A \subset \{ra,\ (r+2)a,\ (r+3)a,\ \cdots,\ (2r-2)a,\ 2ra\}$ and $|A| \leq r-1$, which is impossible. If r is even then $A = \{\frac{r}{2}a,\ ra,\ (r+2)a,\ \cdots,\ (2r-2)a,\ 2ra\}$. We have that $r+3 \leq 2r-2$ and $(r+3)a \in A$ by $r \geq 5$. However $(r+3)a + 2ra = (r+2)a$ and $(r+3)a \neq 2ra$, which is a contradiction.), and exactly one of ja and $(2r-j)a (= (s-1-j)a)$ $(j = 1, 2, \cdots, r-1)$ belongs to A. By claims 1 and 3 we have that $2a,\ (s-2)a \notin A$. Thus $a,\ (s-3)a \in A$. If $(r+1)a \in A$ then $(r+2)a \notin A$. Thus $(r-2)a \in A$. However $(s-3)a + (r+1)a = (r-2)a$ and $(s-3)a \neq (r+1)a$, which is a contradiction. Analogously, the assumption that $(r+1)a \notin A$ provides the same contradiction.

Case 2: $a_r = (2r-1)a,\ s = (2r+1)a$.

We have that $a \notin A$ (by Claim 2) and exactly one of ja and $(s-2-j)a$ ($j = 1,\ 2,\ \cdots,\ r-1$) belongs to A. Thus $(s-3)a \in A$. So $(s-5)a \notin A$, $3a \in A$. Therefore $5a,\ 6a,\ (s-6)a \notin A$, $(s-7)a,\ (s-8)a,\ 4a \in A$. If $r \geq 6$ then $(s-7)a \neq 4a$, which is a contradiction. If $r = 5$ then $\{3a,\ 4a,\ 8a,\ 9a\} \subset A$ and $a,\ 5a,\ 6a \notin A$. Since $4a + 9a = 2a$ and $3a + 4a = 7a$ we have that $2a,\ 7a \notin A$. So $A \not\subset R$. The result holds.

Case 3: $a_r = (2r-2)a$.

Since $ja + (l-j)a = (2r-2)a$ for $j = 1,\ 2,\ \cdots,\ r-1$ we have that $|A \cap (\{ja\} \cup \{(2r-2-j)a\})| \leq 1$. Thus there is exactly one of ja and $(2r-2-j)a$ ($j = 1, 2, \cdots, r-1$) belongs to A and $(r-1)a \in A$. If $ja \in A$ for $1 \leq j \leq r-2$ and $j \neq \frac{r-1}{2}$ then $(r-1)a + ja \notin A$. Since $ja + [(r-1)a - ja] = (r-1)a$ and $j \neq \frac{r-1}{2}$ we have that $(r-1)a - ja \notin A$. However $[\,(r-1)a + ja\,] + [\,(r-1)a - ja\,] = la$, contradicting the fact that there is exactly one of ja and $(2r-2-j)a$ ($j = 1, 2, \cdots, r-1$) belongs to A. So $ja \notin A$ for $1 \leq j \leq r-2$ and $j \neq \frac{r-1}{2}$.

If $s = (2r+1)a$ then $(r+2)a \notin A$(If not, noticing that $(r+2)a + (2r-2)a = (r-1)a$ and $r \geq 5$ we have that $(r+2)a \neq (2r-2)a$, contradicting the fact that $(r+2)a,\ (2r-2)a \in A$.). Therefore, if r is even then $A = \{(r-1)a,\ ra,\ (r+1)a,\ (r+3)a,\ \cdots,\ (2r-2)a\}$ and $|A| = r-1$, which is impossible. If r is odd then $A = \{\frac{r-1}{2}a,\ (r-1)a,\ ra,\ (r+1)a,\ (r+3)a,\ \cdots,\ (2r-2)a\}$. If $r = 5$ then $A = \{2a, 4a, 5a, 6a, 8a\}$. However $2a+4a = 6a$, which is a contradiction. If $r \geq 7$ then $(2r-2)a + (r+3)a = ra$ and $(2r-2)a \neq (r+3)a$, which is also a contradiction.

If $s = (2r-1)a$ then $ra \notin A$(If not, noticing that $ra + (2r-2)a = (r-1)a$ and $r \geq 3$ we have that $ra \neq (2r-2)a$, contradicting the fact that ra, $(2r-2)a \in A$.). Therefore, if r is even then $A = \{(r-1)a, (r+1)a, (r+2)a, \cdots, (2r-2)a\}$ and $|A| = r-1$, which is impossible. If r is odd then $A = \{\frac{r-1}{2}a, (r-1)a, (r+1)a, (r+2)a, \cdots, (2r-2)a\}$. If $r = 3$ then $A = \{a, 2a, 4a\}$ and $m = 5a$. However $2a+4a = a$, which is a contradiction. If $r \geq 5$ then $(r+2)a + (2r-2)a = (r+1)a$ and $(2r-2)a \neq (r+2)a$, which is a contradiction.

From the above we know that the lemma holds. □

Lemma 2.8 *$\rho(K_{r,s}) \leq r$ for $r \leq s \leq 3r-4$.*

Proof. Let

$a_i = (i-1)N + 2,\ i = 1, \cdots, r,$

$b_{1,j} = (j-1)N + 3,\ j = 1, \cdots, s-2r,$

$b_{2,j} = (j-1)N + 5,\ j = 1, \cdots, \min\{r, s-r\},$

$b_{3,j} = (j-1)N + 7,\ j = 1, \cdots, r,$

$c_k = (k-1)N + 9,\ k = 1, \cdots, r,$

and $m = rN$, where $N \geq 18$ is an integer. Let $V(K_{r,s}) = (A, B)$ is the bipartition of $K_{r,s}$, where $A = \{a_1, a_2, \cdots, a_r\}$, $B = \{b_{1,1}, \cdots, b_{1,s-2r}, b_{2,1}, \cdots, b_{2,\min\{r,s-r\}}, b_{3,1}, \cdots, b_{3,r}\}$. Let $C = V(rK_1) = \{c_1, c_2, \cdots, c_r\}$, $S = V(K_{r,s} \cup rK - 1) = A \cup B \cup C$.

