Realizability Results Involving Two Connectivity Parameters

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Abstract

There are networks that can be modeled by simple graphs, where edges are perfectly reliable but nodes are subject to failure, e.g. hardwired computer systems. One measure of the "vulnerability" of the network is the connectivity κ of the graph. Another, somewhat related, vulnerability parameter is the component order connectivity $\kappa_c^{(k)}$, i.e. the smallest number of nodes that must fail in order to ensure that all remaining components have order less than some value k. In this paper we present necessary and sufficient conditions on a 4-tuple (n,k,a,b) for a graph G to exist having n nodes, $\kappa = a$, and $\kappa_c^{(k)} = b$. Sufficiency of the conditions follows from a specific construction described in our work. Using this construction we obtain ranges of values for the number of edges in a graph having n nodes, $\kappa = a$, and $\kappa_c^{(k)} = b$ thereby obtaining sufficient conditions on the 5-tuple (n,e,k,a,b) for a graph to exist having n nodes, e edges, $\kappa = a$, and $\kappa_c^{(k)} = b$. In a limited number of special cases, we show the conditions on (n,e,k,a,b) to be necessary as well.

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0 Introduction

We consider two notions for node failure. Let $G(V, E)$ be a simple undirected graph where $|V| = n$ nodes and $|E| = e$ edges serving as a model for network node failure. The more traditional model uses the parameter connectivity as follows. When nodes fail in $G(V,E)$ and W is the set of surviving nodes then the surviving subgraph is $\langle W \rangle$ where $\langle W \rangle$ denotes the subgraph induced by W. We say $W \subseteq V$ is an **operating state** iff $\langle W \rangle$ is connected and $|W| > 1$; otherwise we say W is a **failure state.**

Definition: The **connectivity** of G, denoted by $\kappa(G)$ or simply κ, is the minimum $|D|$ such that $D \subseteq V$ and $\langle V - D \rangle$ is a **failure state,** i.e., $\langle V - D \rangle$ is disconnected or trivial.

There are some inadequacies inherent in the traditional model. A failure state occurs when the surviving subgraph is either disconnected or trivial. Thus, a failure state can have a large component, a subset of a failure state can be an operating state, and relatively small operating states are tolerated.

In the new model [1,2], which addresses each of these inadequacies, we say that $W \subseteq V$ is an **operating state** if and only if $\langle W \rangle$ contains a component of order $\geq k$, where k is some predetermined number; otherwise we say that W is a **failure state.**

Definition: Given $n \geq k \geq 2$, the **k-component connectivity** or **component order connectivity** of G, denoted by $\kappa_c^{(k)}(G)$ or simply $\kappa_c^{(k)}$, is the minimum $|D|$ such that $D \subseteq V$ and $\langle V - D \rangle$ is a **failure state,** i.e., all components of $\langle V - D \rangle$ have order at most k-1.

There are two ways for a failure state to occur:

(1) $\langle V - D \rangle$ is disconnected and all component orders are less than or equal to $k - 1$;
(2) $\langle V - D \rangle$ is connected but $|V - D| \leq k - 1$ i.e. $|D| \geq n - (k - 1)$.

Since we can always create a failure state by removing $n - (k - 1)$ nodes, it follows that $\kappa_c^{(k)}(\mathrm{G}) \leq n - (k - 1)$.

The following theorem, which relates κ and $\kappa_c^{(k)}$, due to Boesch, Gross, and Suffel is included without proof.

Theorem 0.1 [1] Let G be a graph on n nodes and $2 \le k \le n$.

(1) If $\kappa \ge n-(k-1)$ then $\kappa_c^{(k)} = n-(k-1)$.

(2) If $\kappa \le n-k$ then $\kappa \le \kappa_c^{(k)} \le n-k$.

Although there are some inadequacies in the traditional connectivity model, it does have a very nice routing feature. Namely, by the Menger-Whitney theorem [3], maximizing the connectivity maximizes the number of node disjoint paths over all pairs of non-adjacent nodes in the graph. In view of this it is both a natural and pragmatic question as to the nature of graphs which simultaneously maximize κ and $\kappa_c^{(k)}$ [2]. One approach for obtaining such information is to study the simultaneous realizability of κ and $\kappa_c^{(k)}$. This is the main topic of this work and discussed in the next section.

1 Realizability Results

Recalling Theorem 0.1, we see that two realizability questions arise:
(1) Given $a \ge n-(k-1)$ does there exist a G such that $|V(G)| = n$, $\kappa = a$, and $\kappa_c^{(k)} = n-(k-1)$?
(2) Given $0 < a \le b \le n-k$ does there exist a G such that $\kappa = a$ and $\kappa_c^{(k)} = b$?

