Binomial Identities Generated by Counting Spanning Trees

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Abstract

We show various combinatorial identities that are generated by tree counting arguments. In particular, we give formulas for n^p and $\tau(K_{s,t})$ which establishes an equivalence.

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1 Introduction

We use the standard notation and terminology which can be found, e.g., in [12]. Let $\tau(G)$ denote the number of labelled spanning trees in a graph G. With K_n denoting the complete graph of n vertices and $K_{s,t}$ the complete bipartite graph with partite sets containing s and t vertices, respectively. It is well known [1, 2, 4, 5, 6, 7, 8, 10]:

$$\tau(K_n) = n^{n-2}, \qquad n \geq 2 \tag{1}$$

$$\tau(K_{s,t}) = s^{t-1}t^{s-1}, \qquad s,t \geq 1. \tag{2}$$

We remark that (1) is often referred to as Cayley's theorem. Let $s + t = n$, where $1 \leq s \leq t$. We have the following observation; apparently this was first observed by Austin and Moon [1, 8].

Theorem 1.1. *With $n \geq 2$, any spanning tree T in K_n is a spanning tree in $K_{s,t}$ for a unique pair (s,t) where $1 \leq s \leq t$ and $s + t = n$.*

Proof. Consider a spanning tree T in K_n, then T is a connected bipartite graph and as is well known, necessarily possesses a unique bipartition. So construct this unique bipartition by properly 2-coloring the vertex set of T

with colors red (R) and blue (B). Let the number of red vertices be s and the number of blue vertices be t, w.l.o.g. let $s \leq t$. We then have T is a spanning tree in this $K_{s,t}$. □

The converse is straightforward.

Theorem 1.2. *With $s+t=n$ any spanning tree in $K_{s,t}$ is a spanning tree in K_n.*

Proof. This follows since $K_{s,t}$ is a spanning subgraph of K_n. □

Theorem 1.3. $2\tau(K_n) = \sum_{s=1}^{n-1} \binom{n}{s}\tau(K_{s,n-s})$.

Proof. By combining Theorems 1.1 and 1.2 we see that to find $\tau(K_n)$ we can enumerate all labelled spanning trees in the possible $K_{s,t}$ graphs. □

We rewrite Theorem 1.3 as:

$$\sum_{s=1}^{n-1} \binom{n}{s} \tau(K_{s,n-s}) = 2\tau(K_n). \tag{3}$$

Substituting equations (1) and (2) into (3) yields the identity:

$$\sum_{s=1}^{n-1} \binom{n}{s} s^{n-s-1}(n-s)^{s-1} = 2n^{n-2}. \tag{4}$$

An analytic proof of (4) is forthcoming [3], where we derive the RHS of (4) from the LHS by using partial derivatives and Abel's binomial formula. So in this sense, knowledge of $\tau(K_{s,t})$ implies $\tau(K_n)$, yielding an analytic proof of Cayley's theorem. The ideas in Theorems 1.1, 1.2 are also valid when graphs are unlabelled, since the unique bipartition aspect is a structural property of the graph G. So, for a connected graph G, let $I(G)$ be the number of non-isomorphic spanning trees in G. We have:

Theorem 1.4. $I(K_n) = \sum_{s=1}^{\lfloor n/2 \rfloor} I(K_{s,n-s})$. □

Observational examples of Theorem 1.4 are:

$$\begin{aligned} I(K_6) = 6 &= I(K_{1,5}) + I(K_{2,4}) + I(K_{3,3}) \\ &= 1+2+3 \\ I(K_7) = 11 &= I(K_{1,6}) + I(K_{2,3}) + I(K_{3,4}) \\ &= 1+3+7. \end{aligned}$$

Theorem 1.4 suggests a different approach to enumerating $I(K_n)$ from the usual approach of Polya and Otter. Since a general formula for $I(K_{s,t})$

is unknown, it motivates us to research $I(K_{s,t})$ more deeply. In [7], we have used the automorphism group of $K_{s,t}$ in conjunction with Burnside's formula to establish $I(K_{s,t})$ and hence $I(K_n)$, for $2 \leq n \leq 12$, $s + t = n$.

Getting back to equation (4), one can find a similar formula on Prof. László Székely's home web-page [11], which contains information on Abel's binomial theorem.

$$\text{(Székely)} \qquad \sum_{s=1}^{n-1} \binom{n}{s} s^{s-1}(n-s)^{n-s-1} = 2(n-1)n^{n-2}. \tag{5}$$

Equating (4) and (5) yields an interesting identity:

$$(n-1)\sum_{s=1}^{n-1} \binom{n}{s} s^{n-s-1}(n-s)^{s-1} = \sum_{s=1}^{n-1} \binom{n}{s} s^{s-1}(n-s)^{n-s-1}. \tag{6}$$

We now derive recursive formulas for $\tau(K_{s,t})$ and $\tau(K_n)$ that yield corresponding identities. For a graph G with vertex set $V(G) = \{1, 2, \ldots, n\}$, let A_i denote the set of spanning trees T in G where vertex i is a leaf in T, i.e., $\deg_T(i) = 1$.