It is direct to verify that the above labelling is a mod sum labelling of $K_{r,s} \cup rK_1$ for $r \leq s \leq 3r-4$. □

Theorem 2.1 *For $s \geq r$,*

$$\rho(K_{r,s}) = \begin{cases} 0, & s > r = 1, \text{ or } s = r = 2, \text{ or } s > 3r-4 (r \geq 2), \text{ or} \\ & 3r-4 \geq s > 2r-1,\ s \text{ is even, or } \frac{5}{2}r \leq s \leq 3r-4, \\ & s \text{ is odd and } 5 \mid s, \\ r, & r \leq s \leq 2r-1 (r \neq 2) \text{ or } s = 2r+1 (r \geq 5), \\ 0 \text{ or } r, & 2r+3 \leq s < \frac{5}{2}r,\ s \text{ is odd, or } \frac{5}{2}r \leq s \leq 3r-4,\ s \text{ is} \\ & \text{odd and cannot be divided by } 5. \end{cases}$$

Proof. We only need to prove the last case. Since s satisfies that $r \leq s \leq 3r-4$ we have that $\rho(K_{r,s}) \leq r$ by Lemma 2.8. If $a_i + b_j \notin S - (A \cup B)$ for any $a_i \in A$ and any $b_j \in B$ then $\rho(K_{r,s}) = 0$. If there exist $a_p \in A$ and $b_q \in B$ such that $a_p + b_q \in S - (A \cup B)$ then by Lemmas 2.5 and 2.6 we have that $a_i + b_q \notin A \cup B$ for $a_i \in A$. Therefore $\rho(K_{r,s}) = r$. □

Lemma 2.9 *If there doesn't exist some vertex with degree $|V(G)|-1$, then $\psi(G)=\rho(G)$.*

This lemma is obvious. Thus we have the following corollary.

Corollary 2.1 *For complete bipartite graph $K_{r,s}$ with $s \geq r$ and $r+s \geq 3$, $\psi(K_{r,s})=\rho(K_{r,s})$.*

Proof. If $s \geq r \geq 2$ then we obtain the corollary by Lemma 2.9. If $s \geq r = 1$ and $r+s \geq 3$ then we can obtain $\psi(K_{r,s})=\rho(K_{r,s})$ by $\psi(K_{r,s}) \leq \rho(K_{r,s}) = 0$. □

3 $K_n - E(K_r)$

Let $A = V(K_r) = \{a_1, a_2, \cdots, a_r\}$, $B = V(K_n) - A = \{b_1, b_2, \cdots, b_{n-r}\}$. If $r = n$ it is obvious that $K_n - E(K_r)$ is a mod sum graph. Since $K_n - E(K_r)$ is a tree for $r = n-1$ and $K_n - E(K_r) = K_n$ for $r = 1$ we have that the mod sum number of $K_n - E(K_r)$ is known for $r = n-1$ and $r = 1$. Then, from now on we only need to consider the cases $2 \leq r \leq n-2$.

Lemma 3.1 $\rho(K_n - E(K_r)) \leq r$ *for* $\frac{n}{2} \leq r \leq n-2$.

Proof. Let $a_i = (i-1)N+2$, $i = 1, 2, \cdots, r$, $b_j = (j-1)N+1$, $j = 1, 2, \cdots, n-r$, $c_k = (k-1)N+3$, $k = 1, 2, \cdots, r$, and $m = rN$, where $N \geq 6$ is an integer. Let

$$A = V(K_r) = \{a_1, a_2, \cdots, a_r\},$$
$$B = V(K_n) - V(K_r) = \{b_1, b_2, \cdots, b_{n-r}\},$$
$$C = V(rK_1) = \{c_1, c_2, \cdots, c_r\},$$
$$S = V((K_n - E(K_r)) \cup rK_1) = A \cup B \cup C.$$

It is direct to verify that the above labelling is a mod sum labelling of $(K_n - E(K_r)) \cup rK_1$ for $\frac{n}{2} \leq r \leq n-2$. □

Lemma 3.2 $\rho(K_n - E(K_r)) \leq 2(n-r)$ *for* $2 \leq r < \frac{n}{2}$.

Proof. Let $a_i = (i-1)N+1$, $i = 1, 2, \cdots, r$, $b_j = (j-1)N+3$, $j = 1, 2, \cdots, n-r$, $c_k = (k-1)N+4$, $k = 1, 2, \cdots, n-r$, $d_l = (l-1)N+6$, $l = 1, 2, \cdots, n-r$, and $m = (n-r)N$, where $N \geq 12$ is an integer. Let

$$A = V(K_r) = \{a_1, a_2, \cdots, a_r\},$$
$$B = V(K_n) - V(K_r) = \{b_1, b_2, \cdots, b_{n-r}\},$$

$C = V(2(n-r)K_1) = \{c_1, c_2, \cdots, c_{n-r}, d_1, d_2, \cdots, d_{n-r}\}$,

$S = V((K_n - E(K_r)) \cup 2(n-r)K_1) = A \cup B \cup C$.