In this section, we answer these two questions in the affirmative. Furthermore, we provide graphs in answer to (1) for all possible $e \ge \left\lceil \frac{na}{2} \right\rceil$. The graphs given in answer to (2) include a large range of values for e.

To answer (1) we need the following lemma.

Lemma 1.1 [4] There exists a graph G having n nodes, e edges, and connectivity $\kappa = a$ if and only if either

(1) $n-1 \le e \le \frac{(n-1)(n-2)}{2}+1$ when $a = 1$ or

(2) $\left\lceil \frac{na}{2} \right\rceil \le e \le \frac{(n-1)(n-2)}{2}+a$ when $a > 1$.

Theorem 1.2 The 5-tuple (n,e,k,a,b) where $a \geq n-k+1$ is realizable by a n-node, e-edge graph G with $\kappa(G)=a$ and $\kappa_c^{(k)}(G)=b$ if and only if either

(1) $a=b=1$, $n=k$ and $k-1 \leq e \leq \frac{(k-1)(k-2)}{2}+1$ or

(2) $a \geq 2$, $b=n-k+1$ and $\left\lceil \frac{na}{2} \right\rceil \leq e \leq \frac{(n-1)(n-2)}{2}+a$

Proof: If the graph G has $\kappa(G) \geq n-k+1$ then we know from Theorem 0.1(1) that $\kappa_c^{(k)}(G)=n-k+1$. Therefore, we observe the result due to Lemma 1.1. ■

We now present a construction to show that given $0 \leq a \leq b \leq n-k$, where $2 \leq k \leq n$, there exists a graph G on n nodes having $\kappa = a$ and $\kappa_c^{(k)} = b$.

The Construction

We begin with the node set of our construction G_1 consisting of three pairwise disjoint sets, V, U, and S the orders of which are to be specified.

<u>*Case I:*</u>
For $\alpha \geq 0$ and $k \geq 2$ consider n and β satisfying
(1) $k+\alpha \leq n \leq 2k+\alpha$
(2) $1 \leq \beta \leq \alpha+1$

Then set $|U|=k+\alpha-\beta, |V|=\beta, |S|=n-(k+\alpha)$, and define the edge set $E(G_1)$ to be $\{\{v,s\} \mid v \in U \cup V, s \in S\} \cup E(\langle U \rangle) \cup E(\langle S \rangle)$ (see Figure 1.1) where $\langle U \rangle$ is chosen such that $\kappa(\langle U \rangle) \geq \alpha-\beta+1$, $\langle V \rangle$ is chosen such that $E(\langle V \rangle)=\varnothing$, and $E(\langle S \rangle)$ is arbitrary. Note that in the event that $n=k+\alpha$, $S=\varnothing$ and $E(G_1)=E(\langle U \rangle)$.

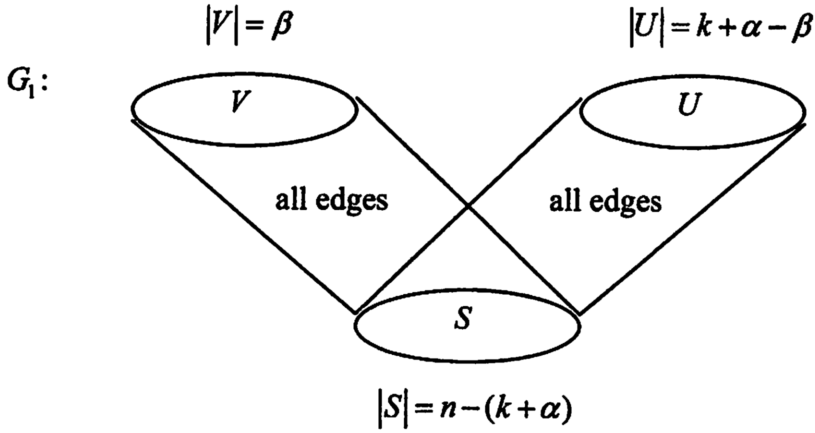

Figure 1.1

Case II:
For $\alpha \geq 0$ and $k \geq 2$ consider n and β satisfying
(1) $n \geq 2k+\alpha+1$
(2) $1 \leq \beta \leq \alpha+1$
Then with $|U|$, $|V|$, and $|S|$ as in Case I, define the edge set $E(G_1)$ to be $\{\{v,s\} | v \in U \cup V, s \in S\} \cup E(\langle U \rangle) \cup E(\langle S \rangle)$ (see Figure 1.1) where $\langle U \rangle, \langle V \rangle$, and $\langle S \rangle$ are chosen such that

- $\kappa(\langle U \rangle) \geq \alpha-\beta+1$
- $\kappa(\langle S \rangle) \geq n-2k-2\alpha$
- $\kappa_c^{(k)}(\langle S \rangle) \geq n-2k-\alpha-\beta+1$
- $E(\langle V \rangle) = \varnothing$.