Theorem 1.5. *Let $s + t = n$, with $2 \leq s \leq t$, then*

$$\tau(K_{s,t}) = \sum_{i=1}^{t-1} (-1)^{i-1} s^{t-1}(t-i)^{s-1}.$$

Proof. For the graph $K_{s,t}$, let X denote the set of s vertices in the one partite set, Y the vertices in the t-set, i.e., $K_{s,t} = K_{|X|,|Y|}$. Since $2 \leq s \leq t$, i.e., $|X| \leq |Y|$, observe that necessarily any spanning tree T in $K_{s,t}$ must contain a leaf vertex $y \in Y$. This follows since otherwise all vertices $y \in Y$ would then have $\deg_T(y) \geq 2$, and

$$e(T) = \sum_{y \in Y} \deg_T(y) \geq 2|Y| \geq |X| + |Y| = n > n-1,$$

which contradicts that T is a tree with $n-1$ edges. Here $e(T)$ denotes the number of edges in T. Let $Y = \{y_1, y_2, \ldots, y_t\}$, then for any spanning tree T in $K_{s,t}$, we have $T \in A_{y_i}$ for some $y_i \in Y$. Consequently $\tau(K_{s,t}) = |A_{y_1} \cup A_{y_2} \cdots \cup A_{y_t}|$. By the principle of inclusion-exclusion, we have;

$$|A_{y_1} \cup \cdots \cup A_{y_t}| = \sum_{i=1}^{t-1} (-1)^{i-1} \binom{t}{i} \tau(K_{s,t-i}) s^i.$$

Using equation (2) for $\tau(K_{s,t-i})$ gives:

$$s^{t-1}t^{s-1} = \tau(K_{s,t}) = \sum_{i=1}^{t-1}(-1)^i\binom{t}{i}s^{t-i-1}(t-i)^{s-1}s^i$$
$$= \sum_{i=1}^{t-1}(-1)^i\binom{t}{i}s^{t-1}(t-i)^{s-1}.$$

□

Since the LHS and RHS of the equation in Theorem 5 both contain the term s^{t-1}, we obtain the identity:

$$\text{for } 2 \le s \le t, \qquad t^{s-1} = \sum_{i=1}^{t-1}(-1)^{i-1}(t-i)^{s-1}\binom{t}{i}. \tag{7}$$

With $t = n$ and $p = s - 1$ we rewrite (7) as:

$$n^p = \sum_{i=1}^{n-1}(-1)^{i-1}(n-i)^p\binom{n}{i}, \text{ for integers } 1 \le p,\ p < n,\ n \ge 2. \tag{8}$$

We remark that by using the principle of inclusion-exclusion and the formula for $\tau(K_{s,t})$ we obtained (8). We can also use (8) to obtain $\tau(K_{s,t})$ by using induction on $n = s + t$ and the inclusion-exclusion identity given in Theorem 1.5. The equation (8) is easily derived by considering the number of surjections from $[p]$ to $[n]$. The number of surjections as an inclusion exclusion formula is well-known, and (8) is produced when $p < n$, i.e., there are no surjections. This establishes the equivalence of (2) and (8).

For the case of K_n, with $V(K_n) = \{1, \ldots, n\}$, and again let A_i be the set of spanning trees T in K_n where vertex i is a leaf in T. We have:

Theorem 1.6. $\tau(K_n) = \sum_{i=1}^{n-1}(-1)^{i-1}\binom{n}{i}(n-i)^{n-2},\ n \ge 3.$

Proof. Since any spanning tree T in K_n with $n \ge 3$ must contain a leaf vertex, we have $\tau(K_n) = |A_1 \cup A_2 \cdots \cup A_n|$. Notice that all n vertices cannot be leafs. By the principle of inclusion-exclusion, we have;

$$|A_1 \cup \cdots \cup A_n| = \sum_{i=1}^{n-1}(-1)^i\binom{n}{i}(n-i)^i\tau(K_{n-i}).$$

Replacing $\tau(K_{n-i})$ with equation (1) yields the theorem. □

Applying equation (1) to the LHS of Theorem 1.6 gives the identity:

$$n^{n-2} = \sum_{i=1}^{n-1}(-1)^{i-i}\binom{n}{i}(n-i)^{n-2}. \tag{9}$$

J.W. Moon [9] also derives (9) by an inclusion-exclusion method. It is interesting that letting $p = n-2$ in (8) gives (8) = (9). This seems surprising since the motivational arguments come from spanning trees in two different families of graphs, namely, K_n and $K_{s,t}$. However, the connection between the two sets of trees, as indicated in Theorem 1.3, perhaps explains this. We would like to derive similar decompositions for graph families other than the complete and complete bipartite graphs (e.g., the hypercube). Also, we would like to extend these results to q-analogues of the counts, arriving at identities involving q-binomial coefficients.

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