It is direct to verify that the above labelling is a mod sum labelling of $(K_n - E(K_r)) \cup 2(n-r)K_1$ for $2 \leq r < \frac{n}{2}$. Therefore $\rho(K_n - E(K_r)) \leq 2(n-r)$ for $2 \leq r < \frac{n}{2}$. □

Let $\rho = \rho(K_n - E(K_r))(2 \leq r \leq n-2)$. We will give some properties of the mod sum graph $(K_n - E(K_r)) \cup \rho K_1$. Let $A = V(K_r) = \{a_1, a_2, \cdots, a_r\}$, $B = V(K_n) - V(K_r) = \{b_1, b_2, \cdots, b_{n-r}\}$, where $a_1 < a_2 < \cdots < a_r$, $b_1 < b_2 < \cdots < b_{n-r}$ and $S = V((K_n - E(K_r)) \cup \rho K_1)$. Suppose the modulus is m. First we consider the cases $3 \leq r \leq n-3$.

Lemma 3.3 *$a_i + b_j \notin A \cup B$ for any $a_i \in A$ and any $b_j \in B$.*

Proof. We will prove that $a_i + b_1 \notin A \cup B$ for any $a_i \in A$ first.

Suppose that there exists $a_p \in A$ such that $a_p + b_1 \in A \cup B$. We will consider the following two cases.

Case 1: $a_p + b_1 \in B$.

Suppose $a_p + b_1 = b_q$. Notice that $b_1 + b_{n-r} \notin B$ and $b_q + b_{n-r} = (b_1 + b_{n-r}) + a_p$. We will consider the following two subcases.

Subcase 1: $q \neq n-r$.

From $b_q + b_{n-r} = (b_1 + b_{n-r}) + a_p \in S$ we have that $a_p = b_1 + b_{n-r}$. Hence there exists at most one such a_p.

Subcase 2: $q = n-r$.

We have that $b_1 + a_p = b_q = b_{n-r}$. Hence $a_p = b_{n-r} - b_1$ and there exists at most one such a_p.

By $r \geq 3$ and the above subcases we have that if $a_p = b_1 + b_{n-r}$, then there exists a vertex $a_k \neq b_{n-r} \pm b_1$. Then $b_1 + a_k \notin B$ and $b_1 + a_k \neq a_p$. Thus $(b_1 + a_k) + a_p = (b_1 + a_p) + a_k = b_q + a_k \notin S$, contradicting the fact that b_q is adjacent to a_k. So the first subcase will not occur. Therefore, by $r \geq 3$ we have that if $a_p = b_{n-r} - b_1$ there exist at least two vertices a_k and a_l with $b_1 + a_k, b_1 + a_l \notin B$. Notice that $(b_1 + a_k) + a_p = (b_1 + a_p) + a_k = b_q + a_k \in S$, $(b_1 + a_l) + a_p = (b_1 + a_p) + a_l = b_q + a_l \in S$ and there exists at least one of $b_1 + a_k$ and $b_1 + a_l$ which is not equal to a_p, a contradiction. Therefore $a_i + b_1 \notin B$ for any $a_i \in A$. (The proof of the result is suitable to $r = n-2 \geq 3$, too.)

Case 2: $a_p + b_1 \in A$.

Suppose $a_p + b_1 = a_q$. Thus $a_q + b_{n-r} = (b_1 + b_{n-r}) + a_p \in S$. Since $b_1 + b_{n-r} \notin B$ we have that $a_p = b_1 + b_{n-r}$. There exists at most one such a_p. By Case 1 we have that $a_i + b_1 \notin B$ for any $a_i \in A$ and $a_i \neq a_p$. Since $b_1 + b_2$, $b_1 + b_3$, $\cdots$, $b_1 + b_{n-r} \notin B$ (if not, suppose that $b_1 + b_j = b_k$ then

$b_k + a_i = (b_1 + a_i) + b_j \in S$, contradicting the fact that $b_1 + a_i \notin A \cup B$.) and $a_q + b_2 = (b_1 + b_2) + a_p \in S$, we have that $a_p = b_1 + b_2$. Thus $b_1 + b_2 = b_1 + b_{n-r}$. So $b_2 = b_{n-r}$, i.e., $n - r = 2$, contradicting $n - r \geq 3$.

From the above we have that $a_i + b_1 \notin A \cup B$ for any $a_i \in A$. We now aim at proving that $a_i + b_j \notin A \cup B$ for any $a_i \in A$ and any $b_j \in B (j \neq 1)$.

Suppose that there exist $a_p \in A$ and $b_q \in B$ such that $a_p + b_q \in A \cup B$. Since $b_1 + (a_p + b_q) = (b_1 + a_p) + b_q \notin S$ we have that $a_p + b_q = b_1$. For a fixed integer q there exists at most one such a_p. For any $a_i \in A$ and $a_i \neq a_p$ we have that $a_i + b_1 = a_i + (a_p + b_q) = (a_i + b_q) + a_p \in S$, contradicting the fact that $a_i + b_q \notin A \cup B$.

This completes the proof of Lemma 3.3. □

Lemma 3.4 *$b_i + b_j \notin B$ for any $b_i,\ b_j \in B (i \neq j)$.*

Proof. Suppose that there exist $b_p,\ b_q \in B (p \neq q)$ such that $b_p + b_q \in B$. Since $a_1 + (b_p + b_q) = (a_1 + b_p) + b_q \in S$ we have that $a_1 + b_p \in A \cup B$, contradicting Lemma 3.3. □

Lemma 3.5 *For $3 \leq r < \frac{n}{2}$ we have that $b_i + b_j \notin A$ for any $b_i,\ b_j \in B\ (i \neq j)$.*

Proof. By the distinctness of $b_1 + b_2, \cdots, b_1 + b_{n-r}, b_2 + b_{n-r}$ and $r < n - r$ we have that the above $n - r$ vertices cannot be all contained in A. Therefore there exist $b_p,\ b_q \in B\ (p \neq q)$ such that $b_p + b_q \notin A$. Notice that $(b_i + b_q) + b_p = b_i + (b_p + b_q) \notin S$ for any $b_i \in B$ and $b_i \neq b_q$. Thus by Lemma 3.4 we have that $b_i + b_q \notin A \cup B$. Since $(b_i + b_j) + b_q = (b_i + b_q) + b_j \notin S$ and $b_i + b_j \in S - B$ for any $b_i,\ b_j \in B - \{b_q\}\ (i \neq j)$ we have that $b_i + b_j \notin A$.