Theorem 1.3 The graph G_1 has *n* nodes, $\kappa(G_1) = n-k-\alpha$, and $\kappa_c^{(k)}(G_1) = n-k-\beta+1$.

Proof: Since G_1 is spanned by a complete bipartite graph with parts *S* and $U \cup V$ any minimum disconnecting set must include either *S* or $U \cup V$. In case I $|S| = n-k-\alpha < k+\alpha = |U \cup V|$ so any disconnecting set which contains $U \cup V$ has more than $n-k-\alpha$ nodes. However, *S* itself is a disconnecting set so it is a minimum disconnecting set. Thus, $\kappa(G_1) = n-k-\alpha$ in case I. In case II removal of S disconnects G_1 so $\kappa(G_1) \le n-k-\alpha$. Any disconnecting set containing $U \cup V$ must also contain at least $n-2k-2\alpha$ nodes from S in order to insure that the surviving subgraph is disconnected. Thus any such disconnecting set contains at least $n-2k-2\alpha+(k+\alpha-\beta+\beta) = n-k-\alpha$ nodes. Hence $\kappa(G_1) = n-k-\alpha$ in case II as well.

We define a $\kappa_c^{(k)}$**-cut** to be a set of nodes whose removal from G_1 creates a surviving subgraph with all component orders less than or equal to $k-1$. In turn, we define a $\kappa_c^{(k)}$**-set** to be a minimum $\kappa_c^{(k)}$-cut.

In either of the two cases $\kappa \le n-k$, so it follows by Theorem 0.1(2) that any $\kappa_c^{(k)}$-set must disconnect G_1 upon removal. Thus, if D is a $\kappa_c^{(k)}$-set then either $S \subseteq D$ or $U \cup V \subseteq D$.

Suppose $S \subseteq D$. If $|D \cap U| \le \alpha - \beta$ then $\langle U-(D\cap U)\rangle$ is connected and has at least k nodes. Thus $|D \cap U| \ge \alpha-\beta+1$ and so

$$\begin{aligned} |D| &\ge |D\cap S| + |D \cap U| \\ &\ge n-k-\alpha+\alpha-\beta+1 \\ &= n-k-\beta+1 \end{aligned}$$

in both cases I and II.

On the other hand if $U \cup V \subseteq D$ then in case I

$$\begin{aligned} |D| &\ge (k+\alpha-\beta)+\beta \\ &= k+\alpha \\ &\ge n-k \\ &\ge n-k-(\beta-1). \end{aligned}$$

In case II $|D \cap S| \ge n-2k-\alpha-\beta+1$ or else removal of D leaves a component within $\langle S\rangle$ of order k or more. Thus

$$\begin{aligned} |D| &\ge |U \cup V| + |D \cap S| \\ &\ge (k+\alpha-\beta)+\beta+n-2k-\alpha-\beta+1 \\ &= n-k-\beta+1. \end{aligned}$$

Hence in both cases $\kappa_c^{(k)}(G_1) \ge n-k-\beta+1$.

Finally, in either case removal of S together with a minimum disconnecting set of $\langle U\rangle$ is a $\kappa_c^{(k)}$-cut of order $n-k-\beta+1$ so $\kappa_c^{(k)}=n-k-\beta+1$ in both cases.

■

Now recall that, in case II, we require a "special" $\langle S\rangle$; the question is whether $\langle S\rangle$ always exists? We answer the question in the affirmative. Indeed, as shown in Theorem 1.4-case I when $\alpha\geq 0$, $k\geq 2$, and $1\leq\beta\leq\alpha+1$ are given there exists a graph on n nodes, where $k+\alpha+1\leq n\leq 2k+\alpha$, such that $\kappa=n-k-\alpha$ and $\kappa_c^{(k)}=n-k-\beta+1$. Thus one can insert a graph obtained in this case for S in the construction of case II to obtain all the constructions required for n where $2k+2\alpha+1\leq n\leq 3k+2\alpha$. One proceeds in an inductive manner to obtain all the constructions needed for values of n that satisfy $ik+i\alpha+1\leq n\leq(i+1)k+i\alpha$ for $i\geq 1$. Indeed, for $i\geq 2$ a graph obtained for $(i-1)k+(i-1)\alpha+1\leq n\leq ik+(i-1)\alpha$ is inserted for S in the construction of case II to create a graph having $ik+i\alpha<n<(i+1)k+i\alpha$.