From the above discussion we have that $b_i + b_j \notin A$ for any $b_i, b_j \in B\ (i \neq j)$. □

Lemma 3.6 *For $3 \leq r \leq n - 3$,*

$$\rho(K_n - E(K_r)) \geq \begin{cases} r, & \frac{n}{2} \leq r \leq n - 3, \\ n - 1, & 3 \leq r < \frac{n}{2}. \end{cases}$$

Proof. For $\frac{n}{2} \leq r \leq n-3$ by Lemma 3.3 and the distinctness of $a_1 + b_1, a_2 + b_1, \cdots, a_r + b_1$ we can obtain the result. For $3 \leq r < \frac{n}{2}$ by Lemmas 3.3, 3.4, 3.5 and the distinctness of $a_1 + b_1, a_2 + b_1, \cdots, a_r + b_1, b_1 + b_2, \cdots, b_1 + b_{n-r}$ we know that the lemma holds. □

Lemma 3.7 *$\rho(K_n - E(K_r)) = r - 1$ for $r = n - 2 \geq 3$.*

Proof. From the proof of Lemma 3.3 we know that $b_1 + a_i \notin B(i = 1, 2, \cdots, r)$ still holds for $r = n-2 \geq 3$. If there exists an integer p $(1 \leq p \leq r)$ such that $b_1 + a_p \in A$, then $(b_1 + a_p) + b_{n-r} = (b_1 + b_{n-r}) + a_p \in S$ and $b_1 + b_{n-r} \notin B$. So $a_p = b_1 + b_{n-r}$. Since there exists at most one such a p, $b_1 + a_i \notin A \cup B$ for any $a_i \in A$ and $a_i \neq a_p$. By the distinctness of $a_1+b_1, \cdots, a_{p-1}+b_1, a_{p+1}+b_1, \cdots, a_r+b_1$ we have that $\rho(K_n - E(K_r)) \geq r-1$. So it suffices to prove that $\rho(K_n - E(K_r)) \leq r-1$ for $r = n-2 \geq 3$. Let

$$a_i = 20(i-1)+1, \quad i = 1, 2, \cdots, r-1, \quad a_r = 34, \quad b_1 = 7, \quad b_2 = 27.$$

$$c_j = 20(j-1)+8, \quad j = 1, 2, \cdots, r-1, \quad \text{and } m = 20(r-1).$$

Let $A = V(K_r) = \{a_1, a_2, \cdots, a_r\}$, $B = V(K_n) - V(K_r) = \{b_1, b_2\}$, $C = V((r-1)K_1) = \{c_1, c_2, \cdots, c_{r-1}\}$, $S = V((K_n - E(K_r)) \cup (r-1)K_1) = A \cup B \cup C$. It is easy to verify the following assertions.

(1) $S \subset Z_m \backslash \{0\}$.

(2) $a_i + a_j \notin S$ for any $a_i, a_j \in A (i \neq j)$.

(3) $b_i + b_j \in S$ for any $b_i, b_j \in B (i \neq j)$.

(4) $c_i + c_j \notin S$ for any $c_i, c_j \in C (i \neq j)$.

(5) $a_i + c_j \notin S$ for any $a_i \in A$ and for any $c_j \in C$.

(6) $b_i + c_j \notin S$ for any $b_i \in B$ and for any $c_j \in C$.

(7) $a_i + b_j \in S$ for any $a_i \in A$ and for any $b_j \in B$.

Thus the above labelling is a mod sum labelling of $(K_n - E(K_r)) \cup (r-1)K_1$ for $r = n-2 \geq 3$. □

Lemma 3.8 $\rho(K_n - E(K_r)) \geq n-1$ *for* $r = 2$ *and* $n \geq 5$.

Proof. Let $\rho = \rho(K_n - E(K_r))$, $A = V(K_n) = \{a_1, a_2, \cdots, a_n\}$, where $a_1 < a_2 < \cdots < a_n$, $V(K_2) = \{a_s, a_t\}$, $S = V((K_n - E(K_r)) \cup \rho K_1)$. There exist $a_p, a_q \in A$ such that $a_p + a_q \in S - A$ (If not, then $\{a_s, a_t\} = \{a_1, a_n\}$ and $\{a_1+a_2, a_1+a_3, a_2+a_3, a_2+a_4, \cdots, a_2+a_n\} = \{a_1, a_2, \cdots, a_n\}$. So $a_2 = a_1 + a_3$, giving that $a_1 < a_1 + a_2 < a_2$, i.e., $a_1 + a_2 \notin A$, a contradiction.) and $|\{p, q\} \cap \{s, t\}| \leq 1$. We may assume without loss of generality that $p \notin \{s, t\}$. If there exists $a_j \in A$, $a_j \neq a_q$ and $\{a_j, a_q\} \neq \{a_s, a_t\}$ such that $a_j + a_q \in A$ then $(a_j + a_q) + a_p = a_j + (a_p + a_q) \notin S$. So $a_j + a_q = a_p$. There exists at most one such an integer j. We will consider the following two cases.

Case 1: $q \notin \{s, t\}$.

We have that $\{a_i, a_q\} \neq \{a_s, a_t\}$ for any $a_i \in A(i \neq q)$. Since $n \geq 5$ we have that there exists $a_k \in A - \{a_p, a_q, a_j\}$ such that $a_k + a_q \in S - A$. Notice that $a_k + a_p = a_j + (a_k + a_q) \in S$, contradicting the fact that $a_k + a_q \in S - A$. So $a_i + a_q \in S - A$ for any $a_i \in A(i \neq q)$. By the distinctness of $a_1 + a_q, \cdots, a_{q-1} + a_q, a_{q+1} + a_q, \cdots, a_n + a_q$ we have that $\rho(K_n - E(K_r)) \geq n - 1$.

Case 2: $q \in \{s, t\}$. We may assume without loss of generality that $q = t$.

Since $n \geq 5$ we have that there exists $a_k \in A - \{a_p, a_q, a_s, a_j\}$ such that $a_k + a_q \in S - A$. Notice that $a_k + a_p = a_j + (a_k + a_q) \in S$, contradicting the fact that $a_k + a_q \in S - A$. So $a_i + a_q \in S - A$ for any $a_i \in A(i \neq s, q)$. If there exist $a_{p_1}, a_{q_1} \in A$ and $a_{p_1}, a_{q_1} \notin \{a_s, a_t\}$ such that $a_{p_1} + a_{q_1} \in S - A$, then a similar argument as case 1 shows that the lemma holds. If not, $a_k + a_l \in A$ for any $a_k, a_l \in A(k \neq l)$ and $a_k, a_l \notin \{a_s, a_t\}$, so $(a_k + a_l) + a_q = a_k + (a_l + a_q) \notin S$, giving $a_k + a_l = a_q$. There exists at most one such an integer k for a fixed integer l, i.e., $a_i + a_l \in S - A$ for any $a_i \in A - \{a_k, a_l, a_s, a_q\}$. Since $n \geq 5$ there exists such an a_i, which is a contradiction. □

Theorem 3.1 $$\rho(K_n - E(K_r)) \begin{cases} = 0, & r = n, \text{ or} \\ & r = n-1 (n \geq 3), \\ = 1, & r = 1,\ 2 \leq n \leq 3, \\ = n, & r = 1,\ n \geq 4, \\ = r, & \frac{n}{2} \leq r \leq n-3, \\ = r-1, & r = n-2 \geq 3, \\ \in [n-1, 2(n-r)], & 2 \leq r < \frac{n}{2},\ n \geq 5, \\ = 2, & r = 2,\ n = 4. \end{cases}$$

Proof. We only need to prove $\rho(K_4 - E(K_2)) \geq 2$.

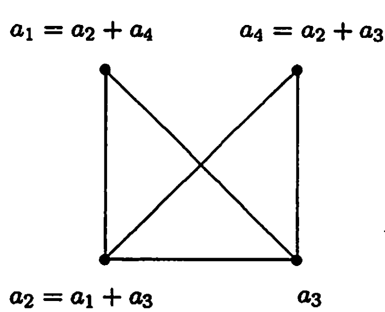

Fig. 1

Let $\rho = \rho(K_4 - E(K_2))$, $A = V(K_4) = \{a_1, a_2, a_3, a_4\}$, $V(K_2) = \{a_1, a_4\}$, $S = V((K_4 - E(K_2)) \cup \rho K_1)$ (See Fig. 1). Similar to the proof of Lemma 3.8 we see that there exists an edge sum that is an isolated vertex. We will consider the following two cases.

Case 1: $a_2 + a_3 \in S - A$.

We have that $(a_2 + a_i) + a_3 = (a_2 + a_3) + a_i \notin S$ $(i = 1, 4)$. Since at least one of $a_2 + a_1$ and $a_2 + a_4$ is not equal to a_3 and so it is an isolated vertex other than $a_2 + a_3$. The theorem holds.

Case 2: $a_1 + a_2 \in S - A$.

We have that $(a_2 + a_i) + a_1 = (a_2 + a_1) + a_i \notin S$, $i = 3, 4$. So $a_2 + a_3 = a_4$, $a_2 + a_4 = a_1$ (If not, there exists an isolated vertex that is not equal to $a_2 + a_1$ and the theorem holds.). Notice that $(a_1 + a_3) + a_2 = (a_1 + a_2) + a_3 \notin S$. If $a_1 + a_3 \neq a_2$ then $a_1 + a_3$ is an isolated vertex that is not equal to $a_1 + a_2$, the theorem holds. If $a_1 + a_3 = a_2$ then $a_1 = a_2 + a_4 = a_1 + a_3 + a_4$. Therefore $a_3 + a_4 = 0 \in S$, which is a contradiction. □

Let $A = V(K_r) = \{a_1, \cdots, a_r\}, B = V(K_n) - V(K_r) = \{b_1, \cdots, b_{n-r}\}$. If $r = n$ or $n - 1$ then $K_n - E(K_r)$ is a mod integral sum graph. If $r = 1$ then $K_n - E(K_r) = K_n$. Thus we only need to consider the case of $n \geq 2$.

Lemma 3.9 [1] *$\psi(K_n) = 0$ for $n \geq 2$.*

Proof. Let $a_i = i - 1$, $i = 1, \cdots, n$, and take the modulus $m = n$. Let $A = V(K_n) = \{a_1, \cdots, a_n\}$. It is easy to verify that the above labelling is a mod integral sum labelling of $K_n (n \geq 2)$. □

Lemma 3.10 *$\psi(K_n - E(K_r)) = 0$ for $r = n - 2 \geq 2$.*

Proof. Let $a_i = (i - 1)N + 1$, $i = 1, \cdots, r$, $b_1 = 0$, $b_2 = N$, and take the modulus $m = rN$, where $N > 3$ is an integer. Let $A = V(K_r) = \{a_1, \cdots, a_r\}, B = V(K_n) - V(K_r) = \{b_1, b_2\}$. It is easy to verify that the following assertions are true.

(1) $A \cup B \subset Z_m$.

(2) $a_i + a_j \notin A \cup B$ for any $a_i, a_j \in A$ $(i \neq j)$.

(3) $a_i + b_j \in A \cup B$ for any $a_i \in A$ and any $b_j \in B$.

(4) $b_i + b_j \in A \cup B$ for any $b_i, b_j \in B$ $(i \neq j)$.

[1]The same result was claimed by Slamin in his Master thesis.