Our next concern is to fill in the remaining intervals, i.e. $ik+(i-1)\alpha+1\leq n\leq ik+i\alpha$, for $i\geq 2$. In order to do this by using the inductive application of case II we need to investigate an additional initial case, i.e. one for which $i=1$, namely $k+1\leq n\leq k+\alpha$. We require a graph that satisfies $n=k+j$ $(1\leq j\leq\alpha)$, $\kappa=0$, and $\kappa_c^{(k)}\geq j$. Indeed choose a graph consisting of a component of order $j+k-1$ and connectivity at least j together with an isolated node. Then, if

$$2k+\alpha+1\leq n\leq 2k+2\alpha$$

it follows that

$$k+1\leq n-(k+\alpha)\leq k+\alpha.$$

With $n=2k+\alpha+j$, where $1\leq j\leq\alpha$, we substitute the graph equal to the one just described for S in the case II construction. Observe that

$$\kappa(\langle S\rangle)=0\geq n-(2k+2\alpha)$$

and

$$\kappa_c^{(k)}(\langle S\rangle)=j=n-(2k+\alpha)\geq n-2k-\alpha-\beta+1.$$

Thus, the case II construction yields a graph satisfying the conditions of Theorem 1.4 for $2k+\alpha+1\leq n\leq 2k+2\alpha$. Continued inductive use of the case II construction by placing each constructed G into the node set S yields all cases satisfying $ik+(i-1)\alpha+1 \le n \le ik+i\alpha$.

Corollary 1.4 Given $0 \le a \le b \le n-k$ there exists a graph G on n nodes such that $\kappa(G)=a$ and $\kappa_c^{(k)}(G)=b$.

Proof: We construct such a graph G as follows:
set

$$a=n-(k+\alpha)$$

to obtain

$$\alpha=(n-k)-a;$$

set

$$b=n-k-\beta+1$$

to obtain

$$\beta=(n-k)-b+1.$$

1) If $a \le k$ let G be a graph obtained in the case I construction where $k+\alpha \le n \le 2k+\alpha$.

2) If $a \ge k+1$ and $\alpha \ge 1$ then first determine whether $a \le \left\lfloor \frac{n}{2} \right\rfloor$.

   (a) If $a \le \left\lfloor \frac{n}{2} \right\rfloor$, so that $2k+\alpha+1 \le n \le 2k+2\alpha$, obtain G from case II construction where $\langle S \rangle$ is a graph on a nodes with a component of order $a-1$ and connectivity of the component being at least $a-k$.

   (b) If $a > \left\lfloor \frac{n}{2} \right\rfloor$ use the division algorithm to determine i such that $i(n-a) < n \le (i+1)(n-a)$ where, of course, $i \ge 2$.

   (*i*) If $i(n-a)+1 < n \le i(n-a)+k$ make $i-1$ successive replacements of the graph obtained from the case II construction beginning with the case I construction on $n-(i-1)(n-a)$ nodes.

   (*ii*) If $i(n-a)+k+1 < n \le (i+1)(n-a)$ make $i-1$ successive replacements of the graph obtained from the case II construction beginning with the graph of 2(a) on $n-(i-1)(n-a)$ nodes.

Having made these specifications, we need only observe that

$\alpha=(n-k)-a \ge 0$ since $a \le n-k$;

$\beta = (n-k) - b + 1 \geq 1$ since $b \leq n-k$; and
$\beta \leq \alpha + 1$ since $a \leq b$
for then G may be constructed by Theorem 1.3. ■

2 Edge Count for Construction G_1

In our previous discussion we established realizability of graphs having n nodes, $\kappa = n-k-\alpha$, and $\kappa_c^{(k)} = n-k-\beta+1$ where $n \geq k+\alpha+1$, $k \geq 2$, and $0 \leq \beta - 1 \leq \alpha$. In this discussion, we indicate the minimum and maximum number of edges obtainable for the construction previously described.