Thus the above labelling is a mod integral sum labelling of $K_n - E(K_r)(r = n-2 \geq 2)$. □

Lemma 3.11 $\psi(K_n - E(K_r)) = 0$ *for* $r = n-3 \geq 2$.

Proof. If $r \geq 3$, let $a_i = (i-1)N+1, i = 1, \cdots, r,\ b_1 = 0,\ b_2 = N,\ b_3 = (r-1)N$, and take the modulus $m = rN$, where $N > 3$ is an integer. Let $A = V(K_r) = \{a_1, \cdots, a_r\}, B = V(K_n) - V(K_r) = \{b_1, b_2, b_3\}$. It is easy to verify that the following assertions are true.

(1) $A \cup B \subset Z_m$.

(2) $a_i + a_j \notin A \cup B$ for any $a_i, a_j \in A$ $(i \neq j)$.

(3) $a_i + b_j \in A \cup B$ for any $a_i \in A$ and any $b_j \in B$.

(4) $b_i + b_j \in A \cup B$ for any $b_i, b_j \in B$ $(i \neq j)$.

Thus the above labelling is a mod integral sum labelling of $K_n - E(K_r)(r = n-3 \geq 3)$.

If $r = 2$, let $a_1 = 3, a_2 = 5, b_1 = 0, b_2 = 1, b_3 = 4$, and take the modulus $m = 6$. Let $A = V(K_r) = \{a_1, a_2\}, B = V(K_n) - V(K_r) = \{b_1, b_2, b_3\}$. It is easy to verify that the above labelling is a mod integral sum labelling of $K_n - E(K_r)(r = n-3 = 2)$. □

Lemma 3.12 $\psi(K_n - E(K_r)) = 0$ *for* $n - r \mid r$.

Proof. Since $n-r \mid r$ we assume that $r = t(n-r)$. Let $a_{i,j} = (j-1)N+2i-1,\ i = 1, \cdots, t,\ j = 1, \cdots, n-r,\ b_k = (k-1)N,\ k = 1, \cdots, n-r$, and take the modulus $m = (n-r)N$, where $N \geq 4t$ is an integer. Let $A = \{\ a_{i,j} \mid i = 1, \cdots, t,\ j = 1, \cdots, n-r\} = V(K_r),\ B = V(K_n) - V(K_r) = \{b_1, \cdots, b_{n-r}\}$. It is easy to verify that the following assertions are true.

(1) $A \cup B \subset Z_m$.

(2) $a_{i,j} + a_{k,l} \notin A \cup B$ for any $a_{i,j}, a_{k,l} \in A$ $(a_{i,j} \neq a_{k,l})$.

(3) $a_{i,j} + b_k \in A \cup B$ for any $a_{i,j} \in A$ and any $b_k \in B$.

(4) $b_i + b_j \in A \cup B$ for any $b_i, b_j \in B$ $(i \neq j)$.

Thus the above labelling is a mod integral sum labelling of $K_n - E(K_r)(n - r \mid r)$. □

From the above discussion we only need to consider the case of $2 \leq r \leq n-4$ and r can not be divided by $n-r$.

Let $\psi = \psi(K_n - E(K_r))(2 \le r \le n-4$ and r can not be divided by $n-r)$. We will give some properties of mod integral sum graph $(K_n - E(K_r)) \cup \psi K_1$. Let $A = V(K_r) = \{a_1, \cdots, a_r\}$, where $a_1 < \cdots < a_r$. $B = V(K_n) - V(K_r) = \{b_1, \cdots, b_{n-r}\}$, where $b_1 < \cdots < b_{n-r}$. $S = V((K_n - E(K_r)) \cup \psi K_1)$, and the modulus be m.

Lemma 3.13 *There exists $b_p, b_q \in B$ and $b_p \neq b_q$ such that $b_p + b_q \notin B$ for $2 \le r \le n-4$ and r can not be divided by $n-r$.*

Proof. If $0 \notin S$ the lemma holds for $b_1 + b_{n-r} \notin B$. If $0 \in S$ then $0 \in B$ and $\psi(K_n - E(K_r)) = 0$, i.e., $S = A \cup B$. We will prove the lemma by contradiction.

If not, we have that $b_i + b_j \in B$ for any $b_i, b_j \in B(i \neq j)$. Since $0 \in B$ we assume that $b_{n-r} = 0$. Thus $\{b_1, b_2, b_1 + b_2, \cdots, b_1 + b_{n-r-1}\} = B$. So $b_i = b_1 + b_{i-1}, i = 3, \cdots, n-r$, i.e., $b_i = (i-2)b_1 + b_2, i = 2, \cdots, n-r$. Since $b_{n-r} = 0$ and $n - r \ge 4$ we have that $(n-r-2)b_1 + b_2 = 0$, i.e., $b_2 = [2-(n-r)]b_1$. So $b_i = [i-(n-r)]b_1, i = 2, \cdots, n-r$. We have that $b_2 \neq b_{n-r-1}$ by $n-r \ge 4$. Thus $b_2 + b_{n-r-1} = b_1$, i.e., $[2-(n-r)]b_1 + [n-r-1-(n-r)]b_1 = b_1$. So $(n-r)b_1 = 0$. Therefore we have that $b_i = ib_1, i = 1, 2, \cdots, n-r$. So $a_i + b_1 \notin B$ for any $a_i \in A$. We have that $a_i + b_1 \in A$, i.e., $\{a_1 + b_1, a_2 + b_1, \cdots, a_r + b_1\} = A$ by $S = A \cup B$ and $a_i + b_1 \in S$. So $rb_1 = 0$. Therefore, $r \ge n-r$ and $n - r \mid r$, which is a contradiction. □

Lemma 3.14 $0 \notin S$ *for $3 \le r \le n-4$ and r can not be divided by $n-r$.*

Proof. There exists $b_p, b_q \in B$ and $b_p \neq b_q$ such that $b_p + b_q \notin B$ by Lemma 3.13. If $0 \in S$ then $S = A \cup B$. Thus $a_i + b_p \in A \cup B$ for any $a_i \in A$. We consider the following two cases, respectively.

Case 1: $a_i + b_p \in B$.

There exists $b_j \in B$ such that $a_i + b_p = b_j$. Notice that $b_p + b_q \notin B$ and $b_j + b_q = (b_p + b_q) + a_i$. If $b_j = b_q$ then $a_i + b_p = b_q$ and there exists at most one such a_i. If $b_j \neq b_q$ then $b_p + b_q = a_i$ and there exists at most one such a_i, too. If $a_i \in A$ satisfies the second case, i.e., $b_p + b_q = a_i$. Since $r \ge 3$ there exists one vertex which doesn't satisfy the above two cases. So $b_p + a_k \notin B$ and $b_p + a_k \neq a_i$. Thus $(b_p + a_k) + a_i = (b_p + a_i) + a_k = b_j + a_k \notin S$, which contradicting that b_j is adjacent to a_k. Therefore, the second case will not occur. Since $r \ge 3$ there exist at least two vertices which don't satisfy the first case, i.e., $b_p + a_k, b_p + a_l \notin B$. For $b_p + a_k \neq b_p + a_l$ there exists at least one which is not equal to a_i. We may assume that $b_p + a_k \neq a_i$. Then $(b_p + a_k) + a_i = (b_p + a_i) + a_k = b_j + a_k \notin S$, which is a contradiction. Thus $a_i + b_p \notin B$ for any $a_i \in A$.

Case 2: $a_i + b_p \in A$.

There exists $a_j \in A$ such that $a_i + b_p = a_j$. So $a_j + b_q = (b_p + b_q) + a_i \in S$. Since $b_p + b_q \notin B$ we have that $b_p + b_q = a_i$ and there exists at most one such a_i. By Case 1 we have that $a_k + b_p \notin A \cup B$ for any $a_k \in A$ and $a_k \neq a_i$. Since $r \geq 3$ we know that there exists such a_k, which is a contradiction. □

Lemma 3.15 $0 \notin S$ *for* $r = 2, n \geq 6$.

Proof. If $0 \in S$ then $\psi(K_n - E(K_r)) = 0$. Let $A = V(K_n) = \{a_1, \cdots, a_n\}$, $E(K_2) = a_s a_t$. So $S = A, 0 \notin \{a_s, a_t\}$. There exists $a_p, a_q \in A - \{a_s, a_t, 0\}$ and $a_p \neq a_q$ such that $a_p + a_q \notin A - \{a_s, a_t\}$ by Lemma 3.13. So $a_p + a_q \in \{a_s, a_t\}$. We may assume $a_s = a_p + a_q$. Then $a_s + a_t = (a_p + a_t) + a_q = (a_q + a_t) + a_p \notin S$. Since $a_p + a_t, a_q + a_t \notin \{a_s, a_t\}$ we have that $a_p + a_t = a_q, a_q + a_t = a_p$. So $m = 2a_t$(where don't take the modulus), $2a_p = 2a_q$. We assume that $ia_p + a_q \in A (i \geq 2)$. If $ia_p + a_q = a_p$ then $(i-1)a_p + a_q = 0$. If $ia_p + a_q \neq a_p$ then $(i+1)a_p + a_q \in A$. Therefore, by the finiteness of A and $a_p + a_q = a_s \in A, 2a_p + a_q = a_p + a_s \in A$ we have that either there exists a $\alpha \geq 2$ such that $\alpha a_p + a_q = 0$ or there exists a $\alpha \geq 3$ such that $\alpha a_p = 0$. We consider the following two cases, respectively.

Case 1: There exists a $\alpha \geq 2$ such that $\alpha a_p + a_q = 0$.

We have that $a_q = -\alpha a_p$. Thus $ia_p + a_q = (i - \alpha)a_p = -(\alpha - i)a_p, i = 1, \cdots, \alpha - 1$, $a_s = a_p + a_q = -(\alpha - 1)a_p$. Since $a_p + a_t = a_q$ we have that $a_p + a_t = -\alpha a_p$, i.e., $a_t = -(\alpha + 1)a_p$. So $a_s + a_t = -2\alpha a_p \notin A$. If $\alpha \neq 2$ then $\alpha \geq 3, a_q = -\alpha a_p \neq -2a_p$. So $-\alpha a_p + (-2a_p) = -(\alpha + 2)a_p \in A$. Since $-(\alpha - 2)a_p \in A$ and $\{-(\alpha - 2)a_p, -(\alpha + 2)a_p\} \neq \{a_s, a_t\}$ we have that $-(\alpha - 2)a_p + [-(\alpha + 2)a_p] = -2\alpha a_p \in A$, which is a contradiction. So $r = 2, a_s = -a_p, a_q = -2a_p, a_t = -3a_p, -4a_p \notin A, -5a_p \in A$. Thus $-5a_p = a_p$(If $-5a_p \neq a_p$ then $-5a_p + a_p = -4a_p \in A$, which is a contradiction.). So $6a_p = 0$. Therefore $a_s = 5a_p, a_q = 4a_p, a_t = 3a_p, 2a_p \notin A$. By $n \geq 6$ we have that there exists $a_i \in A - \{a_p, a_q, a_s, a_t, 0\}$. So $a_i + ja_p \in A, j = 0, 1, \cdots, 5$. Since a_i is not the multiple of a_p we have that $a_i + ja_p \notin \{a_p, a_q, a_s, a_t, 0\} = R, j = 0, 1, \cdots, 5$. So $2a_i + a_p \in A - R, j = 0, 1, \cdots, 5$. If we have known that $ka_i + ja_p \in A - R(k \geq 2), j = 0, 1, \cdots, 5$ then $(k+1)a_i + ja_p \in A (j = 0, 1, \cdots, 5)$ and $(k+1)a_i + ja_p \notin R (j = 0, 1, \cdots, 5)$ (If there exists $(k+1)a_i + ja_p \in R$ then there exists $j' \in \{0, 1, \cdots, 5\}$ such that $(k+1)a_i + j'a_p = 2a_p \notin A$, which is a contradiction.). So $ka_i + ja_p \in A - R$ for any $k \geq 1, 0 \leq j \leq 5$. By the finiteness of A we have that there exist $1 \leq k < l, 0 \leq j_1, j_2 \leq 5$ such that $ka_i + j_1 a_p = la_i + j_2 a_p$. So $(l - k)a_i = (j_1 - j_2)a_p$. Therefore $(l - k)a_i \notin A - R$ and $l - k \geq 1$, which is a contradiction.