For the scenarios of the construction G_1 let e_1 denote the number of edges in G_1. The maximum value of e_1 is determined by setting $\langle U \rangle = K_{k+\alpha-\beta}$ and $\langle S \rangle = K_{n-(k+\alpha)}$. Thus

$$\max e_1 = (k+\alpha)(n-(k+\alpha)) + \left|E(\langle S \rangle)\right| + \left|E(\langle U \rangle)\right|$$

$$= (k+\alpha)(n-(k+\alpha)) + \frac{(n-k-\alpha)(n-k-\alpha-1)}{2} + \frac{(k+\alpha-\beta)(k+\alpha-\beta-1)}{2}.$$

With $a = \kappa$ and $b = \kappa_c^{(k)}$ we obtain

$$\max e_1 = (n-a)a + \frac{a(a-1)}{2} + \frac{(k+b-a-1)(k+b-a-2)}{2}$$

because $k+\alpha = n-a$ and $\alpha - \beta + 1 = b - a$.

Note that when $a = 0$ $\max e_1 = \dfrac{(k+b-1)(k+b-2)}{2}$.

The value of $\min e_1$ is first determined for n in the interval

$$(k+\alpha) \leq n \leq 2(k+\alpha)$$

and then for successive intervals

$$i(k+\alpha)+1 \leq n \leq (i+1)(k+\alpha), \quad i \geq 2$$

by solving a first order linear recurrence relation. The results of this analysis, the details of which may be found in [5], are expressed in terms of n, k, $a=\kappa$ and $b=\kappa_c^{(k)}$ as follows:

i) for $n-a \le n \le 2(n-a)$

$m_1(n) \triangleq \min e_1 =$

- $(n-a)a+\left\lceil \frac{(k+b-a-1)(b-a)}{2} \right\rceil$ for $0 \le a \le k$ and $b-a \ne 1$;
- $(n-a)a+(k-1)$ for $0 \le a \le k$ and $b-a=1$;
- $(n-a)a+\left\lceil \frac{(k+b-a-1)(b-a)}{2} \right\rceil+\left\lceil \frac{(a-1)(a-k)}{2} \right\rceil$ for $k+1<a \le \left\lfloor \frac{n}{2} \right\rfloor$ and $b-a \ne 1$;
- $(n-a)a+k-1+\left\lceil \frac{(a-1)(a-k)}{2} \right\rceil$ for $k+1<a \le \left\lfloor \frac{n}{2} \right\rfloor$ and $b-a=1$;
- $(n-a)a+\left\lceil \frac{(k+b-a-1)(b-a)}{2} \right\rceil+a-2$ for $a=k+1$ and $b-a \ne 1$;
- $(n-a)a+k+a-3$ for $a=k+1$ and $b-a=1$; for $1 \le a \le b \le n-k$.

ii) for $i(n-a)+1 \le n \le (i+1)(n-a)$, $i \ge 2$

$$\min e_1 = \frac{(n-a)^2}{2} i^2 + \left[\frac{(n-a)^2}{2} + s \right] i + m_1(n-(i-1)(n-a)) - (n-a)^2 - s$$

where m_1 is given in *i*) and

$$s = \begin{cases} [n-(i+1)(n-a)](n-a)+\left\lceil \frac{(k+b-a)(b-a)}{2} \right\rceil \text{ for } b-a \ne 1 \\ [n-(i+1)(n-a)](n-a)+k-1 \text{ for } b-a=1 \end{cases}$$

Remark 1 The statement given above is expressed in terms of intervals determined by fixing $n-a=\alpha+k$ since our construction is developed in this way. Perhaps a more "natural" way to arrange the results is to consider n fixed and let a vary over the integers $0,1,\dots,n-1$. Indeed it is the case that

$$n-a \le n \le 2(n-a) \text{ iff } 0 \le a \le \left\lfloor \frac{n}{2} \right\rfloor$$

and, for $i \ge 2$

$$i(n-a)+1 \le n \le (i+1)(n-a) \text{ iff } \left\lfloor \frac{i-1}{i} n \right\rfloor + 1 \le a \le \left\lfloor \frac{i}{i+1} n \right\rfloor .$$

Remark 2 Since the addition of edges to a graph cannot reduce κ or $\kappa_c^{(k)}$ any edge count lying between $\min e_1$ and $\max e_1$ is attainable. We state the complete result next.

Theorem 2.1 If $0 \le a \le b \le n-k$ and $\min e_1 \le e \le \max e_1$, then there exists a G on n nodes and e edges having $\kappa = a$ and $\kappa_c^{(k)} = b$.

Conclusions Given any 5-tuple $(n, e, k, \kappa, \kappa_c^{(k)})$, with each variable within the range of values previously mentioned, there exists a graph having those parameters. The condition on the 4-tuple $(n, k, \kappa, \kappa_c^{(k)})$ is necessary and sufficient but the 5-tuple $(n, e, k, \kappa, \kappa_c^{(k)})$ condition is only sufficient.

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