Case 2: There exists a $\alpha \geq 3$ such that $\alpha a_p = 0$.

We have that $a_p, a_p + a_q, \cdots, (\alpha - 1)a_p + a_q \in A$ and $(\alpha - 1)a_p + a_q = (\alpha - 1)a_p + (a_p + a_t) = a_t$. So $a_s + a_t = 2a_q = 2a_p \notin A$. Notice that $(2a_p + a_q) + [(\alpha - 2)a_p + a_q] = 2a_q \notin S$. So $\alpha = 4$, i.e., $4a_p = 0$. However $0 < 4a_p < 4m$ and $4a_p \neq m$(If not, $2a_p = \frac{m}{2} = a_t \in A$, which is a contradiction.), $4a_p \neq 2m, 4a_p \neq 3m$(where don't take the modulus). So $4a_p \neq 0$, which is a contradiction. □

By Lemmas 3.14 and 3.15 we have that $\psi(K_n - E(K_r)) = \rho(K_n - E(K_r))$ for $2 \leq r \leq n-4$ and r can not be divided by $n-r$. From the above discussion we establish the following theorem.

Theorem 3.2

$$\psi(K_n - E(K_r)) \begin{cases} = 0, & r \geq n-3, \text{ or } r = 1, \\ = 0, & n - r \mid r, \\ = r, & \frac{n}{2} \leq r \leq n-4 \text{ and } r \text{ can not be} \\ & \text{divided by } n-r, \\ \in [n-1, 2(n-r)], & 2 \leq r < \frac{n}{2},\ n \geq 6. \end{cases}$$

4 Wheels

Theorem 4.1 *$\psi(G) \leq \zeta(G)$ for any graph G.*

Proof. We assume S is the integral sum labelling of graph $G \cup \zeta(G)K_1$. Let

$S' = \{\, u \mid u \in S \text{ and } u \geq 0\} \cup \{\, m + u \mid u \in S \text{ and } u < 0\}$,
where m is the modulus and $m > 3 \max \{\, |u| \mid u \in S\}$. We will prove that S' gives a mod integral sum labelling of $G \cup \zeta(G)K_1$. It is easy to verify that the following assertions are true.

(1) $S' \subset Z_m$.

(2) For any $u \in S$ and any $v \in S (v \neq u)$ suppose that there exists $w \in S$ such that $u + v = w$ (where don't take the modulus). We may assume without loss of generality that $u > v$. Then if $u, v \geq 0$ we have that $w \geq 0$, $u, v, w \in S'$ and $u + v = w$; if $u, w \geq 0, v < 0$ we have that $u, m+v, w \in S'$ and $u + (m+v) = w$; if $u \geq 0, v, w < 0$ we have that $u, m+v, m+w \in S'$ and $u + (m+v) = (m+w)$; if $u, v < 0$ we have that $w < 0$, $m+u, m+v, m+w \in S'$ and $(m+u) + (m+v) = m+w$.

(3) For any $u \in S$ and any $v \in S (v \neq u)$ if $u + v \notin S$ (where don't take the modulus) we may assume without loss of generality that $u > v$. If $u, v \geq 0$ we have that $u+v \geq 0$, $u, v \in S'$, and $u+v \notin S'$(If not, then

there exists $w \in S'$ such that $u+v=w$ and $w \notin S$. So $w-m \in S$ and $u+v=w-m$(where don't take the modulus), contradicting the selection of m.); if $u \geq 0, v<0, u+v \geq 0$ we have that $u, m+v \in S'$, and $u+(m+v)=u+v \notin S'$; if $u \geq 0, v<0, u+v<0$ we have that $u, m+v \in S'$, and $u+(m+v)=u+v+m \notin S'$; if $u, v<0$ we have that $u+v<0$, $m+u, m+v \in S'$, and $(m+u)+(m+v)=u+v+m \notin S'$.

From the above we know that $\psi(G) \leq \zeta(G)$ for any graph G. This completes the proof. □

Theorem 4.2 *For any graph G if there doesn't exist some vertex whose degree is $|V(G)|-1$ or $\zeta(G) \neq 0$ then $\rho(G) \leq \zeta(G)$.*

Proof. If there doesn't exist some vertex whose degree is $|V(G)|-1$ then $\psi(G)=\rho(G)$ and $\psi(G)=\rho(G) \leq \zeta(G)$ by Theorem 4.1. If $\zeta(G) \neq 0$ then there doesn't exist 0 in any integral sum labelling of $G \cup \zeta(G)K_1$. Thus in the proof of Theorem 4.1 S' gives a mod sum labelling of $G \cup \zeta(G)K_1$. So $\rho(G) \leq \zeta(G)$. □

Theorem 4.3 [2] *Wheel W_n is a mod integral sum graph.*

Proof. Since $W_2=K_3, W_3=K_4$ and W_4 is a mod sum graph we have that they are mod integral sum graphs. Since $\zeta(W_n)=0$ for $n \geq 5$ we have that $\psi(W_n)=0$ by Theorem 4.1. □

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